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# EIGENVALUES AND EIGENFUNCTIONS OF DISCONTINUOUS STURM-LIOUVILLE PROBLEMS WITH EIGENPARAMETER IN THE BOUNDARY CONDITIONS

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In this study a discontinuous regular Sturm-Liouville problems with eigenvalue parameter at the one of boundary conditions and with transmission conditions at the point of discontinuity are investigated.

We suggest an own approach for finding asymptotic approximation formulas for eigenvalues and eigenfunctions of such discontinuous problems.

In the special case, when our problem is continuous (i.e. when $\delta = 1$ in below) the obtained results are coincided with the corresponding results in [2].

**Key Words : Sturm-Liouville Problems; Transmission Conditions; Asymptotic of Eigenvalues and Eigenfunctions; Discontinuous Boundary-Value Problems.**

## 1. STATEMENT OF THE PROBLEM

We consider the differential equation

$$\tau u := -u'' + q(x)\, u = \lambda u \qquad \text{... (1.1)}$$

on $[-10) \cup (0, 1]$, with boundary conditions

$$L_1(u) := \alpha_1 u(-1) + \alpha_2 u'(-1) = 0 \qquad \text{... (1.2)}$$

and
$$L_2(u) := (\beta_1' \lambda + \beta_1)\, u(1) - (\beta_2' \lambda + \beta_2)\, u'(1) = 0 \qquad \text{... (1.3)}$$

and transmission conditions

$$L_3(u) := u(-0) - \delta u(+0) = 0 \qquad \text{... (1.4)}$$

$$L_4(u) : u'(-0) - \delta u'(+0) = 0, \qquad \text{... (1.5)}$$

where $\lambda$ is complex eigenparameter; the realvalued functin $q(x)$ continuous in $[-1, 0)$ and $(0, 1]$ and has a finite limit $q(\pm 0) := \lim_{x \to \pm 0} q(x)$; $\delta, \alpha_i, \beta_i, \beta_i'$ $(i = 1, 2)$ are real numbers; $|\alpha_1| + |\alpha_2| \neq 0$, $\delta \neq 0$. As a following to [8], everywhere we assume that $\rho := \beta_1' \beta_2 - \beta_2' \beta_1 > 0$.

There are many papers and books, where the spectral properties of the boundary-value problems with eigenparameter depended boundary conditions are investigated (see, for example [1-3, 5 & 6, 8-11] and corresponding bibliography). Also various physical applications of such problems can be found in [2].

Boundary-value problems with transmission conditions arise, as a rule, in the theory of heat and mass transfer and in a varied assortment of physical transfer problems.

## 2. An Operator Formulation in the Adequate Hilbert Space

We introduce the special inner product in the Hilbert space $L_2(-1, 1) \oplus \mathbb{C}$ and a symmetric linear operator $A$ defined on this Hilbert Space such that (1.1)-(1.5) can be considered as the eigenvalue problem of this operator.

Namely, in the Hilbert Space $H_{\delta, \rho} := L_2(-1, 1) \oplus \mathbb{C}$ we define an inner product by

$$\langle F, G \rangle := \int_{-1}^{0} f(x)\overline{g(x)}\,dx + \delta^2 \int_{0}^{1} f(x)\overline{g(x)}\,dx + \frac{\delta^2}{\rho} f_1 \overline{g}_1$$

for

$$F := \begin{pmatrix} f(x) \\ f_1 \end{pmatrix}, G := \begin{pmatrix} g(x) \\ g_1 \end{pmatrix} \in H_{a, \rho}.$$

Following [2] for convenience we shall use the Following notations

$$R_1(u) := \beta_1 u(1) - \beta_2 u'(1)$$

and

$$R_1'(u) := \beta_1' u(1) - \beta_2' u'(1).$$

For functions $f(x)$, which defined on $[-1, 0) \cup (0, 1]$ and has finite limits $f(\pm 0) := \lim_{x \to \pm 0} f(x)$, by $f_{(1)}(x)$ and $f_{(2)}(x)$ we denote the functions

$$f_{(1)}(x) = \begin{cases} f(x), x \in [-1, 0) \\ f(-0), x = 0 \end{cases}, \quad f_{(2)}(x) = \begin{cases} f(+0), & x = 0 \\ f(x), & x \in (0, 1] \end{cases}$$

which are defined on $\Omega_1 := [-1, 0]$ and $\Omega_2 := [0, 1]$ respectively.

In the Hilbert space $H_{\delta,\rho}$ consider the operator $A$ which is defined by the equalities

$$D(A) := \left\{ \begin{pmatrix} f(x) \\ R_1'(f) \end{pmatrix} f_{(i)}(x), f_{(i)}'(x) \text{ are absolutely continuous in} \right.$$

$$\left. \Omega_i\ (i,=1,2)\ \tau f \in L_2[-1,1], L_1 f = L_3 f = L_4 f = 0 \right\},$$

$$A \begin{pmatrix} f(x) \\ R_1'(f) \end{pmatrix} := \begin{pmatrix} \tau f \\ -R_1(f) \end{pmatrix}.$$

Now we can rewrite the considered problem (1.1)-(1.5) in the operator form as

$$AU = \lambda U,$$

where $$U := \begin{pmatrix} u(x) \\ R_1'(u) \end{pmatrix} \in D(A).$$

The eigenvalues and eigenfunctions of the problem (1.1)-(1.5) are defined as the eigenvalues and the first components of the corresponding eigenelements of the operator $A$ respectively.

## 3. Some Basics Properties of Eigenvalues and Eigenfunctions

***Theorem* 3.1** — *The operator A is symmetric.*

Proof : Let $F, G \in D(A)$. By two partial integration we obtain

$$\langle AF, G\rangle = \langle F, AG\rangle + W(f,\bar{g};-0) - W(f,\bar{g};-1) + \delta^2 W(f,\bar{g};1) -$$

$$- \delta^2 W(f,\bar{g};+0) + \frac{\delta^2}{\rho}(R_1'(f)R_1(\bar{g}) - R_1(f)R_1'(\bar{g})) \quad \ldots (3.1)$$

where, as usual, by $W(f, g; x)$ we denote the Wronskian of the functions $f$ and $g$:

$$W(f,g;x) := f(x)g'(x) - f'(x)g(x).$$

Since $f$ and $\bar{g}$ are satisfied the boundary condition (1.2), it follows that

$$W(f,\bar{g};-1) = 0. \quad \ldots (3.2)$$

From the transmission condition (1.4) and (1.5) we get

$$W(f,\bar{g};-0) = \delta^2 W(f,\bar{g};+0). \quad \ldots (3.3)$$

Further, it is easy to verify that

$$R_1'(f)R_1(\bar{g}) - R_1(f)R_1'(\bar{g}) = -\rho W(f,\bar{g};1). \quad \ldots (3.4)$$

Finally, substituting (3.2)-(3.4) in (3.1) then we have

$$\langle AF, G \rangle = \langle F, AG \rangle \; (F, G \in D(A)) \qquad \dots (3.5)$$

so $A$ is symmetric. ■

*Corollary* 3.1 — All eigenvalues of the problem (1.1)-(1.5) are real.

We can now assume that all eigenfunctions of the problem (1.1)-(1.5) are real valued.

*Corollary* 3.2 — Let $\lambda_1$ and $\lambda_2$ be two different eigenvalues of the problem (1.1)-(1.5). Then the corresponding eigenfunctions $u_1$ and $u_2$ of this problem satisfy the following equality

$$\int_{-1}^{0} u_1(x) \cdot u_2(x)\,dx + \delta^2 \int_{0}^{1} u_1(x) \cdot u_2(x)\,dx = -\frac{\delta^2}{\rho} R_1'(u_1) R_1'(u_2). \qquad \dots (3.6)$$

PROOF : The formula (3.6) follows immediately from the orthogonality of corresponding eigenelements

$$U_1 = \begin{pmatrix} u_1(x) \\ R_1'(u_1) \end{pmatrix} \text{ and } U_2 = \begin{pmatrix} u_2(x) \\ R_1'(u_2) \end{pmatrix} \text{ in the Hilbert space } H_{\delta, \rho}.$$ ■

For next consideration, we need the following Lemma, which can be proved by the same technique as in the proof of Theorem 1.5 in [7].

*Lemma* 3.1 — Let the real valued function $q(x)$ be continuous in $[a, b]$ and $f(\lambda), g(\lambda)$ are given entire functions. Then fọr any $\lambda \in \mathbb{C}$ the equation

$$-u'' + q(x)\,u = \lambda u, \; x \in [a, b]$$

has a unique solution $u = u(x, \lambda)$ satisfying the initial conditions

$$u(a) = f(\lambda), u'(a) = g(\lambda) \; (\text{or } u(b) = f(\lambda), u'(b) = g(\lambda)).$$

For each fixed $x \in [a, b]$, $u(x, \lambda)$ is an entire function of $\lambda$.

Let $\phi_{1\lambda}(x) := \phi_1(x, \lambda)$ be the solution of eq. (1.1) of [–1, 0], satisfying the initial conditions

$$u(-1) = \alpha_2, u'(-1) = -\alpha_1. \qquad \dots (3.7)$$

After defining above solution we shall define the solution $\phi_{2\lambda}(x) := \phi_2(x, \lambda)$ of eq. (1.1) on [0, 1] by means of the solution $\phi_1(x, \lambda)$ by the initial conditions

$$u(0) = \delta^{-1} \phi_1(0, \lambda), u'(0) = \delta^{-1} \phi_1'(0, \lambda). \qquad \dots (3.8)$$

Consequently, the function $\phi_\lambda(x) := \phi(x, \lambda)$ defined on $[-1, 0) \cup (0, 1]$ by the equality

$$\phi(x, \lambda) = \begin{cases} \phi_1(x, \lambda), x \in [-1, 0) \\ \phi_2(x, \lambda), x \in (0, 1] \end{cases}$$

is a such solution of the eq. (1.1) on $[-1, 0) \cup (0, 1]$, which satisfies one of the boundary conditions (namely (1.2)) and the both transmission conditions (1.4) and (1.5).

Analogically first we define the solution $\chi_{2\lambda}(x) := \chi_2(x, \lambda)$ on [0, 1] by initial conditions

$$u(1) = \beta_2' \lambda + \beta_2,\ u'(1) = \beta_1' \lambda + \beta_1. \quad \ldots (3.9)$$

After defining the above solution, we shall define the solution $\chi_{1\lambda}(x) := \chi_1(x, \lambda)$ of eq. (1.1) on [0, 1] by means of the solution $\chi_{2\lambda}(x)$ by the initial conditions

$$u(0) = \delta\chi_2(0, \lambda),\ u'(0) = \delta\chi_2'(0, \lambda). \quad \ldots (3.10)$$

Consequently, the function $\chi_\lambda(x) = \chi(x, \lambda)$ defined on $[-1, 0) \cup (0, 1]$ by the equality

$$\chi(x, \lambda) = \begin{cases} \chi_1(x, \lambda), & x \in [-1, 0) \\ \chi_2(x, \lambda), & x \in (0, 1] \end{cases}$$

is a solution of the equality (1.1) on $[-1, 0) \cup (0, 1]$, which satisfies the other boundary condition (1.3) and the both transmission conditions (1.4) and (1.5).

It is obvious that the Wronskians

$$\omega_i(\lambda) := W_\lambda(\phi_i, \chi_i; x)$$

$$:= \phi_i(x, \lambda)\chi_i'(x, \lambda) - \phi_i'(x, \lambda)\chi_i(x, \lambda), x \in \Omega_i \ (i = 1, 2)$$

are independent of $x \in \Omega_1$ and are entire functions.

*Lemma* 3.2 — For each $\lambda \in \mathbb{C}$ $\omega_1(\lambda) = \delta^2 \omega_2(\lambda)$.

PROOF : Because of (3.8) and (3.10) the short calculation gives

$$W_\lambda(\phi_1, \chi_1; 0) = \delta^2 W_\lambda(\phi_2, \chi_2, 0), \text{ so } \omega_1(\lambda) = \delta^2 \omega_2(\lambda).$$ ■

Now we may introduce to the consideration the characteristic function $\omega(\lambda)$ as

$$\omega(\lambda) := \omega_1(\lambda) = \delta^2 \omega_2(\lambda).$$

***Theorem* 3.2** — *The eigenvalues of the problem (1.1)-(1.5) are coincided zeros of the function* $\omega(\lambda)$.

PROOF : Let $\omega(\lambda_0) = 0$. Then $W_{\lambda_0}(\phi_1, \chi_1; x) = 0$ and therefore the functions $\phi_1(x, \lambda)$ and $\chi_1(x, \lambda)$ linearly depended, i.e.,

$$\chi_1(x, \lambda_0) = k_1 \phi_1(x, \lambda_0), x \in [-1, 0]$$

for some $k_1 \neq 0$. Consequently, the function $\chi(x, \lambda_0)$ also satisfied the boundary condition (1.3), so $\chi(x, \lambda_0)$ is an eigenfunction of the problem (1.1)-(1.5) corresponding to the eigenvalue $\lambda_0$.

Now let $u_0(x)$ be any eigenfunction corresponding to eigenvalue $\lambda_0$, but $\omega(\lambda_0) \neq 0$. Then the functions $\phi_1, \chi_1$ and $\phi_2, \chi_2$ would be linearly independent on [– 1, 0] and [0, 1] respectively. Therefore, $u_0(x)$ may be represented as in the form

$$u_0(x) = \begin{cases} c_1 \phi_1(x, \lambda_0) + c_2 \chi_1(x, \lambda_0), x \in [-1, 0) \\ c_3 \phi_2(x, \lambda_0) + c_4 \chi_2(x, \lambda_0), x \in (0, 1], \end{cases}$$

where at least one of the constants $c_1, c_2, c_3, c_4$ is not zero.

Considering the true equations

$$L_\nu(u_0(x)) = 0, \nu = 1, 4 \qquad \text{... (3.11)}$$

as the homogenous system of linear equations of the variables $c_1, c_2, c_3, c_4$, and taking into account (3.8) and (3.10), it follows that the determinant of this syste,

$$\begin{vmatrix} 0 & \omega_1(\lambda_0) & 0 & 0 \\ \phi_{1\lambda_0}(0) & \chi_{1\lambda_0}(0) & -\delta\phi_{2\lambda_0}(0) & -\delta\chi_{2\lambda_0}(0) \\ \phi'_{1\lambda_0}(0) & \chi'_{1\lambda_0}(0) & -\delta\phi'_{2\lambda_0}(0) & -\delta\chi'_{2\lambda_0}(0) \\ 0 & 0 & \omega_2(\lambda_0) & 0 \end{vmatrix}$$

$$= \omega_1(\lambda_0)\,\omega_2(\lambda_0) \begin{vmatrix} \phi_{1\lambda_0}(0) & -\delta\chi_{2\lambda_0}(0) \\ \phi'_{1\lambda_0}(0) & -\delta\chi'_{2\lambda_0}(0) \end{vmatrix}$$

$$= \omega_1(\lambda_0)\,\omega_2(\lambda_0) \begin{vmatrix} \delta\phi_{2\lambda_0}(0) & -\delta\chi_{2\lambda_0}(0) \\ \delta\phi'_{2\lambda_0}(0) & -\delta\chi'_{2\lambda_0}(0) \end{vmatrix}$$

$$= -\delta^2\,\omega_1(\lambda_0)\,\omega_2^2(\lambda_0) \neq 0.$$

Therefore, the system (3.11) has the only trivial solution $c_1 = c_2 = c_3 = c_4 = 0$. Thus we get contradiction, which completes the proof. ∎

*Lemma* 3.3 — If $\lambda = \lambda_0$ is an eigenvalue, then $\phi(x, \lambda_0)$ and $\chi(x, \lambda_0)$ are linearly dependent.

PROOF : By virtue of Theorem 3.2

$$W(\phi_i(x, \lambda_0), \chi_i(x, \lambda_0)) = \omega_i(\lambda_0) = 0$$

and therefore

$$\chi_i(x, \lambda_0) = k_i\,\phi_i(x, \lambda_0) \quad (i = 1, 2) \qquad \text{... (3.12)}$$

for some $k_1 \neq 0$ and $k_2 \neq 0$. We must show that $k_1 = k_2$. Suppose if possible that $k_1 \neq k_2$. Taking into account the definitions of the solution $\phi_i(x, \lambda)$ and $\chi_i(x, \lambda)$ from the equalities (3.12) we have

$$L_3(\chi_{\lambda_0}) = \chi_{\lambda_0}(-0) - \delta_{\chi_{\lambda_0}}(+0) = \chi_{1\lambda_0}(0) - \delta_f \chi_{2\lambda_0}(0) = k_1 \phi_{1\lambda_0}(0) - \delta k_2 \phi_{2\lambda_0}(0)$$

$$= k_1 \delta \phi_{2\lambda_0}(0) - k_2 \delta \phi_{2\lambda_0}(0) = \delta(k_1 - k_2) \phi_{2\lambda_0}(0).$$

Since $L_3(\chi_{\lambda_0}) = 0$ and $\delta(k_1 - k_2) \neq 0$ it follows that

$$\phi_{2\lambda_0}(0) = 0. \qquad \text{... (3.13)}$$

By the same procedure from the equality $L_4(\chi_{\lambda_0}) = 0$ we can derive that

$$\phi'_{2\lambda_0}(0) = 0. \qquad \text{... (3.14)}$$

From the fact that $\phi_{2\lambda_0}(x)$ is a solution of the differential eq. (1.1) on [0, 1] and satisfied the initial conditions (3.13) and (3.14) it follows that $\phi_{2\lambda}(x) = 0$ identically on [0, 1], because of the well-known existence and uniqueness theorem for the initial-value problems of the ordinary linear differential equations.

By using (3.8), (3.13) and (3.14) we may also find

$$\phi_{1\lambda_0}(0) = \phi'_{1\lambda_0}(0) = 0.$$

From latter discussion for $\phi_{2\lambda_0}(x)$, it follows that $\phi_{1\lambda_0}(x) = 0$ identically on [− 1, 0]. Hence, $\phi(x, \lambda_0) = 0$ identically on $[-1, 0) \cup (0, 1]$. But this is contradict with (3.7), which completes the proof. ■

*Corollary* 3.3 — If $\lambda = \lambda_0$ is an eigenvalue, then both $\phi(x, \lambda_0)$ and $\chi(x, \lambda_0)$ would be eigenfunctions corresponding to this eigenvalue.

*Lemma*3.4 — All eigenvalues $\lambda_n$ are simple zeros of $\omega(\lambda)$.

PROOF : Using the well-known Lagrange's formula [cf. 4, pp. 6-7] it can be shown that

$$(\lambda - \lambda_0)\left(\int_{-1}^{0} \phi_\lambda(x)\, \phi_{\lambda_0}(x)\, dx + \delta^2 \int_{0}^{1} \phi_\lambda(x)\, \phi_{\lambda_0}(x)\, dx\right) = \delta^2 W(\phi_\lambda, \phi_{\lambda_0}; 1) \qquad \text{... (3.15)}$$

for any $\lambda$. Since

$$\chi_{\lambda_0}(x) = k_0 \phi_{\lambda_0}(x), x \in [-1, 0) \cup (0, 1]$$

for some $k_0 \neq 0$, the direct calculations gives that

$$W(\phi_\lambda, \phi_{\lambda_n}; 1) = \frac{1}{k_n} W(\phi_\lambda, \phi_{\lambda_n}; 1) = \frac{1}{k_n} (\lambda_n R_1'(\phi_\lambda) + R_1(\phi_\lambda))$$

$$= \frac{1}{k_n} [\omega(\lambda) - (\lambda - \lambda_n) R_1'(\phi_\lambda)] = (\lambda - \lambda_n) \frac{1}{k_n} \left[ \frac{\omega(\lambda)}{\lambda - \lambda_n} = - R_1'(\phi_\lambda) \right].$$

Substituting this formula in (3.15) and letting $\lambda \to \lambda_n$ we get

$$\int_{-1}^{0} (\phi_{\lambda_n}(x))^2 dx + \delta^2 \int_{0}^{1} (\phi_{\lambda_n}(x))^2 dx = \frac{\delta^2}{k_n} \omega'(\lambda_n) - \frac{\delta^2}{k_n} R_1'(\phi_{\lambda_n}). \quad \dots (3.16)$$

Now putting

$$R_1'(\phi_{\lambda_n}) = \frac{1}{k_n} R_1'(\chi_{\lambda_n}) = \frac{\rho}{k_n}$$

in (3.16) it yields $\omega'(\lambda_n) \neq 0$. ■

## 4. Asymptotic Approximate Formulas for $\omega(\lambda)$

We begin by proving two Lemmas.

*Lemma* 4.1 — Let $\phi(x, \lambda)$ be the solutions of eq. (1.1) defined in section3 and let $\lambda = s^2$. Then the following integral equations hold for $k = 0$ and $k = 1$ :

$$\frac{d^k}{dx^k} \phi_{1\lambda}(x) = \alpha_2 \frac{d^k}{dx^k} \cos[s(x+1)] - \alpha_1 \frac{1}{s} \frac{d^k}{dx^k} \sin[s(x+1)]$$

$$+ \frac{1}{s} \int_{-1}^{x} \frac{d^k}{dx^k} \sin[s(x-y)]\, q(y)\, \phi_{1\lambda}(y)\, dy$$

$$\frac{d^k}{dx^k} \phi_{2\lambda}(x) = \frac{1}{\delta} \phi_{1\lambda}(0) \frac{d^k}{dx^k} \cos(sx) + \frac{1}{s} \frac{1}{\delta} \phi_{1\lambda}(0) \frac{d^k}{dx^k} \sin(sx)$$

$$+ \frac{1}{s} \int_{0}^{x} \frac{d^k}{dx^k} \sin[s(x-y)]\, q(y)\, \phi_{2\lambda}(y)\, dy \quad \dots (4.2)$$

Proof : For proving it is enough substitute $s^2 \phi_{1\lambda}(y) + \phi_{1\lambda}''(y)$ and $s^2 \phi_\lambda(y) + \phi_{2\lambda}''(y)$ instead of $q(y)\phi_{1\lambda} y$ and $q(y)\phi_{2\lambda}(y)$ in the integral terms of the (4.1) and (4.2) respectively and integrate by parts twice. ■

*Lemma* 4.2 — Let $\lambda = s^2$, $\text{Im}s = t$. For $|\lambda| \to \infty$, the functions $\phi_{i\lambda}(x)$ have the following asymptotic representations which hold uniformly for $x \in \Omega_i$ (for $i = 1, 2$), for :

$$\frac{d^k}{dx^k}\phi_{1\lambda}(x) = \alpha_2 \frac{d^k}{dx^k}\cos[s(x+1)] + O(|s|^{k-1}e^{|t|(x+1)}) \qquad \dots (4.3)$$

$$\frac{d^k}{dx^k}\phi_{2\lambda}(x) = \frac{1}{\delta}\alpha_2 \frac{d^k}{dx^k}\cos[s(x+1)] + O(|s|^{k-1}e^{|t|(x+1)}) \qquad \dots (4.4)$$

if $\alpha_2 \neq 0$,

$$\frac{d^k}{dx^k}\phi_{1\lambda}(x) = -\frac{\alpha_1}{s}\frac{d^k}{dx^k}\sin[s(x+1)] + O(|s|^{k-2}e^{|t|(x+1)}) \qquad \dots (4.5)$$

$$\frac{d^k}{dx^k}\phi_{2\lambda}(x) = -\frac{1}{\delta}\frac{1}{s}\frac{d^k}{dx^k}\sin[s(x+1)] + O(|s|^{k-2}e^{|t|(x+1)}) \qquad \dots (4.6)$$

while if $\alpha_2 = 0$.

PROOF : First we must note that all formulas for the solution $\phi_{1\lambda}(x)$ are totally analogical to the corresponding formulas in [7] for $\phi_\lambda(x)$. But the similar formulas for $\phi_{2\lambda}(x)$ needed individual consideration, since the last solutions are defined by the initial conditions having special forms in terms of $\phi_{1\lambda}(x)$. Therefore, we shall prove only the formula (4.4) (since (4.6) may be proved analogically to (4.4).

Let $\alpha_2 \neq 0$. It follows from (4.3) that

$$\phi_{1\lambda}(0) = \alpha_2 \cos s + O\left(\frac{1}{s}\right) \qquad \dots (4.7)$$

and

$$\phi'_{1\lambda}(0) = -s\,\alpha_2 \sin s + O(1). \qquad \dots (4.8)$$

Putting (4.7) and (4.8) in (4.2) (for $k = 0$) we have

$$\phi_{2\lambda}(x) = \frac{\alpha_2}{\delta}\cos[s(x+1)] + \frac{1}{s}\int_0^x \sin[s(x-y)]\,q(y)\,\phi_{2\lambda}(y)\,dy + O\left(\frac{e^{|t|x}}{|s|}\right) \qquad \dots (4.9)$$

Now it will be convenient to use the function

$$F_{2\lambda}(x) := e^{-|t|(x+1)}\phi_{2\lambda}(x), \qquad \dots (4.10)$$

which by virtue of (4.5) satisfied the integral equation

$$F_{2\lambda}(x) = \frac{\alpha_2}{\delta}e^{-|t|(x+1)}\cos[s(x+1)] + \frac{1}{s}$$

$$\int_0^x \sin[s(x-y)]\, q(y)\, e^{-|t|(x-y)} F_{2\lambda}(y)\, dy + O\left(\frac{e^{-|t|}}{|s|}\right)$$

Defining $M(\lambda) = \max_{0 \le x \le 1} |F_{2\lambda}(x)|$, from the last integral equation we have

$$M(\lambda) \le \frac{|\alpha_2|}{\delta} + \frac{M(\lambda)}{|s|} \int_0^1 |q(y)|\, dy + \frac{M_0}{|s|},$$

where $M_0 > 0$ some constant, from which it follows that

$$M(\lambda) = O(1), |\lambda| \to \infty.$$

Comparing with (4.10) we get the asymptotic representation

$$\phi_{2\lambda}(x) = O(e^{|t|(x+1)}).$$

Substituting in the integral on the right of (4.9) we have

$$\phi_{2\lambda}(x) = \frac{\alpha_2}{s} \cos[s(x+1)] + O\left(\frac{e^{|t|(x+1)}}{s}\right),$$

so the formula (4.6) follows for $k = 0$. Analogically, putting (4.7) and (4.8) in (4.2) for $k = 0$ and following the same technique we can verify that the formula (4.6) is true also for $k = 1$. ■

***Theorem* 4.1** — *Let $\lambda = s^2$, $t = Ims$. Then the characteristic function $\omega(\lambda)$ has the following asymptotic representations :*

*Case* 1 — If

$$\beta_2' \ne 0, \alpha_2 \ne 0, \text{ then } \omega(\lambda) = \alpha_2 \beta_2' \delta s^3 \sin(2s) + O(|s|^2 e^{2|t|}) \quad \text{... (4.11)}$$

*Case* 2 — If

$$\beta_2' \ne 0, \alpha_2 \ne 0, \text{ then } \omega(\lambda) = \beta_2' \alpha_1 \delta s^2 \cos(2s) + O(|s| e^{2|t|}) \quad \text{... (4.12)}$$

*Case* 3 — If

$$\beta_2' \ne 0, \alpha_2 \ne 0, \text{ then } \omega(\lambda) = \beta_1' \alpha_2 \delta s^2 \cos(2s) + O(|s| e^{2|t|}) \quad \text{... (4.13)}$$

*Case* 4 — If

$$\beta_2' \ne 0, \alpha_2 \ne 0, \text{ then } \omega(\lambda) = -\beta_1' \alpha_1 \delta s \sin(2s) + o(e^{2|t|}). \quad \text{... (4.14)}$$

PROOF : The proof is completed immediately by substituting (4.6) in the next representation of characteristic function $\omega(\lambda)$ :

$$\omega(\lambda) = \delta^2 [\phi_{2\lambda}(1) \chi'_{2\lambda}(1) - \phi'_{2\lambda}(1) \chi_{2\lambda}(1)]$$

$$= \delta^2 [(\lambda \beta_1' + \beta_1) \phi_{2\lambda}(1) - (\lambda \beta_2' + \beta_2) \phi_{2\lambda}(1)]$$

$$= \lambda \delta^2 (\beta_1' \phi_{2\lambda}(1) - \beta_2' \phi_{2\lambda}'(1)) + \delta^2 (\beta_1 \phi_{2\lambda}(1) - \beta_2 (\phi_{2\lambda}'(1)). \quad \text{... (4.15)} ■$$

*Corollary* 4.1 — The eigenvalues of the problem (1.1)-(1.5) is bounded below.

PROOF : Putting $s = it$ $(t > 0)$ in the above formulas it folows that $\omega(-t^2) \to \infty$ as $t \to \infty$. Hence, $\omega(\lambda) \neq 0$ for $\lambda$ negative and sufficiently large in moduli. ■

## 5. ASYMPTOTIC FORMULAE FOR EIGENVALUES

We are now ready to find the asymptotic approximation formulas for the eigenvalues of the considered problem (1.1)-(1.5).

Since the eigenvalues are coincided with the zeros of the entire function $\omega(\lambda)$, it follows that they have no finite limit. Moreover, all eigenvalues are real and bounded below by the Corollaries 3.1 and 4.1. Therefore, we may renumber them as $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$, which counted according to their multiplicitly. Below we shall denote $\lambda_n = s_n^2$ for sufficiently large $n$.

***Theorem*** 5.1 — *For the eigenvalues $\lambda_n$, $n = 0, 1, 2, \ldots$, of the problem (1.1)-(1.5) take place the following asymptotic representation for $n \to \infty$ :*

*Case* 1 — If $\beta_2' \neq 0, \alpha_2 \neq 0$, then

$$s_n = \frac{(n-1)\pi}{2} + O\left(\frac{1}{n}\right) \quad \text{... (5.1)}$$

*Case* 2 — If $\beta_2' \neq 0, \alpha_2 = 0$, then

$$s_n = \frac{(n-1/2)\pi}{2} + O\left(\frac{1}{n}\right) \quad \text{... (5.2)}$$

*Case* 3 — If $\beta_2' = 0, \alpha_2 \neq 0$, then

$$s_n = \frac{(n-1/2)\pi}{2} + O\left(\frac{1}{n}\right) \quad \text{... (5.3)}$$

*Case* 4 — If $\beta_2' = 0, \alpha_2 = 0$, then

$$s_n = \frac{\pi n}{2} + O\left(\frac{1}{n}\right). \quad \text{... (5.4)}$$

PROOF : We shall consider Case 1 only.

Denoting by $\omega_1(s)$ and $\omega_2(s)$ the first and the O-term of the right of (4.11) respectively we shall apply the well-known Rouche theorem which assert that if $f(s)$ and $g(s)$ analytic inside and on a closed contous $C$, and $|g(s)| < |f(s)|$ on $C$, then $f(s)$ and $f(s) + g(s)$ have the same number zeros inside $C$, provided that each zeros are counted according to their multiplicity.

It is readily shown that $|\omega_1(s)| > |\omega_2(s)|$ on the contours

$$C_n : \left\{ s \in \mathbb{C} \mid |s| = \frac{(n+1/2)\pi}{2} \right\}$$

for sufficiently large $n$.

Let $\lambda_0 \le \lambda_1 \le \ldots$ are zeros of $\omega(\lambda)$ and $\lambda_n = s_n^2$.

Since inside the contour $C_n$ the function $\omega_1(s)$ has zeros at points $s = 0$ (with multiplicity 4) and $s = \frac{\pi k}{2}, k = \pm 1, \pm 2, \ldots, \pm n$ (with multiplicity 1), and so the number $2n + 4$, it follows that

$$s_n = \frac{(n-1)\pi}{2} + \delta_n \qquad \ldots (5.5)$$

where $\delta_n = O(1)$, more previsely $|\delta_n| < \frac{\pi}{4}$ for sufficiently large $n$.

By putting this in (4.11) we derive that $\delta_n = O\left(\frac{1}{n}\right)$ which completes the proof for the case 1. The other cases may be considered analogically. ■

## 6. Next Approximation for the Eigenvalues

The next approximation for the eigenvalues may be obtained by following Tichmarsh's procedure as in the continuous case [cf. 7 p. 19]. For this we shall suppose that $q(y)$ is of bounded variation in $[-1, 1]$.

Again we consider Case 1 only. Putting $x = 0$ in (4.1) and then substituting in (4.2) we derive that

$$\phi_{2\lambda}'(1) = -\frac{\alpha_2}{\delta} s \sin(2s) - \frac{\alpha_1}{\delta} \cos(2s)$$
$$+ \frac{1}{\delta} \int_{-1}^{0} \cos[s(1-y)]\, q(y)\, \phi_{1\lambda}(y)\, dy + \int_{0}^{1} \cos[s(1-y)]\, q(y)\, \phi_{2\lambda}(y)\, dy.$$

We can find the asymptotic expression of $\phi_{2\lambda}'(1)$ to within $O(|s|^{-1} e^{2|t|})$, by putting (4.3) and (4.4) in the last integral equality :

$$\phi_{2\lambda}'(1) = -\frac{\alpha_2}{\delta} s \sin(2s) - \frac{\alpha_1}{\delta} \cos(2s) + \frac{\alpha_2}{\delta}$$

$$\int_{-1}^{0} \cos[s(1-y)]\cos[s(1+y)]\,q(y)\,dy$$

$$+\frac{\alpha}{\delta}\int_{0}^{1} \cos[s(1-y)]\cos[s(1+y)]\,q(y)\,dy + O(|s|^{-1}e^{2|t|}).$$

On the other hand from (4.4) it follows that

$$\phi_{2\lambda}(1) = \frac{\alpha_2}{\delta}\cos(2s) + O(|s|^{-1}e^{2|t|}).$$

Putting in (4.15) we get

$$\omega(\lambda) = \alpha_2\beta_2'\delta s^3 \sin 2s + s^2\Bigg[\alpha_2\delta\beta_1'\cos 2s + \beta_2'\delta^2\frac{\alpha_1}{\delta}\cos 2s$$

$$-\delta^2\beta_2'\frac{\alpha_2}{\delta}\int_{-1}^{1}\cos[s(1-y)\cos[s(1+y)]\,q(y)\,dy\Bigg] + O(|s|e^{2(|t|})$$

$$= s^3\alpha_2\beta_2'\delta\sin 2s + s^2\delta\Bigg[(\alpha_2\beta_1' + \alpha_1\beta_2')\cos 2s$$

$$-\alpha_2\beta_2'\int_{-1}^{1}\cos[s(1-y)]\cos[s(1+y)\,q(y)\,dy\Bigg] + O(|s|e^{2|t|})$$

$$= s^3\alpha_2\beta_2'\delta\sin 2s + s^2\delta\Bigg[(\alpha_2\beta_1' + \alpha_1\beta_2')\cos 2s - \frac{1}{2}\alpha_2\beta_2'\cos 2s\int_{-1}^{1}q(y)\,dy$$

$$-\frac{1}{2}\alpha_2\beta_2'\int_{-1}^{1}\cos(2sy)\,q(y)\,dy\Bigg] + O(|s|e^{2|t|}).$$

Putting (5.1) in the last equality we find that

$$\sin 2\delta_n = \frac{\cos 2\delta_n}{s_n}\Bigg[-\frac{\beta_1'}{\beta_2'} - \frac{\alpha_1}{\alpha_2} + \frac{1}{2\delta}$$

$$\int_{-1}^{1}q(y)\,dy + \frac{1}{2\delta}\int_{-1}^{1}\cos(2s_n y)\,q(y)\,dy\Bigg] + O\left(\frac{1}{n^2}\right) \qquad \text{... (6.1)}$$

Recalling that $q(y)$ is of bounded variation in $[-1, 1]$ and applying the well-known Riemann-Lebesque Lemma [cf. 12, p. 48, Theorem 4.12] to the second integral on the right side of

(6.1) it seems that this term is $O\left(\frac{1}{n}\right)$. Consequently, from (6.1) it follows that

$$\delta_n = \frac{1}{(n-1)\pi}\left[-\frac{\beta_1'}{\beta_2'} - \frac{\alpha_2}{\alpha_1} + \frac{1}{2\delta}\int_{-1}^{1} q(y)\,dy\right] + O\left(\frac{1}{n^2}\right).$$

Substituting in (5.5) we have

$$s_n = \frac{\pi(n-1)}{2} + \frac{1}{(n-1)\pi}\left[-\frac{\beta_1'}{\beta_2'} - \frac{\alpha_2}{\alpha_1} + \frac{1}{2\delta}\int_{-1}^{1} q(y)\,dy\right] + O\left(\frac{1}{n^2}\right).$$

Similar formulae in the other cases are as follows :

*In case* 2 —

$$s_n = \frac{(n-1/2)\pi}{2} + \frac{1}{(n-1/2\pi}\left[\frac{\beta_1'}{\beta_2'} - \frac{1}{2\delta}\int_{-1}^{1} q(y)\,dy\right] + O\left(\frac{1}{n^2}\right)$$

*In case* 3 —

$$s_n = \frac{(n-1/2)\pi}{2} + \frac{1}{(n-1/2)\pi}\left[\frac{\beta_1'}{\beta_2'} - \frac{\alpha_2}{\alpha_1} + \frac{1}{2\delta}\int_{-1}^{1} q(y)\,dy\right] + O\left(\frac{1}{n^2}\right).$$

*In case* 4 —

$$s_n = \frac{\pi n}{2} + \frac{1}{n\pi}\left[\frac{\beta_2}{\beta_1'} + \frac{1}{2\delta}\int_{-1}^{1} q(y)\,dy\right] + O\left(\frac{1}{n^2}\right).$$

## 7. Asymptotic Approximation Formulae for the Eigenfunctions

Let $\phi(x,\lambda)$ be defined as in section 3 and let $\beta_2' \neq 0, \alpha_2 \neq 0$ (case 1). We already know that $\phi(x,\lambda_n)$ is an eigenfunction corresponding to the eigenvalue $\lambda_n$ for any $n$ = 0, 1, 2, ... (see Corollary 3.3).

By putting (5.1) in the (4.3) and (4.4) we derive that

$$\phi_{1\lambda_n}(x) = \alpha_2 \cos\left(\frac{\pi(n-1)(x+1)}{2}\right) + O\left(\frac{1}{n}\right)$$

and
$$\phi_{2\lambda_n}(x) = \frac{\alpha_2}{\delta}\cos\left(\frac{\pi(n-1)(x+1)}{2}\right) + O\left(\frac{1}{n}\right).$$

Hence, the eigenfunctions $\phi(x, \lambda_n)$ has the following asympttic representation:

$$\phi(x, \lambda_n) = \begin{cases} \alpha_2 \cos\left(\dfrac{\pi(n-1)(x+1)}{2}\right) + O\,\dfrac{1}{n} & \text{for } x \in [-1, 0) \\ \dfrac{1}{\delta}\alpha_2 \cos\left(\dfrac{\pi(n-1)(x+1)}{2}\right) + O\,\dfrac{1}{n} & \text{for } x \in (0, 1] \end{cases}$$

Similar formulae in the other cases are as follows :

*In case* 2 —

$$\phi(x, \lambda_n) = \begin{cases} -\dfrac{2\alpha_1}{\pi(n-1/2)} \sin\left(\dfrac{\pi(n-1/2)(x+1)}{2}\right) + O\,\dfrac{1}{n} & \text{for } x - [-1, 0) \\ -\dfrac{1}{\delta}\dfrac{2}{\pi(n-1/2)} \sin\left(\dfrac{\pi(n-1/2)(x+1)}{2}\right) + O\,\dfrac{1}{n} & \text{for } x \in (0, 1] \end{cases}$$

*In case* 3 —

$$\phi(x, \lambda_n) = \begin{cases} \alpha_2 \cos\left(\dfrac{\pi(n-1/2)(x+1)}{2}\right) + O\,\dfrac{1}{n} & \text{for } x \in [-1, 0) \\ \dfrac{1}{\delta}\alpha_2 \cos\left(\dfrac{\pi(n-1/2)(x+1)}{2}\right) + O\,\dfrac{1}{n} & \text{for } x \in (0, 1] \end{cases}$$

*In case* 4 —

$$\phi(x, \lambda_n) = \begin{cases} -\dfrac{2\alpha_1}{\pi n} \sin\left(\dfrac{\pi n(x+1)}{2}\right) + O\,\dfrac{1}{n} & \text{for } x \in [-1, 0) \\ -\dfrac{1}{\delta}\dfrac{2}{\pi n} \sin\left(\dfrac{\pi n(x+1)}{2}\right) + O\,\dfrac{1}{n} & \text{for } x \in (0, 1] \end{cases}$$

All this asymptotic approximations are hold uniformly for $x \in [-1, 0) \cup (0, 1]$.

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## REFERENCES

1. G. D. Birkhoff, *Trans. Amer. Soc.*, **9** (1908) 219-31.
2. C. T. Fulton, *Proc. Roy. Soc. Edin.* **77**A (1977) 293-308.
3. D. B. Hinton, *Quart. J. Math. Oxford*, **30** (1979) 33-42.
4. M. N. Naimark, *Linear Differential Opearaors*, Ungar, New York, 1967.
5. A. Schneider, *Math. Z* **136** (1974) 163-67.
6. A. A. Shkalikov, *Trudy Sem. Imeny I. G. Petrowsgo*, **9** (1983) 190-229.

7. E. C. Titchmarsh, *Eigenfunctions Expansion Associated With Second Order Differential Equations I'* (2nd Edn), Oxford Univ. Press, London.

8. J. Walter, *Math. Z.* **133** 301-12.

9. S. Yakybov, *Completeness of Root Functions of Regular Differential Opearators*, Longman, Scientific Technical, New York, 1994.

10. S. Yakubov and Y. Yakubov, *Math. Nachr*, **197** (1999) 157-87.

11. S. Yakubov and Y. Yakubov, *Differential-Operator Equations. Ordinary and Partial Differential Equations*, Chapman and Hall/CRC, Boca Raton, 2000, p. 568.

12. A. Zygmund, *Trigonometric Series I*, (2nd Edn.) London : Pergamon, 1959.