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THE SEQUENCE SPACE $l_M(p, q, s)$ ON SEMINORMED SPACES

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In this paper we define the sequence space $l_M(p, q, s)$ on a semi-normed complex linear space, by using Orlicz function and we give various properties and some inclusion relations on this space.

Key Words : Orlicz Functions; Sequence Spaces

1. INTRODUCTION

Lindenstrauss and Tzafriri[3] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

The space l_M becomes a Banach space, with the norm

$$\| x \| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

which is called an Orlicz space. The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$.

An Orlicz function is a function $M : [0, \infty) - [0, \infty)$, which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by

$$M(x + y) \leq M(x) + M(y)$$

then this function is called modulus function, defined and discussed by Ruckle[8] and Maddox[2].

An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists constant $K > 0$, such that $M(2u) \leq KM(u)$ $(u \geq 0)$. The Δ_2-condition is equivalent to the satisfaction of inequality $M(lu) \leq KlM(u)$ for all values of u and for $l > 1$[4].

Let X be a complex linear space with zero element θ and $X = (X, q)$ be a seminormed space with the seminorm q. By $S(X)$ we denote the linear space of all sequences $x = (x_k)$ with $(x_k) \in X$ and the usual coordinatewise operations: $\alpha x = (\alpha x_k)$ and $x + y = (x_k + y_k)$, for each $\alpha \in C$ where C denotes the set of complex numbers. If $\lambda = (\lambda_k)$ is a scalar sequence and $x \in S(X)$ then we shall write $\lambda x = (\lambda_k x_k)$.

Let M be an Orlicz function, X be a seminormed space with seminorm q, $s \geq 0$ a real number and let $p = (p_k)$ be a sequence of positive real numbers. Then we define

$$l_M(p, q, s) = \left\{ x \in S(X) : \sum_{k=1}^{\infty} k^{-s} \left[M\left(q\left(\frac{x_k}{\rho} \right) \right) \right]^{p_k} < \infty, s \geq 0, \text{ for some } \rho > 0 \right\}.$$

The following inequality and $p = (p_k)$ sequence will be used frequently throughout this paper.

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\}, \quad \text{... (1)}$$

where $a_k, b_k \in C, 0 < p_k \leq \sup_k p_k = H, D = \max(1, 2^{H-1})$[1]

We will now give the theorems that characterize the structure of the sequence $l_M(p, q, s)$

***Theorem* 3** — *Let $H = \sup_k p_k$, then $l_M(p, q, s)$ is a linear space over the field C complex numbers.*

PROOF : Let $x, y \in l_M(p, q, s)$ and $\alpha, \beta \in C$. In order to prove the result we need to find some ρ_3 such that

$$\sum_{k=1}^{\infty} k^{-s} \left[M\left(q\left(\frac{\alpha x_k + \beta y_k}{\rho_3} \right) \right) \right]^{p_k} < \infty.$$

Since $x, y \in l_M(p, q, s)$, there exist some positive ρ_1 and ρ_2 such that

$$\sum_{k=1}^{\infty} k^{-s} \left[M\left(q\left(\frac{x_k}{\rho_1} \right) \right) \right]^{p_k} < \infty$$

and

$$\sum_{k=1}^{\infty} k^{-s} \left[M\left(q\left(\frac{y_k}{\rho_2} \right) \right) \right]^{p_k} < \infty.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is non-decreasing and convex, and since q is a seminorm,

$$\sum_{k=1}^{\infty} k^{-s} \left[M\left(q\left(\frac{\alpha x_k + \beta y_k}{\rho_3} \right) \right) \right]^{p_k} \leq \sum_{k=1}^{\infty} k^{-s} \left[M\left(q\left(\frac{\alpha x_k}{\rho_3} \right) + q\left(\frac{\beta y_k}{\rho_3} \right) \right) \right]$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{2^{p_k}} k^{-s} \left[M\left(q\left(\frac{x_k}{\rho_1} \right) \right) + M\left(q\left(\frac{y_k}{\rho_2} \right) \right) \right]^{p_k} \quad \text{... (2)}$$

$$\leq \sum_{k=1}^{\infty} k^{-s} \left[M\left(q\left(\frac{x_k}{\rho_1} \right) \right) + M\left(q\left(\frac{y_k}{\rho_2} \right) \right) \right]^{p_k}$$

$$\leq D \sum_{k=1}^{\infty} k^{-s} \left[M\left(q\left(\frac{x_k}{\rho_1} \right) \right) \right]^{p_k} + D \sum_{k=1}^{\infty} k^{-s} \left[M\left(q\left(\frac{y_k}{\rho_2} \right) \right) \right]^{p_k} < \infty$$

where $D = \max(1, 2^{H-1})$. This proves that $l_M(p, q, s)$ is a linear space.

***Theorem* 2** — *$l_M(p, q, s)$ is paranormed (need not total paranorm) space with*

$$g(x) = \inf\left\{ \rho^{p_n/H} : \left(\sum_{k=1}^{\infty} k^{-s}\left[M\left(q\left(\frac{x_k}{\rho} \right) \right) \right]^{p_k} \right)^{1/H} \leq 1,\ n = 1, 2, 3, \ldots \right\}$$

where $H = \max(1, \sup_k p_k)$.

PROOF : Clearly $g(x) = g(-x)$. Taking $\alpha = \beta = 1$ in (2), we may write

$$\sum_{k=1}^{\infty} k^{-s}\left[M\left(q\left(\frac{\alpha x_k + \beta y_k}{\rho_3} \right) \right) \right]^{p_k} \leq \sum_{k=1}^{\infty} k^{-s}\left[M\left(q\left(\frac{x_k}{\rho_1} \right) \right) + M\left(q\left(\frac{y_k}{\rho_2} \right) \right) \right]^{p_k}$$

and then using Minkowski's inequality we get $g(x+y) \leq g(x) + g(y)$.

Since $q(\theta) = 0$ and $M(0) = 0$, we get $\inf\{\rho^{p_n/H}\} = 0$ for $x = \theta$.

Finally, we prove that scalar multiplication is continuous. Let λ be any number. Since

$$g(\lambda x) = \inf\left\{ \rho^{p_n/H} : \left(\sum_{k=1}^{\infty} k^{-s}\left[M\left(q\left(\frac{\lambda x_k}{\rho} \right) \right) \right]^{p_k} \right)^{1/H} \leq 1, n = 1, 2, 3, \ldots \right\},$$

we may write

$$g(\lambda x) = \inf$$

$$\left\{ (\lambda r)^{p_n/H} : \left(\sum_{k=1}^{\infty} k^{-s}\left[M\left(q\left(\frac{x_k}{r} \right) \right) \right]^{p_k} \right)^{1/H} \leq 1,\ n = 1, 2, 3, \ldots \right\}.$$

where $r = \rho/\lambda$. Since $|\lambda|^{p_k} \leq \max(1, |\lambda|^H)$, then $|\lambda|^{p_k/H} \leq (\max(1, |\lambda|^H))^{1/H}$. Hence

$$g(\lambda x) = (\max(1, |\lambda|^H))^{1/H}$$

$$\inf\left\{ (r)^{p_n/H} : \left(\sum_{k=1}^{\infty} k^{-s}\left[M\left(q\left(\frac{x_k}{r} \right) \right) \right]^{p_k} \right)^{1/H} \leq 1, n = 1, 2, 3, \ldots \right\}$$

which converges to zero as $g(x)$ converges to zero in $l_M(p, q, s)$. Now suppose that $\lambda_n \to 0$ as $n \to \infty$ and let $x \in l_M(p, q, s)$. Let $\varepsilon > 0$ be given and let N be a positive integer such that

$$\sum_{k=N+1}^{\infty} k^{-s}\left[M\left(q\left(\frac{x_k}{\rho} \right) \right) \right]^{p_k} < \frac{\varepsilon}{2}$$

for some $\rho > 0$. This implies that

$$\left(\sum_{k=N+1}^{\infty} k^{-s}\left[M\left(q\left(\frac{x_k}{\rho} \right) \right) \right]^{p_k} \right)^{1/H} \leq \frac{\varepsilon}{2}.$$

If $0<|\lambda|<1$, then using convexity of M we get

$$\sum_{k=N+1}^{\infty} k^{-s}\left[M\left(q\left(\frac{\lambda x_k}{\rho}\right)\right)\right]^{p_k} < \sum_{k=N+1}^{\infty} k^{-s}\left[|\lambda| M\left(q\left(\frac{x_k}{\rho}\right)\right)\right]^{p_k} < \left(\frac{\varepsilon}{2}\right)^H.$$

Since M is continuous everywhere in $[0, \infty)$, then

$$f(t) = \sum_{k=1}^{N} k^{-s}\left[M\left(q\left(\frac{t x_k}{\rho}\right)\right)\right]$$

is continuous at 0. So there is $1>\delta>0$ such that $|f(t)|<\frac{\varepsilon}{2}$ for $0<t<\delta$. Let K be such that $|\lambda_n|<\delta$ for $n > K$, then for $n > K$ we have

$$\left(\sum_{k=1}^{N} k^{-s}\left[M\left(q\left(\frac{\lambda_n x_k}{\rho}\right)\right)\right]^{p_k}\right)^{1/H} < \frac{\varepsilon}{2}.$$

Thus
$$\left(\sum_{k=1}^{\infty} k^{-s}\left[M\left(q\left(\frac{\lambda_n x_k}{\rho}\right)\right)\right]^{p_k}\right)^{1/H} < \varepsilon, \text{ for } n > K.$$

***Theorem* 3** — *Let M, M_1, M_2 be Orlicz functions which satisfy Δ_2-condition and let s, s_1, s_2 be non-negative real numbers.*

(*i*) If $s > 1$, then $l_{M_1}(p, q, s) \subseteq l_{M \circ M_1}(p, q, s)$.

(*ii*) $l_{M_1}(p, q, s) \cap l_{M_2}(p, q, s) \subseteq l_{M_1+M_2}(p, q, s)$.

(*iii*) If $s_1 \leq s_2$, then $l_M(p, q, s_1) \subseteq l_M(p, q, s_2)$.

PROOF : (i) Let $\varepsilon>0$ and choose δ with $0<\delta<1$ such that $M(t)<\varepsilon$ for $0\leq t\leq \delta$. Write $y_k = M_1\left(q\left(\frac{x_k}{\rho}\right)\right)$ and consider

$$\sum_{k=1}^{\infty} k^{-s}[M(y_k)]^{p_k} = \sum_{1} k^{-s}[M(y_k)]^{p_k} + \sum_{2} k^{-s}[M(y_k)]^{p_k}$$

where the first summation is over $y_k \leq \delta$ and the second over $y_k > \delta$. Since M is continuous, we have

$$\sum_{1} k^{-s}[M(y_k)]^{p_k} < \max(1, \varepsilon^H) \sum_{k=1}^{\infty} k^{-s} < \infty.$$

For $y_k > \delta$, we use the fact that

$$y_k < \frac{y_k}{\delta} \leq 1 + \left(\frac{y_k}{\delta}\right).$$

Since M is non-decreasing and convex, it follows that

$$M(y_k) < M\left(1+\frac{y_k}{\delta}\right) < \frac{1}{2}M(2) + \frac{1}{2}M\left(2\frac{y_k}{\delta}\right).$$

Since M satisfies Δ_2-condition, we have

$$M(y_k) < \frac{1}{2}K\frac{y_k}{\delta}M(2) + \frac{1}{2}\frac{y_k}{\delta}M(2) = K\,y_k\,\delta^{-1}M(2).$$

Hence

$$\sum_{2} k^{-s}[M(y_k)]^{p_k} \le \max\left(1, (K\delta^{-1}M(2))^H\right) \sum_{k=1}^{\infty} k^{-s}[y_k]^{p_k} < \infty.$$

$x \in l_M(p, q, s)$ and $s > 1$ gives

$$\sum_{k=1}^{\infty} k^{-s}[M(y_k)]^{p_k} = \sum_{1} k^{-s}[M(y_k)]^{p_k} + \sum_{2} k^{-s}[M(y_k)]^{p_k}$$

$$\le \max(1, \varepsilon^H) \sum_{k=1}^{\infty} k^{-s} + \max\left(1, (K\delta^{-1}M(2))^H\right) \sum_{k=1}^{\infty} k^{-s}[y_k]^{p_k} < \infty.$$

(*ii*) From (1) we have

$$k^{-s}\left[(M_1+M_2)\left(q\left(\frac{x_k}{\rho}\right)\right)\right]^{p_k} = k^{-s}\left[M_1\left(q\left(\frac{x_k}{\rho}\right)\right) + M_2\left(q\left(\frac{x_k}{\rho}\right)\right)\right]^{p_k}$$

$$\le Dk^{-s}\left[M_1\left(q\left(\frac{x_k}{\rho}\right)\right)\right]^{p_k} + Dk^{-s}\left[M_2\left(q\left(\frac{x_k}{\rho}\right)\right)\right]^{p_k}. \qquad \text{... (3)}$$

Let $x = (x_k) \in l_{M_1}(p, q, s) \cap l_{M_2}(p, q, s)$. Taking summation from $k = 1$ to ∞ in (3), we get $x = (x_k) \in l_{M_1+M_2}(p, q, s)$.

(*iii*) Let $s_1 \le s_2$ and $x \in l_M(p, q, s_1)$. Since $k^{-s_2} \le k^{-s_1}$, we have $x \in l_M(p, q, s_2)$.

We get the following sequence spaces from $l_M(p, q, s)$ by choosing some of the special p and s. If we take $p_k = 1$ for all k, then we have

$$l_M(q, s) = \left\{ x \in S(X) : \sum_{k=1}^{\infty} k^{-s}\left[M\left(q\left(\frac{x_k}{\rho}\right)\right)\right] < \infty, s \ge 0, \rho > 0 \right\}.$$

If we take $s = 0$, then we have

$$l_M(p, q) = \left\{ x \in S(X) : \sum_{k=1}^{\infty} \left[M\left(q\left(\frac{x_k}{\rho}\right)\right)\right]^{p_k} < \infty, \rho > 0 \right\}.$$

If we take $p_k = 1$ for all k and $s = 0$, then we have

$$l_M(q) = \left\{ x \in S(X) : \sum_{k=1}^{\infty} \left[M\left(q\left(\frac{x_k}{\rho}\right)\right)\right] < \infty, \rho > 0 \right\}.$$

***Theorem* 4** — *(i) Suppose that* $0 < r_k \le t_k < \infty$ *for each k. Then* $l_M(r, q) \subseteq l_M(t, q)$.

(ii) $l_M(q) \subseteq l_M(q, s)$.

(iii) $l_M(p, q) \subseteq l_M(p, q, s)$.

PROOF : *(i)* Let $x \in l_M(r, q)$. Then there exists some $\rho > 0$ such that

$$\sum_{k=1}^{\infty} \left[M\left(q\left(\frac{x_k}{\rho} \right) \right) \right]^{r_k} < \infty.$$

This implies that $M\left(q\left(\frac{x_i}{\rho} \right) \right) \le 1$ for sufficiently large values of i.

Since M is non-decreasing, we get

$$\sum_{k=1}^{\infty} \left[M\left(q\left(\frac{x_k}{\rho} \right) \right) \right]^{t_k} \le \sum_{k=1}^{\infty} \left[M\left(q\left(\frac{x_k}{\rho} \right) \right) \right]^{r_k} < \infty.$$

Hence $x \in l_M(t, q)$.

The proof of *(ii)* and *(iii)* is trivial.

***Theorem* 5** — *(i) If* $0 < p_k \le 1$ *for all k, then* $l_M(p, q) \subseteq l_M(q)$.

(ii) If $p_k \ge 1$ for all k, then $l_M(q) \subseteq l_M(p, q)$.

PROOF : *(i)* If $r_k = p_k$ and $t_k = 1$ for all k, in Theorem 4 *(i)*, then $l_M(p, q) \subseteq l_M(q)$.

(ii) If $t_k = p_k$ and $r_k = 1$ for all k, in Theorem 4 *(i)*, then $l_M(q) \subseteq l_M(p, q)$.

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