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CERTAIN SUBCLASSES OF MULTIVALENT ANALYTIC FUNCTIONS

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The object of the piesent paper is to derive some properties of certain general class $S_{n,p}(A, B)$ of multivalent analytic functions involving a linear operator in the open unit disk. Relevant connections of the results presented here with those obtained in earlier works are pointed out.

Key Words : Analytic Functions; Starlike; Convex; Integral Operator; Polylogarithm; Hadamard Product (or Convolution); Subordination

1. INTRODUCTION

Let $\mathcal{A}_p$ be the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \qquad \dots (1.1)$$

which are analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in \mathcal{A}_p$ is said to be p-valently starlike of order α if it satisfies the condition

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \le \alpha < p, z \in E)$$

On the other hand, a function $f \in \mathcal{A}_p$ is said to be p-valently convex of order α if it satisfies the condition

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (0 \le \alpha < p; z \in E)$$

For $f \in \mathcal{A}_p$ given by (1.1), the generalized Bernardi-Libera-Livingston integral operatr F_c is defined by

$$F_c(z) = \frac{c+p}{c+p+k} a_{p+k} z^{p+k} \quad (c+p > 0; z \in E). \qquad \dots (1.2)$$

It follows from (1.2) that $F_c \in \mathcal{A}_p$. For an analytic function g, defned in E by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}$$

and a real number η. Flett[1] defined the multiplier transform I^η by

$$I^\eta g(z) = \sum_{k=0}^{\infty} (p+k+1)^{-\eta} b_{p+k} z^{p+k} \ (z \in E).$$

Clearly, the function $I^\eta g$ is analytic in E and

$$I^\eta (I^\mu g(z)) = I^{\eta+\mu} g(z)$$

for all real numbers η and μ.

For any integer n, let the operator D^n be defined, for an analytic function f given by (1.1), by

$$D^n f(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{p+k+1}{1+p}\right)^{-n} a_{p+k} z^{p+k}$$

$$= f(z) * p^{p-1} \left[z + \sum_{k=1}^{\infty} \left(\frac{k+1+p}{1+p}\right)^{-n} z^{k+1} \right] (z \in E$$

Lemma 2.4 — If $\psi(z)$ is analytic in E, $\phi(z)$ and $h(z)$ are convex (univalent in E such that $\psi(z) < (z)$, then $\psi(z) * h(z) < \phi(z) * h(z), z \in E$ where the symbol "*" stands for the Hadamard product or convolution.

Ponnusamy and Sabapathy[7] defined the generalized polylogarithm $\Phi_{r,s}(a, b; z)$ by

$$\Phi_{r,s}(a, b\, z) = \sum_{k=1}^{\infty} \frac{(1+a)^r (1+b)^2}{(k+a)^r (k+b)^s} z^k = z + \ldots \ (z \in E)$$

where any term $k + a = 0$, $k + b = 0$ are excluded and r, s are the complex number such that Re $r > 0$ and Re $s > 0$ and $a, b > -1$. The case a, b with $r + s = n > 0$ yields

$$\Phi_{r,s}(a, a; z) := \Phi_n(a; z) = \sum_{k=1}^{\infty} \frac{(1=a)^n}{(k+a)^n} z^k, z \in E,$$

where $(1+a)^{-n} \Phi_n(a; z)$ is well known Lerch[3] function which is a generalized Riemann zeta function. In forms of this notation, we can reqrite (1.3) by

$$D^n f(z) = f(z) * z^{p-1} \Phi_n(p, z). \qquad \text{... (1.4)}$$

This suggests that our results can further be generalized for the convolution with eneralized polylogarithm[7]. It follows from (1.4) that

$$z(D^n (f(z))' = (p+1) D^{n-1} f(z) - D^n f(z). \qquad \text{... (1.5)}$$

We also have

$$D^0 f(z) = f(z), D^{-1} f(z) = \frac{zf'(z) + f(z)}{(p+1)} \text{ and } D^{-2} f(z) = \frac{z^2 f''(z) + 3zf'(z) + f(z)}{(p+1)^2}$$

Making use of the operator notation D^n, we introduce a subclass of $\mathcal{A}_p$ as follows :

Definition — For any integer n and $-1 \le B < A \le 1$, a function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}_{n,p}(A, B)$, if and only if

$$\frac{z(D^n f(z))'}{D^n f(z)} < \frac{p(1+Az)}{1+Bz} \quad (z \in E), \qquad \text{... (1.6)}$$

where < denotes subordination.

For convenience, we write

$$\mathcal{S}_{n,p}\left(1 - \frac{2\alpha}{p}, -1\right) \equiv \mathcal{S}_{n,p}(\alpha),$$

where $\mathcal{S}_{n,p}(\alpha)$ denotes the class of functions $f \in \mathcal{A}_p$ satisfying the inequality

$$\text{Re}\left\{\frac{(p+1) D^{n-1} f(z)}{D^n f(z)} - 1\right\} > \alpha \quad (0 \le \alpha < p; z \in E).$$

We also note that $\mathcal{S}_{n,1}(\alpha) \equiv \mathcal{S}_n(\alpha)$ is the class introduced and studied by Uralegaddi and Somanatha[9] and $\mathcal{S}_{0,p}(\alpha) \equiv \mathcal{S}_p^*(\alpha)$ $(0 \le \alpha < p)$ is the class of p-valently starlike functions of order α.

In the present paper, we derive some properties of the general calss $\mathcal{S}_{n,p}(A, B)$ by using the techniques of Briot-Bouquet differential subordination. The results obtained here improve the corresponding results of Uralegaddi and Somanatha[9].

2. Preliminaries

In our present investigation of the general class $\mathcal{S}_{n,p}(A, B)$, we shall require the following lemmas.

Lemma 2 — If $-1 \le B < A \le 1, \beta > 0$ and the complex number γ satisfy *Re* $(\gamma) \ge -\beta(1-A)/(1-B)$, then the differential equation

$$q(z) + \frac{z\, q'(z)}{\beta\, q(z) + \gamma} = \frac{1 + Az}{1 + Bz}$$

has a univalent solution in E given by

$$q(z) = \begin{cases} \dfrac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta \int_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta(A-B)/B}\, dt} - \dfrac{\gamma}{\beta}, & B \neq 0, \\ \dfrac{z^{\beta+\gamma}\exp(\beta Az)}{\beta \int_0^z t^{\beta+\gamma-1}\exp(\beta At)\, dt} - \dfrac{\gamma}{\beta}, & B = 0. \end{cases} \quad \text{... (2.1)}$$

If $\phi(z)$ is analytic in E and satisfies

$$\phi(z) + \frac{z\, \phi'(z)}{\beta(\phi(z) + \gamma} < \frac{1 + Az}{1 + Bz} \quad (z \in E)$$

then $\phi(z) < q(z) < (1 + Az)/(1 + Bz)$ and $q(z)$ is the best dominant.

The above result can be found in Miller and Mocanu[6].

Lemma 2.2 — Let μ be a positive measure on the unit interval $I = [0, 1]$. Let $g(t, z)$ be a function analytic in E for each $t \in I$, and integrable in t for each $z \in E$ and for almost all $t \in I$, and suppose that Re $Pg(t, z) > 0$ in E, $g(t, -r)$ is real for real r and Re $\{1/g, (t, z)\} \geq 1/g(t, -r)$ for $|z| \leq r < 1$ and $t \in I$. If $g(z) = \int_I g(t, z)\, d\mu(t)$, then Re $\{1/g(z)\} \geq 1/g(-r)$ for $|z| \leq r$.

The above lemma is due to Wilken and Feng[11].

For real or complex numbers a, b and c $(c \neq 0, -1, -2, ...)$, the hypergeometric function is defined by

$$_2F_1(a, b; c; z) = 1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a+1) \cdot b(b+1)}{2!\, c(c+1)} z^2 + ... \quad \text{... (2.2)}$$

We note that the series in (2.2) converges absolutely for $z \in E$.

The following identities are well-known[10]

Lemma 2.3 — For real or complex numbers a, b and c $(c \neq 0, -1, -2, ...)$,

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt = \frac{\Gamma(b)\,\Gamma(c-b)}{\Gamma(c)}\, {_2F_1}(a, b; c; z)$$

$(\text{Re}\,(c) > \text{Re}\,(b) > 0)$. ... (2.3)

$$_2F_1(a, b; c; z) = (1-z)^{-a}\, {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right). \quad \text{... (2.4)}$$

$$_2F_1(a, b\, c; z) = {}_2F_1(b, a; c; z). \quad \text{... (2.5)}$$

The following lemma is due to Ruscheweyh and Sheil-Small[8].

3. Main Results

***Theorem* 3.1** — *Let n be any integer and* $-1 \le B < A \le 1$. *If* $f \in S_{n,p}(A, B)$, *then*

$$\frac{x\,(D^{n+1} f(z))'}{D^{n+1} f(z)} < \frac{1}{Q(z)} - 1 = \tilde{q}(z) < \frac{p(1+Az)}{1+Bz} \quad (z \in E), \quad \text{... (3.1)}$$

where
$$Q(z) = \begin{cases} \displaystyle\int_0^1 t^p \left(\frac{1+Btz}{1+Bz}\right)^{p(A-B)/B} dt, & B \neq 0 \\ \displaystyle\int_0^1 t^p \exp(p(t-1)Az)\, dt, & B = 0 \end{cases} \quad \text{... (3.2)}$$

and $\tilde{q}(z)$ is the best dominant of (3.1). Furthermore, if $-1 \le B < 0, B < A \le \min\{1, -2B/p\}$ then $f \in S_{n+1,p}(\rho(p, A, B))$, where

$$\rho(p, A, B) = \frac{p+1}{{}_2F_1(1, p(B-A)/B; p+2; B/(B-1))} - 1.$$

The result is best possible.

Proof : Let $g(z) = z\,[D^{n+1} f(z)/z^p]^{1/p}$ and $R = \sup\{r : g(z) \neq 0, 0 < |z| < 1|\}$. Then $g(z)$ is single valued and analytic in $|z| < R$. Further, ϕ defined by

$$\phi(z) = \frac{z\,g'(z)}{g(z)} = \frac{z\,(D^{n+1} f(z))'}{p\,D^{n+1} f(z)} \quad \text{... (3.3)}$$

is analytic in $|z| < R$ and $\phi(0) = 1$. Using the identity (1.5) in (3.3) and differentiating the resulting equation, we get

$$\frac{z\,(D^n f(z))'}{p\,D^n f(z)} = \phi(z) + \frac{z\,\phi'(z)}{p\,\phi(z) + 1} < \frac{1+Az}{1+Bz} \quad (|z| < R). \quad \text{... (3.4)}$$

Thus, by using Lemma 2.1 (for $\beta = p$ and $\gamma = 1$), we deduce that

$$\phi(z) < \tilde{q}(z) < \frac{1+Az}{1+Bz} \quad (|z| < R),$$

where $Q(z)$ is given by (3.2).

By (3.3) and (3.5), we note that $g(z)$ is starlike in $|z|<R$. Thus, it is not possible that $g(z)$ vanishes in $|z|<R$ if $R < 1$. So, we conclude that $R = 1$ and hence $\phi(z)$ in analytic in E. This proves (3.1).

Next, we show that

$$\inf_{|z|<1} \left\{ \operatorname{Re}(\tilde{q}(z)) \right\} = \tilde{q}(-1). \quad \ldots (3.6)$$

Setting $a = p(B-A)/B, b = p+1, c = p+2$ (so that $c > b > 0$) and by using (2.3), (2.4) and (2.5) in (3.2) we see that for $B \neq 0$

$$Q(z) = (1+Bz)^a \int_a^1 t^{b-1} (1+Btz)^{-a} \, dt = \frac{\Gamma(b)}{\Gamma(c)} {}_2F_1\left(1, a; c; \frac{B}{Bz+1}\right). \quad \ldots (3.7)$$

To prove (3.6), we show that Re $\{1/Q(z)\} \geq 1/Q(-1), z \in E$. Again (3.2), by (3.7) for $A < -2B/p$ (so that $c > a > 0$), can be written as

$$Q(z) = \int_0^1 g(t, z) \, d\mu(t),$$

where $$g(t, z) = \frac{1+Bz}{1+(1-t)Bz} \text{ and } d\mu(t) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)} t^{a-1} (1-t)^{c-a-1)} \, dt$$

is a positive measure on [0, 1].

For $-1 \leq B < 0$, it may be noted that Re $\{g(t, z)\} > 0$, $g(t-r)$ is real for

$0 \leq r < 1, t \in [0, 1]$

and $$\operatorname{Re}\left\{\frac{1}{t} g(t, z)\right\} = \operatorname{Re}\left\{\frac{1+(1-t)Bz}{1+Bz}\right\} \geq \frac{1-(1-t)Br}{1-Br} = \frac{1}{g(t, -r)}$$

for $|z| \leq r < 1$ and $t \in [0, 1]$. Now, by using Lemma 2.2 and letting $r \to 1^-$ we obtain $\operatorname{Re}\{1/Q(z)\} \geq 1/Q(-1), z \in E$. In the case $A = -2B/p$ and we obtain (3.3) by letting $A \to (-2B/p)^+$. This proves the assertion of Theorem 3.1. The result is best possible because of the best dominant property of $\tilde{q}(z)$.

Putting $A = 1 - (2\alpha/p)$ and $B = -1$ in Theorem 3.1, we have

Corollary 3.1 — For any integer n and $\max\{0, (p-2)/2\} \leq \alpha < p$, we have

$$S_{n,p}(\alpha) \subset S_{n+1,p}(\rho(p, \alpha)),$$

where $$\rho(p,\alpha)=\frac{p+1}{{}_2F_1(1,2(p-\alpha);p+2;1/2)}-1. \qquad \text{... (3.8)}$$

The result is best possible.

Taking $p = 1$ in Corollary 3.1, we get the following result which improves the corresponding work due to Uralgaddi and Somanatha[9].

Corollary 3.2 — For any integer n and $0 \le \alpha < 1$, we have

$$S_n(\alpha) \subset S_{n+1}(\delta(\alpha)),$$

$$\text{where } \delta(\alpha)=\begin{cases} \dfrac{\alpha(2\alpha-1)}{2(2^{-2\alpha}+\alpha-1)}-1, & \alpha\neq\dfrac{1}{2} \text{ and } \alpha\neq 0,\\ \dfrac{1}{2(1-\ln 2)}-1=0.629\ldots, & \alpha=\dfrac{1}{2},\\ \dfrac{1}{2(2\ln 2-1)}-1=0.294\ldots, & \alpha=0. \end{cases} \qquad \text{... (3.9)}$$

The result is best possible.

Remark : It follows from Corollary 3.1 that

$$S_{n,p}(\alpha) \subset S_{n+1,p}(\alpha) \subset \ldots . \subset S_{0,p}(\alpha) = S_p^*(\alpha)$$

for all non-positive integer n, which means that all functions in $S_{n,p}(\alpha)$ are p-valently starlike and hence p-valent in E.

***Theorem* 3.2** — *For any integer* n *and* $0 \le \alpha < p$, *if* $f \in S_{n=1,p}(\alpha)$, *then* $f \in S_{n=1,p}(\alpha)$ *for* $|z| < \xi(p,\alpha)$, *where* $\xi(p,\alpha)$ *is the smallest positive root in (0, 1) of the equation*

$$(p-2\alpha-1)r^2-2(p-\alpha+1)r+p+1=0.$$

The result is best possible.

PROOF : Since $f \in S_{n+1,p}(\alpha)$, we have

$$\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}=\alpha+(p-\alpha)w(z), \qquad \text{... (3.10)}$$

where $w(z)=1+w_1z+w_2z^2+\ldots.$ is analytic and has a positive real part in E. Making use of the logarithmic differentiation in (3.10) and using the identity (1.5) in the resulting equation, we get

$$\mathrm{Re}\left\{\frac{z(D^n(f(z))'}{D^nf(z)}-\alpha\right\}=(p-\alpha)\left\{w(z)+\frac{zw'(z)}{(\alpha+1)+(p-\alpha)w(z)}\right\}. \qquad \text{... (3.11)}$$

Now, using the well-known estimates[4]

$$\frac{|zw'(z)|}{\mathrm{Re}\,(w(z))} \le \frac{2r}{1-r^2} \text{ and } \mathrm{Re}\,(w(z)) \ge \frac{1-r}{1+r}\,(|z| = r < 1) \qquad \dots (3.12)$$

in (3.11), we deduce that

$$\mathrm{Re}\left\{\frac{z\,(D^n\,(f(z))'}{D^n f(z)} - \alpha\right\} = (p-\alpha)\,\mathrm{Re}\,(w(z))\left\{1 - \frac{2r}{(\alpha+1)(1-r^2)+(p-\alpha)(1-r)^2}\right\}..$$

It is easily seen that the right-hand side of the above expression is positive if $|z| < \xi(p, \alpha)$ for $\xi(p, \alpha)$ given as in Theorem 3.2. Hence, $f \in \mathcal{S}_{n,p}(\alpha)$ for $|z| < \xi(p, \alpha)$.

To show that the bound $\xi(p, \alpha)$ is best possible, we consider the function $f \in \mathcal{A}_p$ defined by

$$\frac{z\,(D^{n+1} f(z))'}{D^{n+1} f(z)} = \alpha\,(p-\alpha)\,\frac{1-z}{1+z} \quad (z \in E).$$

Noting that

$$\frac{z\,(D^n f(z))'}{D^n f(z)} - \alpha = \frac{1-z}{1+z} + \frac{2z}{(1-z)\,((p+1)+(p-2\,\alpha-1)\,z^2)} = 0$$

for $z = \xi(p, \alpha)$, we complete the proof of Theorem 3.2.

***Theorem* 3.3** — *Let* $-1 \le B < A \le 1$ *and* c *be a real number satisfying*

$$c \ge -\frac{p\,(1-A)}{1-B} \qquad \dots (3.13)$$

(*a*) If $f \in \mathcal{S}_{n,p}(A, B)$, then the function F_c defined by (1.2) belongs to $\mathcal{S}_{n,p}(A, B)$. Furthermore, we have

$$\frac{z\,(D^n F_c(z))'}{D^n F_c(z)} < \frac{1}{Q(z)} - c \equiv \tilde{q}(z) < \frac{p\,(1+Az)}{1+Bz} \quad (z \in E), \qquad \dots (3.14)$$

where

$$Q(z) = \begin{cases} \displaystyle\int_0^1 t^{p+c-1}\left(\frac{1+Btz}{1+Az}\right)^{p(A-B)/B} dt, & B \ne 0, \\ \displaystyle\int_0^1 t^{p+c-1} \exp\,(pA\,(t-1)\,z)\,dt, & B = 0. \end{cases} \qquad \dots (3.15)$$

(*b*) If in addition to (3.13), $\dfrac{A}{B} > -\dfrac{c+1}{p}$ with $B < 0$, then $f \in \mathcal{S}_{n,p}(A, B)$ implies that $F_c \in \mathcal{S}_{n,p}(\eta\,(p, c, A, B))$, where

$$\eta(p, c, A, B) = \frac{p+c}{{}_2F_1(1\, p(B-A)/B; p+c+1; B/(B-1))} - c.$$

The result is best possible.

PROOF : Letting $g(z) = z\,[D^n F_c(z)/z^p]^{1/p}$ and $R = \sup\{r : g(z) \neq 0, 0 < |z| < r < 1\}$, we see that $g(z)$ is single valued and analytic in $|z| < R$. Further,

$$\phi(z) = \frac{z\,g'(z)}{g(z)} = \frac{z\,(D^n F_c(z))'}{pD^n F_c(z)} \qquad \text{... (3.16)}$$

is analytic in $|z| < R$ and $\phi(0) = 1$. Using (1.4) and the identity

$$z\,(D^n F_c(z))' = (p+c)\,D^n f(z) - cD^n F_c(z) \qquad \text{... (3.17)}$$

in (3.16), we get

$$\frac{D^n F_c(z)}{D^n f(z)} = \frac{p+c}{p\,\phi(z) + c}$$

from which it follows that

$$\frac{z\,(D^n f(z))'}{D^n f(z)} = \phi(z) + \frac{z\,\phi'(z)}{p\,\phi(z) + c} \prec \frac{1+Az}{1+Bz} \; (|z| < R). \qquad \text{... (3.18)}$$

Since $f \in S_{n,p}(A, B)$, by using Lemma 2.1 (for $\beta = p$ and $\gamma = c$), we deduce that

$$\phi(z) \prec \tilde{q}(z) = \frac{1}{p}\left[\frac{1}{Q(z)} - c\right] \prec \frac{1+Az}{1+Bz} \; (|z| < R),$$

where $Q(z)$ is given by (3.15) and $\tilde{q}(z)$ is the best dominant of (3.18). We note that $R = 1$. Therefore, $\phi(z)$ is analytic in E. This proves the first part of the theorem. Proceeding as in Theorem 3.1 the second part follows.

Putting $A = 1 - (2\alpha/p)$ and $B = -1$ in Theorem 3.2, we obtain

Corollary 3.3 — Let n be any integer and $0 \leq \alpha < p$.

(*i*) If $f \in S_{n,p}(\alpha)$ and c is a real number satisfying $c \geq -\alpha$, then the function F_c defined by (1.2) belongs to the class $S_{n,p}(\alpha)$. Furthermore, we have

$$\frac{z\,(D^n F_c(z))'}{D^n F_c(z)} \prec \frac{1}{Q(z)} - c \equiv \tilde{q}(z) \quad (z \in E)$$

where $Q(z)$ is obtained from (3.15) with $A = 1 - (2\alpha/p)$ and $B = -1$.

(*ii*) If c is a real number satisfying $c \geq \max\{-\alpha, p - 2\alpha - 1\}$ and $f \in S_{n,p}(\alpha)$ then $F_c \in S_{n,p}(\mathcal{H}(p, c, \alpha))$, where

$$\mathcal{H}(p, c, \alpha) = \frac{p+c}{{}_2F_1(1, 2(p-\alpha); p+c+1; 1/2)} - c$$

The result is best possible.

Setting $n = 0$ and $c = p = 1$ in Theorem 3.2, we get the following result which improves the corresponding work obtained by Uralegaddi and Somanatha[9].

Corollary 3.4 — If $f \in S_n(\alpha)$ for $0 \le \alpha < 1$, then the function

$$G(z) = \frac{2}{z} \int_0^z f(t)\, dt$$

belongs to the class $S_n(\delta(\alpha))$, where $\delta(\alpha)$ is given by (3.9). The result is best possible.

Theorem 3.4 — *Let* $-1 \le B < A \le 1$ *and* $h(z)\, z^p\, {}_2F_1(1, p+1; p+2; z), z \in E$. *Then*

$$f \in S_{n+1, p}(A, B) \Leftrightarrow F_1 \in S_{n, p}(A, B) \Leftrightarrow f * h \in S_{n, p}(A, B),$$

where F_1 is obtained from (1.2).

PROOF : Using the identity (1.4) for F_1 and (3.17), we get

$$D^{n+1} f(z) = D^n F_1(z) \quad (z \in E). \qquad \text{... (3.19)}$$

Again, for f given by (1.1), we have

$$F_1(z) = z^p + \sum_{k=1}^{\infty} \frac{p+1}{p+k+1} a_{p+k} z^{p+k} = f(z) * z^p \sum_{k=0}^{\infty} \frac{(p+1)_k (1)_k}{(p+2)_k\, k!} z^k$$

$$= f(z) * z^p\, {}_2F_1(1, p+1; p+2; z). \qquad \text{... (3.20)}$$

Now the assertion of the theorem follows by using (3.19) and (3.20) in (1.5).

Theorem 3.5 — *For any integer n and* $0 \le \alpha < p$, *if* $F_c \in S_{n, p}(\alpha)$ *then the function f defined by (1.1) belongs to* $S_{n, p}(\alpha)$ *for* $|z| < \xi^*(p, c, \alpha)$, *where* $\xi^*(p, c, \alpha)$ *is the smallest positive root in (0, 1) of the equation*

$$(p - 2\alpha - c)\, r^2 - 2(p - \alpha + 1)\, r + p + c) = 0.$$

The result is best possible.

PROOF : Since $F_c \in S_{n,p}(\alpha)$, we write

$$\frac{z\,(D^n F_c(z))'}{D^n F_c(z)} = \alpha + (p-\alpha)\,u(z), \qquad \text{... (3.21)}$$

where $u(z)$ is analytic, $u(0) = 1$ and Re $u(z)) > 0$ in E. Using (3.17) in (3.21) and differentiating the resulting equation, we obtain

$$\text{Re}\left\{\frac{z\,(D^n f(z))'}{D^n f(z)} - \alpha\right\} = (p-\alpha)\,\text{Re}\left\{u(z) + \frac{z\,u'(z)}{(p-\alpha)\,u(z)+\alpha+c}\right\} \qquad \text{... (3.22)}$$

Now, by using (3.12) for $u(z)$ in (3.22) and following the lines of proof of Theorem 3.2, we get the assertion of Theorem 3.4.

The bound $\xi^*(p, c, \alpha)$ is best possible for the function F_c defined in E by

$$\frac{z\,(D^n F_c(z))'}{D^n F_c(z)} = \alpha + (p-\alpha)\frac{1+z}{1-z} \quad (z \in E).$$

This completes the proof of Theorem 3.4.

Theorem 3.6 — *If* $f \in S_{n,p}(A, B)$, *then for all* $|s| \leq 1, |t| \leq 1$ $(s \neq t)$,

$$\frac{t^p D^n f(sz)}{s^p D^n f(tz)} \prec \begin{cases} \left(\dfrac{1+Bsz}{(1+Btz}\right)^{p(A-B)/B} & , \; B \neq 0, \\ \exp\,(pA\,(s-t)\,z), & B = 0. \end{cases} \qquad \text{... (3.23)}$$

PROOF : From (1.5), we deduce that

$$\frac{z\,(D^n f(z))'}{D^n f(z)} - p \prec \frac{p\,(A-B)\,z}{1+Bz} \quad (z \in E).$$

Since $z/(1+Bz)$ and the function

$$h(z) = \int_0^z \left(\frac{s}{1-su} - \frac{t}{1-tu}\right) du \quad (|s| <= 1, |t| \leq 1 \; (s \neq t))$$

are convex (univalent) in E, Lemma 2.4 yields

$$\left(\frac{z\,(D^n f(z))'}{D^n f(z)} - p\right) * h(z) \prec \frac{p\,(A-B)\,z}{1+Bz} * h(z). \qquad \text{... (3.24)}$$

As for every analytic function $\psi(z)$ in E with $\psi(0) = 0$

$$\psi(z) * h(z) = \int_{tz}^{sz} \frac{\psi(u)}{u}\, du.$$

Eq. (3.24) can be written as

$$\int_{tz}^{sz} \left(\frac{u\,(D^n f(u))'}{D^n f(u)} - p \right) \frac{du}{u} < p\,(A-B) \int_{tz}^{sz} \frac{du}{1+Bu}.$$

This proves the assertion (3.23) of Theorem 3.5.

Theorem 3.7 — *If* $f \in S_{n,p}(A, B)$, *then for* $|z| = r < 1$

$$|D^n f(z)| \le \begin{cases} r^p (1+Br)^{p(A-B)/B}, & B \ne 0, \\ r^p \exp(pAr); & B = 0, \end{cases} \qquad \text{... (3.25)}$$

$$|D^n f(z)| \le \begin{cases} r^p (1-Br)^{p(A-B)/B}, & B \ne 0, \\ r^p \exp(-pAr); & B = 0, \end{cases} \qquad \text{... (3.26)}$$

and

$$\left| arg\left(\frac{D^n f(z)}{z^p} \right) \right| \le \begin{cases} \frac{p(A-B)}{B} \sin^{-1}(Br), & B \ne 0, \\ pAr, & B = 0. \end{cases} \qquad \text{... (3.27)}$$

All the estimates are sharp.

PROOF : Taking $s = 1$, $t = 0$ in (3.23) and using the definition of subordination, it follows that

$$\frac{D^n f(z)}{z^p} = \begin{cases} (1+Bw(z))^{p(A-B)/B} & B \ne 0, \\ \exp(pAw(z)), & B = 0, \end{cases} \qquad \text{... (3.28)}$$

where $w(z)$ is analytic in E with $w(0) = 0$ and $|w(z)| \le |z|$ for $z \in E$.

(*i*) If $B > 0$, then from (3.28), we get

$$\left| \frac{D^n f(z)}{z^p} \right| = \left| (1+Bw(z))^{p(A-B)/B} \right|$$

$$= \left| \exp\left\{ \frac{p(A-B)}{B} \log(1+Bw(z)) \right\} \right|$$

$$= \exp\left[\mathrm{Re}\left\{ \frac{p(A-B)}{B} \log(1+Bw(z)) \right\} \right]$$

$$= \exp\left[\frac{p(A-B)}{B} \log|1+Bw(z)| \right]$$

$$\le |1+Bw(z)|^{p(A-B)/B} \le (1+Br)^{p(A-B)/B}.$$

(*ii*) If $B < 0$, we put $B = -C, C > 0$ so that

$$|1 + Bw(z)|^{p(A-B)/B} = |(1 - Cw(z))^{-1}|^{p(A-B)/C}$$

$$\leq (1 - Cr)^{-p(A-B)/B} = (1 + Br)^{p(A-B)/B}$$

Similarly, we can prove the other inequalities in (3.25), (3.26) amd (3.27) for $B = 0$.

For $|z| = r$ and $B \neq 0$ by (3.28), we get

$$\left| arg \frac{D^n f(z)}{z^p} \right| = \frac{p(A-B)}{B} |arg(1 + Bw(z))|$$

$$\leq \frac{p(A-B)}{B} \sin^{-1}(Br).$$

For $B = 0$, (3.27) is a direct consequence of (3.28).

It is easily seen that all the estimates are sharp, being attained by the function f_0 defined in E by

$$f_0(z) = \begin{cases} z^p (1 + B\varepsilon z)^{p(A-B)/B}, & B \neq 0, \\ z^p \exp(pA\varepsilon z), & B = 0, |\varepsilon| = 1 \end{cases} \qquad \text{... (3.29)}$$

***Theorem* 3.8** — *If* $f \in S_{n,p}(A, B)$, *then for* $|z| = r < 1$

$$|(D^n f(z))'| \leq \begin{cases} pr^{p-1}(1 + Ar)(1 = Br)^{(pA-(p+1)B)/B}, & B \neq 0, \\ pr^{p-1}(1 + Ar)\exp(pAr), & B = 0, \end{cases}$$

$$|(D^n f(z))'| \leq \begin{cases} pr^{p-1}(1 - Ar)(1 - Br)^{(pA-(p+1)B)/B}, & B \neq 0, \\ pr^{p-1}(1 - Ar)\exp(-pAr), & B = 0, \end{cases}$$

and

$$\left| arg \frac{(D^n f(z))'}{z^{p-1}} \right| \leq \begin{cases} \frac{p(A-B)}{B} \sin^{-1}(Br) + \sin^{-1}\left[\frac{(A-B)r}{1 - ABr^2}\right], & B \neq 0, \\ pAr + \sin^{-1}(Ar), & B = 0. \end{cases}$$

All the estimates are sharp.

PROOF : From (1.5), we have

$$(D^n f(z))' = p \frac{D^n f(z)}{z} \phi(z), \qquad \text{... (3.30)}$$

where $\phi(z)$ is analytic in E with $\phi(0) = 1$ and $\phi(z) < (1 + Az)/(1 = Bz), z \in E$. It is known[2] that the function ϕ satisfies the sharp estimates

$$\frac{1 - Ar}{1 - Br} \leq |\phi(z)| \leq \frac{1 + Ar}{1 + Br} \qquad (3.31)$$

and
$$\left| \phi(z) - \frac{1 - ABr^2}{1 - B^2 r^2} \right| \le \frac{(A - B)\, r}{1 - B^2 r^2} \; (|z| = r < 1)$$

so that
$$|\phi(z)| \le \sin^{-1}\left[\frac{(A - B)\, r}{1 - B r^2} \right]. \quad \ldots (3.32)$$

Using Theorem 3.7 and the estimates (3.31) and (3.32) in (3.30), we get the assertion of Theorem 3.8.

Equality signs are attained by the function f_0 given by (3.29).

Putting $n = 0$, $A = 1 - (2\alpha/p)$ and $B = -1$ in Theorem 3,7 and Theorem 3.8, we have

Corollary 3.6 — If $f \in S_p^*(\alpha)$ for $0 \le \alpha < p$, then for $|z| = r < 1$

$$\frac{r^p}{(1 + r)^{2(p - \alpha)}} \le |f(z)| \le \frac{r^p}{(1 - r)^{2(p - \alpha)}},$$

$$\frac{r^{p-1}(p(p - 2\alpha) r)}{(1 + r)^{2p + 1 - 2\alpha}} \le |f'(z)| \le \frac{r^{p-1}(p + (p - 2\alpha) r)}{(1 - r)^{2p + 1 - 2\alpha}},$$

$$\left| arg \frac{f(z)}{z^p} \right| \le 2(p - \alpha) \sin^{-1}(r),$$

and
$$\left| arg \frac{f'(z)}{z^{p-1}} \right| \le \sin^{-1}\left[\frac{2(p - \alpha) r}{p + (p - 2\alpha) r^2} \right] = 2(p - \alpha) \sin^{-1}(r).$$

All the estimates are sharp.

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