Indian J. pure appl. Math., **34**(4) : 523-527, April 2003

EXPLICIT NORM ATTAINING POLYNOMIALS

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(*Received* 3 *May* 2001; *after final revision* 5 *February* 2002; *accepted* 6 *May* 2002)

In the early sixties Bishop and Phelps showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. Very recently the problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials.

The first results about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Wener[1], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and the author[3] showed the Radon-Noikodym property is also sufficient for the denseness of norm attaining polynomials. Recently Jimenez-Sevilla and Paya[7] studied the denseness of norm attaining polynomials on preduals of Lorentz sequence spaces.

In this paper, for $p = 1, 2, \infty$, we give explicit descriptions of 2-homogeneous polynomials in $\mathcal{P}(^2 l_p^2)$ attaining their norms at a given unit vector of l_p^2 in terms of their Coefficients.

Recall the following definitions. Throughout this paper E and F will be real Banach spaces. Let m be a positive integer. A function $P : E \to F$ is a continuous m-homogeneous polynomial provided that there exists a m-linear mapping A such that $P(x) = A(x, ..., x)$; $\mathcal{P}(^m E; F)$ will denote the space of F-valued a continuous m-homogeneous polynomials on E, endowed with the norm $\|P\| = \sup_{\|x\| \le 1} \|P(x)\|$ (See [3, 4] for a general reference on infinite dimensional polynomials). P is called *norm attaining* if $\|P\| = \|P(x_0)\|$ for some unit vector $x_0 \in E$. When F is the scalar field, we omit F in the notation $\mathcal{P}(^m E; F)$. Let $P(x, y) = a x^2 + b y^2 + cxy$ denote a 2-homogeneous polynomial on the 2-dimensional real Banach space l_p^2 with real coefficients a, b and c for $p = 1, 2, \infty$.

Lemma 1 [2, Theorem 2.4] — Let $a, b, c \in \mathbf{R}, |a| \le 1, |b| \le 1, 2 < |c| \le 4$. Suppose $P(x, y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2 l_1^2)$. Then :

$$\|P\| = 1 \text{ if and only if } 4|c| - c^2 = 4(|a + b| - ab).$$

***Theorem* 2** — *Let* $(x_0, y_0) \in l_1^2$ *be a point of norm 1, i.e.,* $\|(x_0, y_0)\| = |x_0| + |y_0| = 1$. *Then a polynomial* $P(x, y) = ax^2 + by^2 + cxy$ *in* $\mathcal{P}(^2 l_1^2)$ *with* $\|P\| = 1$ *attaining the norm at* (x_0, y_0) *is one of the following forms*:

*The author wishes to acknowledge the financial support of Korea Research Foundation Grant (ICRF-2002-07)0-C0006

Case 1 — $x_0 y_0 \neq 0$

(*A*) For each $\dfrac{-x_0^2+y_0^2}{x_0^2+y_0^2} \leq \lambda < 1$,

$$P(x,y) = \pm\left\{\left(1-\left(\frac{y_0}{x_0}\right)^2(1-\lambda)\right)x^2 + \lambda y^2 + \left(\pm 2 + 2\left(\frac{y_0}{x_0}\right)(1-\lambda)\right)xy\right\};$$

(*B*) For each $\dfrac{x_0^2-y_0^2}{x_0^2+y_0^2} \leq \beta < 1$,

$$P(x,y) = \pm\left\{\beta x^2 + \left(1-\left(\frac{x_0}{y_0}\right)^2(1-\beta)\right)y^2 + \left(\pm 2 + 2\left(\frac{x_0}{y_0}\right)(1-\beta)\right)xy\right\};$$

(*C*) $P(x,y) = \pm\left\{\left(\dfrac{x_0^2-y_0^2}{x_0^2+y_0^2}\right)x^2 + \left(\dfrac{-x_0^2+y_0^2}{x_0^2+y_0^2}\right)y^2 + \left(\dfrac{\pm 2}{x_0^2+y_0^2}\right)xy\right\};$

(*D*) $P(x,y) = \pm(x^2+y^2+2xy)$.

Case 2 — $x_0 y_0 = 0$

(*E*) For $|b| \leq 1, |c| \leq 2, P(x,y) = \pm x^2 + by^2 + cxy$;

(*F*) For $|a| \leq 1, |c| \leq 2, P(x,y) = ax^2 \pm y^2 + cxy$.

PROOF : Let $P(x,y) = ax^2 + by^2 + cxy$ be a polynomial in $\mathcal{P}(^2 l_1^2)$ with $\|P\| = 1$ attaining the norm at (x_0, y_0). It is clear that if $|a| \leq 1, |b| \leq 1, 0 \leq |c| \leq 2$, then $P(x, y)$ is of the form (*D*). It is easy to show that if $x_0 y_0 = 0$, then $P(x, y)$ is one of the forms (*E*) and (*F*). So we assume that $x_0 y_0 \neq 0$. Since $P(x, y)$ attains its norm at (x_0, y_0), we have $1 = |P(x_0, y_0)| = ax_0^2 + by_0^2 + cx_0 y_0$. First we find coefficients a, b, c such that $|a| \leq 1, |b| \leq 1, 2 < |c| \leq 4$, when $1 = P(x_0, y_0)$. If $x_0 y_0 \geq 0$ and $a + b \geq 0$, using Lemma 1 we can get : For each $\dfrac{-x_0^2+y_0^2}{x_0^2+y_0^2} \leq \lambda \leq 1$,

$$a = 1-\left(\frac{y_0}{x_0}\right)^2(1-\lambda),\ b = \lambda,\ c = 2+2\left(\frac{y_0}{x_0}\right)(1-\lambda)$$

or for each $\dfrac{x_0^2-y_0^2}{x_0^2+y_0^2} \leq \beta \leq 1$, $a = \beta, b = 1-\left(\dfrac{x_0}{y_0}\right)^2(1-\beta), c = 2+2\left(\dfrac{x_0}{y_0}\right)(1-\beta)$,

showing $P(x, y)$ is one of the forms (*A*) and (*B*). If $x_0 y_0 \geq 0$ and $a + b < 0$, again using Lemma 1 we can get :

$$a = \frac{-4b-4c+c^2}{4(1+b)},\ c = 2-2\left(\frac{y_0}{x_0}\right)(1+b) + \frac{\sqrt{2((1+b)((x_0-y_0)^2+1))}}{x_0}.$$

Using $a + b < 0$ and after some calculation we can get :

$$a = \frac{x_0^2 - y_0^2}{x_0^2 + y_0^2},\ b = \frac{-x_0^2 + y_0^2}{x_0^2 + y_0^2},\ c = \frac{2}{x_0^2 + y_0^2},$$

showing $P(x, y)$ is of the form (C). In the case $x_0 y_0 < 0$ using similar arguments of the above case $x_0 y_0 \geq 0$ we can show that $P(x, y)$ is one of the other parts of the forms (A) – (C). Similarly to the case that $|a| \leq 1, |b| \leq 1, 2 < |c| \leq 4$ and $1 = P(x_0, y_0)$, if $|a| \leq 1, |b| \leq 1, 2 < |c| \leq 4$, and $-1 = P(x_0, y_0)$, after some computation we can show that $P(x, y)$ is one of the forms (A) – (C). ∎

Lemma 3 [4, Lemma 2.1] — Let $a, b, c \in \boldsymbol{R}, |a| \leq 1, |b| \leq 1, |c| \leq 2$. Suppose $P(x, y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2 l_2^2)$. Then :

$$\| P \| = 1 \text{ if and only if } 4 - c^2 = 4(|a + b| - ab).$$

***Theorem* 4** — *Let* $(x_0, y_0) \in l_2^2$ *be a point of norm 1, i.e.,* $\| (x_0, y_0) \| = \sqrt{x_0^2 + y_0^2} = 1$. *Then a polynomial* $P(x, y) = ax^2 + by^2 + cxy$ *in* $\mathcal{P}(^2 l_2^2)$ *with* $\| P \| = 1$ *attaining the norm at* (x_0, y_0) *is one of the following forms :*

Case 1 — $x_0 y_0 \neq 0$

(A) For each $-x_0^2 + y_0^2 \leq \lambda < 1$,

$$P(x, y) = \pm \left\{ \left(1 - \left(\frac{y_0}{x_0} \right)^2 (1 - \lambda) \right) x^2 + \lambda y^2 + 2 \left(\frac{y_0}{x_0} \right) (1 - \lambda) xy \right\};$$

(B) For each $x_0^2 - y_0^2 \leq \beta < 1$,

$$P(x, y) = \pm \left\{ \beta x^2 + \left(1 - \left(\frac{x_0}{y_0} \right)^2 (1 - \beta) \right) y^2 + 2 \left(\frac{x_0}{y_0} \right) (1 - \beta) xy \right\}.$$

Case 2 — $x_0 y_0 = 0$

(C) For $|b| \leq 1, |c| \leq 2, P(x, y) = \pm x^2 + by^2 + cxy$;

(D) For $|a| \leq 1, |c| \leq 2, P(x, y) = ax^2 \pm y^2 + cxy$.

PROOF : Let $P(x, y) = ax^2 + by^2 + cxy$ be a polynomial in $\mathcal{P}(^2 l_2^2)$ with $\| P \| = 1$ attaining its norm at (x_0, y_0). It is easy to show if $x_0 y_0 = 0$, then $P(x, y)$ is one of the forms (C) and (D). So we assume that $x_0 y_0 \neq 0$ and $a + b \geq 0$. Since $P(x, y)$ attains its norm at (x_0, y_0), we have $1 = |P(x_0, y_0)| = ax_0^2 + by_0^2 + cx_0 y_0$. We claim that $1 = P(x_0, y_0)$. Indeed, if $-1 = P(x_0 y_0)$, we can get using Lemma 3 when $-x_0^2 + y_0^2 \leq \lambda \leq 1$:

$$a = \frac{-1 + (b - 2) y_0^2 + \sqrt{2} \sqrt{1 - b} \, |y_0|}{x_0^2}.$$

Note that using the arithmetic and geometric means inequality

$$a+b=\frac{(-1+b)-2y_0^2+\sqrt{1-b}\,(\sqrt{2}\,|y_0|)}{x_0^2}\le\frac{-((1-b)+2y_0^2)+2\sqrt{1-b}\,(\sqrt{2}\,|y_0|)}{x_0^2}<0,$$

which is a contradiction of $a+b\ge 0$. Using the fact that $1 = P(x_0, y_0)$ and Lemma 3, we get : $a=1-\left(\frac{y_0}{x_0}\right)^2(1-\lambda)$, $b=\lambda$, $c=2\left(\frac{y_0}{x_0}\right)(1-\lambda)$ when $-x_0^2+y_0^2\le\lambda\le 1$, showing $P(x, y)$ is of the form (A). In the case $a + b < 0$ using similar arguments of the case $a+b\ge 0$ we can show that $P(x, y)$ is of the form (B). ■

Theorem 5 — *Let* $(u_0, v_0)\in l_\infty^2$ *be a point of norm 1, i.e.* $\|(u_0, v_0)\| = \max\{|u_0|, |v_0|\} = 1$. *Then a polynomial* $Q(u, v)\in\mathcal{P}(^2 l_\infty^2)$ *with* $\|Q\|=1$ *attaining the norm at* (u_0, v_0) *is one of the following forms;*

Case 1 — $|u_0| = 1, |v_0| < 0$

(A) For each $\frac{2u_0 v_0}{1+v_0^2}\le\lambda<1,$

$$Q(u, v)=\pm\left\{\left(\frac{v_0^2(\lambda-1)}{(u_0-v_0)^2}+1\right)u^2+\left(\frac{\lambda-1}{(u_0-v_0)^2}+1\right)v^2+\frac{2u_0 v_0(1-\lambda)}{(u_0-v_0)^2}uv\right\};$$

(B) For each $\frac{-2u_0 v_0}{1+v_0^2}\le\beta<1,$

$$Q(u, v)=\pm\left\{\left(\frac{v_0^2(\beta-1)}{(u_0+v_0)^2}+1\right)u^2+\left(\frac{\beta-1}{(u_0+v_0)^2}+1\right)v^2+\frac{2u_0 v_0(1-\beta)}{(u_0+v_0)^2}uv\right\};$$

(C)
$$Q(u, v)=\pm\left\{\left(\frac{1}{1+v_0^2}\right)u^2+\left(\frac{-1}{1+v_0^2}\right)v^2+\left(\frac{2u_0 v_0}{1+v_0^2}\right)uv\right\}.$$

Case 2 — $|v_0| = 1, |u_0| < 1$

(D) For each $\frac{2u_0 v_0}{1+u_0^2}\le\lambda<1,$

$$Q(u, v)=\pm\left\{\left(\frac{\lambda-1}{(u_0-v_0)^2}+1\right)u^2+\left(\frac{u_0^2(\lambda-1)}{(u_0-v_0)^2}+1\right)v^2+\frac{2u_0 v_0(1-\lambda)}{(u_0-v_0)^2}uv\right\};$$

(E) For each $\frac{-2u_0 v_0}{1+u_0^2}\le\beta<1,$

$$Q(u, v) = \pm\left\{\left(\frac{\beta-1}{(u_0-v_0)^2}+1\right)u^2+\left(\frac{u_0^2(\beta-1)}{(u_0-v_0)^2}+1\right)v^2+\frac{2u_0 v_0(1-\beta)}{(u_0-v_0)^2}uv\right\}$$

(*F*) $$Q(u, v) = \pm\left\{\left(\frac{1}{1+u_0^2}\right)u^2+\left(\frac{-1}{1+u_0^2}\right)v^2+\left(\frac{2u_0 v_0}{1+u_0^2}uv\right)\right\}.$$

Case 3 — $|v_0| = 1 = |u_0|$

(*G*) For each $|b| \leq 1, |c| \leq 2$,

$$Q(u, v) = \pm\left\{\frac{1}{4}(1+b+c)\,u^2+\frac{1}{4}(1+b-c)\,v^2 \pm \frac{1}{2}(b-1)\,uv\right\}.$$

PROOF : Let $Q(u, v) \in \mathcal{P}(^2 l_\infty^2)$ be such that $\|Q\| = 1 = |Q(u_0, v_0)|$. Consider the isometry $\pi : l_\infty^2 \to l_1^2$ be such that $\pi(u, v) = \left(\frac{1}{2}(u-v), \frac{1}{2}(u+v)\right)$. Let $x_0 = \frac{1}{2}(u_0 - v_0)$ and $y_0 = \frac{1}{2}(u_0 + v_0)$ and $P(x, y) = (Q \circ \pi^{-1})(x, y) = ax^2 + by^2 + cxy$. Then $P \in \mathcal{P}(^2 l_1^2)$ and $1 = \|P\| = |P(x_0, y_0)|$. Note that $Q(u, v) = \frac{1}{4}(a+b+c)\,u^2 + \frac{1}{4}(a+b-c)\,v^2 + \frac{1}{2}(b-a)\,uv$. From Theorem 2 it is easy to conclude the proof. ■

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