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WEIGHTED APPROXIMATION OF UNBOUNDED CONTINUOUS FUNCTIONS BY SEQUENCES OF LINEAR POSITIVE OPERATORS

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We prove two weighted Korovkin type theorem on the approximation of unbounded continuous functions by linear positive operators on the whole real axis. A construction of a sequence of linear positive operators is given which satisfies our results by applying to Gadjev-Ibragimov operators.

Key Words : Weighted Spaces; Korovkin Type Theorem; Linear Positive Operators; Gadjiev-Ibragimov Operators

1. INTRODUCTION

The problem of the approximation of unbounded continuous functions by sequences of linear positive operators have been investigated in many papers. We refer to some papers[6, 7, 10, 11, 13 & 14] and the monography[1]. Korovkin type theorems on the weighted approximation of unbounded continuous functions on unbounded sets with a single weight function were first established by A. D.Gadjiev in [6] and [7]. In this paper as a generalization of[6&7] we give two weighted Korovkin type theorems. We deal with the following weighted spaces of functions. Let $\rho_k \geq 1$ for $k = 1, 2$ be continuous, unbounded functions on $I\!R$ and $B_{\rho k} := \left\{ f : \| f(x) | \leq M_f \cdot \rho_k(x), -\infty < x < \infty \right\}$, C_{ρ_k} : $\{f : f \in B_{\rho_k}$ and f continuous$\}$ are the spaces of functions which are defined on unbounded regions.

First we give the following properties of positive linear operators acting between these spaces[3]:

1. A positive linear operator L, defined on C_{ρ_1}, maps C_{ρ_1} into B_{ρ_2} iff the inequality

   $\| L(\rho_1, x) \|_{\rho_2} \leq M_1$ holds.

2. Let $L : C_{\rho_1}(I\!R) \rightarrow B_{\rho_2}(I\!R)$ be a positive linear operator. Then

   $$\| L \|_{C_{\rho_1} \rightarrow B_{\rho_2}} = \| L(\rho_1, x) \|_{\rho_2}$$

and therefore for all $f \in C_{\rho_1}$, the inequality

$$\| L(f, x) \|_{\rho_2} \leq \| L(\rho_1, x) \|_{\rho_2} \| f \|_{\rho_1} \text{ holds.}$$

3. For every $n \geq 1$ let

$$A_n : C_{\rho_1} \to B_{\rho_2}$$

be a positive linear operator. Suppose that there exists $M > 0$ such that for all $x \in I\!R$, $\rho_1(x) \leq M, \rho_2(x)$. If

$$\lim_{n \to \infty} \| A_n(\rho_1, x) - \rho_1(x) \|_{\rho_2} = 0,$$

then the sequence of norms $\| A_n \|_{C_{\rho_1} \to B_{\rho_2}}$ is uniformly bounded.

Remark 1.1 : From the condition $\rho_1(x) \leq M \rho_2(x)$ it follows that $C_{\rho_1} \subset C_{\rho_2} \subset B_{\rho_2}$.

2. Main Results

In this section we prove some theorems on weighted approximation of continuous functions by sequences of linear positive operators, mapping C_{ρ_1} into B_{ρ_2}.

Theorem 2.1 — *Suppose* $\lim_{x \to \infty} \frac{\rho_1(x)}{\rho_2(x)} = 0$. *Then, for a sequence of positive linear operators* $A_n : C_{\rho_1} \to B_{\rho_2}$ *we have*

$$\lim_{n \to \infty} \| A_n(f, x) - f(x) \|_{\rho_2} = 0$$

$$\text{iff} \lim_{n \to \infty} \| A_n(F_j, x) - F_j(x) \|_{\rho_2} = 0, \; j = 0, 1, 2, \qquad \text{... (1)}$$

where $$F_j(x) := \frac{x^j}{1 + x^2} \rho_1(x) \quad (x \in I\!R).$$

PROOF : For $j = 0, 1, 2$ and $x \in I\!R$, we have $\frac{x^j}{1+x^2} \leq 1$; therefore, $| F_j(x) | \leq \rho_1(x)$.

This implies $F_j \in C_{\rho_1}$. This shows the first part of the statement.

Conversely, suppose that $\lim_{n\to\infty} \| A_n(F_j, x) - F_j(x) \|_{\rho_2} = 0$ for $j = 0, 1, 2$. We have to show that for all $f \in C_{\rho_1}$

$$\lim_{n\to\infty} \| A_n(f, x) - f(x) \|_{\rho_2} = 0.$$

To see this, as in[4], we must show that the sequence of $\| A_n \|_{C_{\rho_1} \to B_{\rho_2}}$ of operators norms are uniformly bounded and also for any arbitrary s_0, with $|x| \le s_0$

$$\lim_{n\to\infty} | A_n(f, x) - f(x) | = 0.$$

Since $\lim_{x\to\infty} \dfrac{\rho_1(x)}{\rho_2(x)} = 0$, for $M > 0$ we have

$$\| \rho_1(x) \|_{\rho_2} = \sup_{x \in \mathbb{R}} \frac{\rho_1(x)}{\rho_2(x)} \le M.$$

Using the Property 2 we obtain

$$\| A_n \|_{C_{\rho_1} \to B_{\rho_2}} = \| A_n(\rho_1, x) \|_{\rho_2}$$

$$\le \left\| A_n\left(\frac{1}{1+t^2} \rho_1, x \right) - \frac{1}{1+x^2} \rho_1(x) \right\|_{\rho_2}$$

$$+ \left\| A_n\left(\frac{t^2}{1+t^2} \rho_1, x \right) - \frac{x^2}{1+x^2} \rho_1(x) \right\|_{\rho_2} + M$$

$$= \| A_n(F_0, x) - F_0(x) \|_{\rho_2} + \| A_n(F_2, x) - F_2(x) \|_{\rho_2} + M$$

By using the assumption (1), there exists an $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$

$$\| A_n \|_{C_{\rho_1} \to B_{\rho_2}} \le 2 + M.$$

If we denote

$$K := \left\{ \| A_1 \|_{C_{\rho_1} \to B_{\rho_2}}, \ldots, \| A_{n_0} - 1 \|_{C_{\rho_1} \to B_{\rho_2}}, 2 + M \right\}$$

then for all $n \in \mathbb{N}$ we have

$$\| A_n \|_{C_{\rho_1} \to B_{\rho_2}} \le K.$$

This means that the sequence $\left(\| A_n \|_{C_{\rho_1} \to B_{\rho_2}} \right)_{n \in \mathbb{N}}$ is uniformly bounded.

Next we will show that for an arbitrary s_0, with $|x| \le s_0$ and for all $f \in C_{\rho_1}$

$$\lim_{n \to \infty} |A_n(f, x) - f(x)| = 0.$$

Since A_n is a positive linear operator for each $n \in \mathbb{N}$ we get

$$|A_n(f, x) - f(x)| \le A_n(|f(t) - f(x)|, x) + |f(x)|\,|A_n(1, x) - 1|$$

Now let

$$S_n'(x) := A_n(|f(t) - f(x)|, x) \text{ and } S_n''(x) := |f(x)|\,|A_n(1, x) - 1| \qquad \dots (2)$$

By the definition of F_j

$$A_n((t-x)^2 F_0(t), x) \le |A_n(F_2, x) - F_2(x)| + 2|x|\,|A_n(F_1, x) - F_1(x)|$$
$$+ x^2 |A_n(F_0, x) - F_0(x)|.$$

Using the assumption (1) we can write

$$|A_n(F_j, x) - F_j(x)| \le \varepsilon_n \rho_2(x) \qquad \dots (3)$$

where $(\varepsilon_n)_{n \in \mathbb{N}}$ is a zero sequence. So we obtain

$$A_n((t-x)^2 F_0(t), x) \le \varepsilon_n \rho_2(x)(1 + 2|x| + x^2) \le \varepsilon_n \rho_2(x)\, 2(1 + x^2).$$

This implies

$$\lim_{n \to \infty} A_n((t-x)^2 F_0(t), x) = 0. \qquad \dots (4)$$

Since f is continuous on $\mathbb{R}$ for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(t) - f(x)| < \varepsilon$$

for all $|x| \le s_0$ for which $|t - x| < \delta$. If $|t - x| \ge \delta$, we obtain

$$|f(t) - f(x)| \le 4M_f \rho_1(x) F_0(t) \left[\frac{|t-x|^2}{\delta^2}(1 + |x|^2) + |t - x|^2 \right]$$
$$= 4M_f \rho_1(x) F_0(t) |t-x|^2 \left[\frac{1 + |x|^2}{\delta^2} + 1 \right].$$

If we denote

$$K_{\rho_1}(x) := 4M_f \rho_1(x)\left(\frac{1+|x|^2}{\delta^2}+1\right)$$

for all $t \in I\!R$ and $x \in I\!R$ with $|x| \le s_0$ we get

$$|f(t)-f(x)| \le \varepsilon + K_{\rho_1}(x)(t-x)^2 F_0(t). \qquad \text{... (5)}$$

Firstly we examine $S_n'(x)$. By using (5) we obtain

$$S_n'(x) = A_n(|f(t)-f(x)|, x)$$

$$\le \varepsilon A_n(1,x) + K_{\rho_1}(x) A_n((t-x)^2 F_0(t)\, x)$$

On the other hand we have the following inequality

$$0 \le \limsup_{n\to\infty} S_n'(x) \le \lim_{n\to\infty} (\varepsilon A_n(1,x) + K_{\rho_1}(x) A_n((t-x)^2 F_0(t), x)).$$

By definition, $\rho_1(x) \ge 1$ and so $A_n(1,x) \le A_n(\rho_1, x)$ and also for an $M_1 > 0$, since $\| A_n(\rho_1, x)\|_{\rho_2} \le M_1$ we have $A_n(\rho_1, x) \le M_1 \rho_2(x)$. So, by using (4) and for all $|x| \le s_0$ we have

$$0 \le \limsup_{n\to\infty} S_n'(x) \le \varepsilon M_1 \rho_2(x).$$

Since $\varepsilon > 0$ is given arbitrarily

$$\lim_{n\to\infty} S_n'(x) = 0.$$

Finally, let's examine $S_n''(x)$. Note that since f is continuous, and so it is bounded on the closed interval $|x| \le s_0$. Since $S_n''(x) = |f(x)|\,|A_n(1,x) - 1|$, therefore it is enough to show that

$$\lim_{n\to\infty} |A_n(1,x) - 1| = 0.$$

Then, by using the facty that $|a+b| \ge |a| - |b|$ we get

$$|A_n(F_0, x) - F_0(x)| = |A_n(F_0(t) - F_0(x) + F_0(x), x) - F_0(x)|$$

$$\ge |F_0(x)|\,|A_n(1,x) - 1| - |A_n(F_0(t) - F_0(x), x)|.$$

So by (4) we obtain

$$|F_0(x)|\,|A_n(1,x) - 1| \le |A_n(F_0, x) - F_0(x)| + |A_n(F_0(t) - F_0(x), x)|$$

$$\leq \varepsilon_n \rho_2(x) + A_n(|F_0(t) - F_0(x)|, x).$$

Since $F_0(x) \in C_{\rho_1}$ and using the inequality (5) we get

$$|F_0(t) - F_0(x)| \leq \varepsilon + K_{\rho_1}(x)(t-x)^2 F_0(t)$$

and hence

$$|F_0(x)||A_n(1,x) - 1| \leq \varepsilon_n \rho_2(x) + \varepsilon A_n(1,x) + K_{\rho_1}(x) A_n((t-x)^2 F_0(t), x).$$

So for $|x| \leq s_0$, by the same method of finding the limit of $S_n'(x)$ we have

$$|F_0(x)| \lim_{n \to \infty} |A_n(1,x) - 1| = 0.$$

This implies, for $|x| \leq s_0$, $\lim_{n \to \infty} |A_n(1,x) - 1| = 0$, since $|F_0(x)| \neq 0$. That means that $\lim_{n \to \infty} S_n''(x) = 0$ for $|x| \leq s_0$. Thus using (2), for $|x| \leq s_0$ we have

$$\lim_{n \to \infty} |A_n(f,x) - f(x)| = \lim_{n \to \infty} S_n'(x) + \lim_{n \to \infty} S_n''(x) = 0$$

and this proves the assertation. □

Theorem 2.2 — *Suppose $\chi(x) = 1 + x^2$ and $L_n : C_\chi \to B_\chi$ is a sequence of linear positive operators such that*

$$\lim_{n \to \infty} \| L_n(t^\nu, x) - x^\nu \|_\chi = 0,$$

where ν = 0, 1, 2. If $\lim_{x \to \infty} \frac{\rho_1(x)}{\rho_2(x)} = 0$ then the sequence of positive linear operators A_n defined by

$$A_n(f,x) := \frac{\rho_2(x)}{\chi(x)} L_n\left(\frac{\chi(t)}{\rho_1(t)} f(t), x\right)$$

maps C_{ρ_1} into B_{ρ_2} and further $\lim_{n \to \infty} \| A_n(F_j, x) - F_j(x) \|_{\rho_2} = 0$,

where $$F_j(x) = \frac{x^j}{\chi(x)} \rho_1(x) \text{ and } j = 0, 1, 2.$$

PROOF : It is obvious that A_n is linear and positive for each $n \in I\!N$. Since positive linear operator L_n goes from C_χ to B_χ, there exists a $K > 0$ such that

$$\| L_n(\chi, x) \|_\chi \leq K$$

and also by hypothesis, there exists an $M > 0$ such that $\sup_{x \in I\!R} \frac{\rho_1(x)}{\rho_2(x)} \leq M$. On the other hand the sequence L_n is strictly increasing,

$$\| A_n(\rho_1, x) \|_{\rho_2} = \sup_{x \in I\!R} \frac{A_n(\rho_1, x)}{\rho_2(x)}$$

$$= \| L_n(\chi, x) \|_{\chi} \leq K$$

and using the Property 1 the positive linear operator A_n maps C_{ρ_1} to B_{ρ_2} for each $n \in I\!N$. Since $F_j \in C_{\rho_1}$ we have

$$A_n(F_j, x) - F_j(x) = \frac{\rho_2(x)}{\chi(x)} [L_n(t^j, x) - x^j]$$

and then we get the equality

$$\| A_n(F_j, x) - F_j(x) \|_{\rho_2} = \| L_n(t^j, x) - x^j \|_{\chi}.$$

This completes the proof.

3. Construction of Linear Positive Operators

In the last section we give an application of our main result to the sequence of Gadjiyev-Ibragimov operators[4,5,8&12].

Assume $(\varphi_n(t))_{n \in I\!N}$ and $(\psi_n(t)_{n \in I\!N}$ are sequences of functions on $C(0, \infty)$ and $(\alpha_n)_{n \in N}$ is a sequence of positive numbers satisfying the following conditions :

$$\varphi_n(0) = 0, \ \psi_n(0) \neq 0, n \in I\!N, \ \lim_{n \to \infty} \frac{\alpha_n}{n} = 1, \ \lim_{n \to \infty} \frac{1}{n^2 \psi_n(0)} = 0.$$

Further let $(K_n(x, t, u))_{n \in I\!N}$ be a sequence of functions with three variables x, t, u where $x, t \in [0, \infty)$ and $u \in I\!R$ satisfying the following conditions :

(*i*) For each $x, t \in [0, \infty)$ and for each $n \in I\!N$, K_n is analytic with respect to variable u.

(*ii*) For all $x \in [0, \infty)$, $K_n(x, 0, 0) = 1$ for $n \in I\!N$.

$$(iii)\ \left\{ (-1)^{\nu} \left[\frac{\partial^{\nu}}{\partial u^{\nu}} K_n(x,t,u) \right]_{u=u_1;\, t=0} \right\} \geq 0, \text{ for } \nu, n \in I\!N,\ x \in [0,\infty).$$

(*iv*) There exist a number m making $m + n = 0$ or $m + n \in I\!N$ such that

$$-\frac{\partial^{\nu}}{\partial u^{\nu}} K_n(x,t,u)\,|_{u=u_1;\, t=0} = nx \left[\frac{\partial^{\nu-1}}{\partial u^{\nu-1}} K_{n+m}(x,t,u) \right]_{u=u_1;\, t=0}$$

Now we use the following sequence of operators defined in the paper of Ibrahimov-Gadjiev[12]

$$L_n(f,x) = \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{n^2 \psi_n(0)} \right) \left\{ \left[\frac{\partial^{\nu}}{\partial u^{\nu}} K_n(x,t,u) \right]_{u=\alpha_n \psi_n(0);\, t=0} \right\} \cdot \frac{(-\alpha_n \psi_n(0))^{\nu}}{\nu!}.$$

If we apply the condition (iv) on this sequence of operators ν-times we obtain the equality

$$L_n(f,x) = \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{n^2 \psi_n(0)} \right) \frac{n\cdot(n+m)\,.....\,(n+(\nu-1)\,m)}{\nu!}$$

$$\cdot [\alpha_n \psi_n(0)]^{\nu} K_{n+\nu m}(x, 0, \alpha_n \psi_n(0))\, x^{\nu}. \qquad ... (6)$$

By the condition (iii), we see that $L_n(f,x)$ is positive and linear for each $n \in I\!N$. Additionally by [9] these operators satisfy

$$\lim_{n\to\infty} \| L_n(t^{\nu}, x) - x^{\nu} \|_{\chi} = 0 \quad \nu = 0, 1, 2.$$

So we can make the following remarks.

Remark 3.1 : The operator given in (6) maps C_{χ} to B_{χ} for $\nu = 0, 1, 2$ satisfies the condition

$$\lim_{n\to\infty} \| L_n(t^{\nu}, x) - x^{\nu} \|_{\chi} = 0$$

in Theorem 2.1. That is operator given in eqn. (6) satisfies the conditions of Theorem 2.2

Remark 3.2 : The operator

$$A_n(f,x) = \frac{\rho_2(x)}{1+x^2} \sum_{\nu=0}^{\infty} \frac{1+\frac{\nu^2}{n^2}}{\rho_1\left(\frac{\nu}{n}\right)} f\left(\frac{\nu}{n}\right) \left[\frac{\partial^{\nu}}{\partial u^{\nu}} K_n(x,t,u)\,|_{u=\alpha_n \psi_n(0),\, t=0} \right]$$

$$\frac{(-\alpha_n \psi_n(0))^{\nu}}{(\nu)!},$$

obtained from the operator given in eqn. (6) by the method of Theorem 2.2 maps C_{ρ_1} to B_{ρ_2} and satisfies

$$\lim_{n \to \infty} \| A_n (F_j, x) - F_j (x) \|_{\rho_2} = 0,$$

where $$F_j (x) = \frac{x^j}{1 + x^j}, \; j = 0, 1, 2.$$

Remark 3.3 : The operator A_n constructed in Remark 3.2 satisfies the conditions of Theorem 2.1. Hence we see that the operator obtained from the operator given in eq. (6) satisfies condition of Theorem 2.1.

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