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VALUE DISTRIBUTION OF MEROMORPHIC FUNCTION CONCERNING SHARED-VALUES*

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In this paper we study the value distribution of meromorphic functions concerning shared-values. Moreover, some criteria for normality of families of meromorphic functions are obtained, which extend a result established by Pang and Zalcman.

Key Words : Shared-Value; Normal Family; Meromorphic Function

1. INTRODUCTION AND MAIN RESULTS

Let D be a domain in the plane C, f be a meromorphic function on D and a be finite complex number. Set

$$\overline{E}(a,f)=f^{-1}(\{a\})\cap D=\{z\in D: f(z)=a\}.$$

We say two meromorphic functions f and g share value a on D if $\overline{E}(a,f)=\overline{E}(a,g)$.

In 1959, W. K. Hayman[1] proved that if $f(z)$ is a transcendental meromorphic (entire) function and n is a positive integer satisfying $n\geq 4$ $(n\geq 3)$, then $(f^n)'$ assumes every finite non-zero value infinitely often. He conjectured in [1] that it remains true for $n = 2$ and $n = 3$. This conjecture was completely solved by W. Bergweiler and A. Eremenko [2], H. H. Chen and M. L. Fang [3], and L. Zalcman [4] independently in 1995. In fact, they proved the following :

Theorem A (see [3]) — *Let $f(z)$ be transcendental meromorphic function and n be a positive integer greater than 1, then $(f^n)'$ assumes every finite non-zero value infinitely often.*

Remark : The condition that n is a positive integer greater than 1 can not be omitted. For example, $f(z)=e^z+z$, but $f'(z)\neq 1$.

In 1998, Y. Wang and M. Fang [5] generalized Theorem A by allowing f to have only multiple zero points and pole points. In fact, Y. Wang and M. Fang proved the following.

Theorem B (see [5]) — *Let $f(z)$ be a transcendental meromorphic function. If f has only zeros of order at least 2 and poles of order at least 2, then f' assumes every finite non-zero value infinitely often.*

In view of the shared-value theory, we prove the following :

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***Theorem* 1** — *Let $f(z)$ be a transcendental meromorphic function in C and a be a finite value. If f and f' share a in C, then $f'(z)$ assumes every finite value infinitely often except possibly one value b, where $b \in \left\{0, \frac{a}{k+1}\right\}$, k is a positive integer.*

According to Nevanlinna's famous five-point theorem and Montel's theorem, many authors proved normality criteria for families of meromorphic functions concerning shared-values. A first attempt to these was made by W. Schwick [6] in the form of the following :

Theorem C — *Let F be a family of meromorphic functions in a domain D and a_1, a_2, a_3 distinct finite complex numbers. If f and f' share a_1, a_2, a_3 on D for every $f \in F$, then F is normal on D.*

Recently, X. Pang and L. Zalcman [7] obtained :

Theorem D — *Let F be a family of meromorphic functions on the unit disc Δ, and let a_1 and a_2 be distinct complex numbers. If f and f' share a_1 and a_2 on Δ for every $f \in F$, then F is normal on Δ.*

In this paper we prove the following.

***Theorem* 2** — *Let F be a family of meromorphic functions on the unit disc Δ. If there exist finite complex numbers a and b ($b \neq 0, \frac{a}{b}$ is not any positive integer) such that for any $f \in F$, f and f' share a on Δ, and $|f(z) - a| \geq \varepsilon$ holds if $f'(z) = b$, where ε is a positive number, then F is normal on Δ.*

Remark : It is easily seen that Theorem D is a consequence of Theorem 2, if we assume $a = a_1, b = a_2, \varepsilon = |b - a|$ and $|a_1| \leq |a_2|$.

From Theorem 2, we immediately have the following result :

***Theorem* 3** — *Let F be a family of meromorphic functions on the unit disc Δ. If there exist finite complex numbers a and b ($b \neq 0, \frac{a}{b}$ is not any positive integer) such that for any $f \in F, f$ and f' share a on Δ and $f'(z) \neq b$, then F is normal on Δ.*

2. Some Lemmas

For the proof of our theorems, we need the following definition and lemmas

Definition 1 (see [7]) — A meromorphic function f on C is called a normal function if there exists a positive number M such that

$$f(z) \leq M,$$

where, as usual, $f(z) = |f'(z)|/(1 + |f(z)|^2)$ denotes the spherical derivative.

From Definition 1, we obtain :

Lemma 1 — A normal meromorphic function has order at most 2.

Lemma 2[2] — Let $g(z)$ be a transcendental meromorphic function with finite order. If $g(z)$ has only finitely many critical values, then $g(z)$ has only finitely many asymptotic values.

Lemma 3[9, 10] — Let $g(z)$ be a transcendental meromorphic function. Suppose that $g(0) \neq \infty$ and the set of finite critical and asymptotic values of $g(z)$ is bounded. Then there exists $R > 0$ such that

$$|g'(z)| \geq \frac{|g(z)|}{2\pi|z|} \log \frac{|g(z)|}{R}$$

holds for any $z \in C - \{0\}$ which is not a pole of $g(z)$.

Lemma 4 — Let f be a nonconstant meromorphic function with finite order, and let a and b be distinct non-zero numbers. If $\overline{E}(f, 0) = \overline{E}(f', a), f'(z) \neq b$, then

$$f(z) = b(z-c) + \frac{A}{(z-c)^k}$$

and $a = (k+1)b$, where A and c are complex numbers and k is a positive integer.

PROOF : Define $g(z) = f(z) - bz$, then $g'(z) = f'(z) - b \neq 0$. We claim that $g(z)$ is not a transcendental meromorphic function, that is, $g(z)$ is a rational function. Suppose that $g(z)$ is a transcendental meromorphic function, then $f(z)$ is also a transcendental meromorphic function. Since $f' \neq b$, by Hayman's inequality (see [12, Theorem 4.5], we obtain that $f(z)$ has infinitely many zeros, $z_1, z_2 \ldots$, and $\lim_{j \to \infty} z_j = \infty$.

Noting that $g'(z) \neq 0$, by Lemma 2 we know that $g(z)$ has only finitely many asymptotic values. Without loss of generality, we assume that $f(0) \neq \infty$, then by Lemma 3 we deduce that

$$\frac{|z_j g'(z_j)|}{|g(z_j)|} \geq \frac{1}{2\pi} \log \frac{|g(z_j)|}{R} = \frac{1}{2\pi} \log \frac{|b z_j|}{R}.$$

Particularly, $\frac{|z_j g'(z_j)|}{|g(z_j)|} \to \infty$ as $j \to \infty$. On the other hand $\frac{|z_j g'(z_j)|}{|g(z_j)|} = \frac{|z_j (b-a)|}{|bz_j|} = \frac{|b-a|}{|b|}$. This is a contradiction. Hence we deduce that $g(z)$ and $f(z)$ are rational functions.

Suppose that $g(z)$ is a polynomial. It follows from $g'(z) \neq 0$ that $f(z) = g(z) + bz = Bz + c$, where $B (\neq b, 0)$ and c are complex numbers, which contradicts $\overline{E}(f, 0) = \overline{E}(f', a)$. Therefore, $g(z)$ is not a polynomial. Let

$$g(z) = \alpha + \frac{\beta}{\gamma},$$

where α, β and γ are polynomials, β and γ are coprime, and $\deg \beta < \deg \gamma$.

It follows from $g'(z) \neq 0$ that $\alpha' \equiv 0$. Thus α is a constant, and

$$g'(z) = \frac{\beta'\gamma - \gamma'\beta}{\gamma^2}. \qquad \ldots (1)$$

Since $g'(z) \neq 0$, we deduce from (1) that the zeros of $\beta'\gamma - \gamma'\beta$ are the zeros of γ^2. We denote the zeros of $\beta'\gamma - \gamma'\beta$ by $\omega_1, \omega_2, \ldots, \omega_m$, and the related orders by $l_1, l_2, \ldots, l_m$. Since β and γ are coprime, we see from (1) that ω_i is the zero of γ with order $l_i + 1$ ($i = 1, 2, \ldots, m$). Hence we have

$$\deg \gamma + \deg \beta - 1 = \deg (\beta'\gamma - \gamma'\beta) = \sum_{i=1}^{m} (l_i + 1) - m \leq \deg \gamma - m,$$

which implies that $m = 1$ and $\deg \beta = 0$. Therefore

$$f(z) = b(z-c) + \frac{A}{(z-c)^k} + d, \qquad \text{... (2)}$$

where $A\ (\neq 0)$, c and d are complex numbers, k is a positive integer.

As $\overline{E}(f, 0) = \overline{E}(f', a)$, it follows from (2) that $a = (k+1)\,b$ and $d = 0$. Thus the proof of the lemma is complete.

Remark : Pang and Zalcman gave a lemma in [6, Lemma 5] in which the same result as Lemma 4. is obtained under an additional condition that f has poles of order at least 2 and $f''(z) \not\equiv 0$. As shown above, the method used in our proof is very simple and completely different from that in [6].

Lemma 5[2] — Let f be a transcendental meromorphic function with finite order. If f has only zeros of order at least 2, then $f'(z)$ assumes every finite non-zero value infinitely often.

Lemma 6 — Let f be a transcendental meromorphic function with finite order, and let a be a finite complex numbers. If $\overline{E}(f, 0) = \overline{E}(f', a)$, then $f'(z)$ assumes any finite non-zero value infinitely often.

PROOF : In fact, in the case $a \neq 0$, we can derive a contradiction that f is a rational function if $f'(z) - b$ has only finitely many zeros in the same way as in the proof of Lemma 4. And in the case $a = 0$, we may find that the zeros of $f(z)$ are of multiplicity ≥ 2. Therefore, we conclude Lemma 6 and Lemma 5.

As an immediate consequence of Lemma 4 and Lemma 6, we have the following result.

Lemma 7 — Let $f(z)$ be a nonconstant meromorphic function of finite order, and let a be a finite complex number. If $\overline{E}(f, 0) = \overline{E}(f', a)$, then every finite value in C can be taken by $f'(z)$ at most except one value which is 0 or $\frac{a}{k+1}$, where k is a positive integer.

Lemma 8[8] — Let F be a family of meromorphic functions on the unit disc Δ, all of whose zeros have multiplicity at least k, and suppose there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, $f \in F$. Then if F is not normal, for each $\alpha, 0 \leq \alpha \leq k$, there exist

a) a number r, $0 < r < 1$;

b) points z_n, $|z_n| < r$;

c) functions $f_n \in F$; and

d) positive numbers $\rho_n \to 0$;

such that

$$\frac{f_n(z_n + \rho_n \xi)}{\rho_n^{\alpha}} = g_n(\xi) \to g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a meromorphic function on C such that $g^{\#}(\xi) \leq g^{\#}(0) = kA + 1$.

3. PROOF OF THE THEOREMS

PROOF OF THEOREM 1

We define the family $F = \{f(z + z') - a, z \in C\}_{z' \in C}$ and consider two cases.

Case 1 — F is normal in the plane. Using Marty's normality criterion, we deduce that $f - a$ is a normal function on C. Therefore, by Lemma 1, $f(z)$ has order at most 2. The conclusion of Theorem 3 follows from Lemma 6.

Case 2 — F is not normal at a point z_0. Without loss of generality, we may assume $z_0 = 0$. By Lemma 8, we can find that there exists one sequence of positive numbers $\rho_n \to 0$, two sequences of complex numbers z_n' and $z_n, z_n \to z_0^* \, (z_0^* \in \Delta)$, such that

$$g_n(\xi) = \frac{f(z_n + z_n' + \rho_n \xi) - a}{\rho_n} \to g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on C and satisfies $g^{\#}(\xi) \le g^{\#}(0) = (|a| + 1) + 1 = |a| + 2$. By Lemma 1, the order of g is at most 2.

We claim that

$$\overline{E}(g, 0) = \overline{E}(g', a), \quad \lim_{n \to \infty} z_n' = \infty.$$

In fact, suppose $g(\xi_0) = 0$. Since g is not constant, there exist $\xi_n, \xi_n \to \xi_0$, such that (for sufficiently large n)

$$g_n(\xi_n) = \frac{f(z_n + z_n' + \rho_n \xi_n) - a}{\rho_n} = 0.$$

Thus $f(z_n + z_n + \rho_n \xi_n) = a$. It follows from $\overline{E}(f, a) = \overline{E}(f', a)$ that $f'(z_n + z_n + \rho_n \xi_n) = a$, hence $g'(\xi_0) = \lim_{n \to \infty} g_n'(\xi_n) = \lim_{n \to \infty} f'(z_n + z_n' + \rho_n \xi_n) = a$. We have established $\overline{E}(g, 0) \subseteq \overline{E}(g', a)$.

Suppose that w_0 is a point such that $g'(w_0) = a$. We claim that $g'(\xi) \not\equiv a$. In fact, if $g'(\xi) \equiv a$, then $g^{\#}(\xi) \le |a|$, which contradicts $g^{\#}(0) = |a| + 2$. Since $g'(w_0) = a$ and $g'(\xi) \not\equiv a$, there exist $w_n, w_n \to w_0$, such that, for sufficiently large n,

$$g_n(w_n) = f'(z_n + z_n' + \rho_n w_n) = a.$$

Hence

$$g_n(w_n) = \frac{f(z_n + z_n' + \rho_n w_n) - a}{\rho_n} = 0.$$

Letting $n \to \infty$, we obtain $g(w_0) = 0$. It follows that $\overline{E}(g', a) \subseteq \overline{E}(g, 0)$. Thus $\overline{E}(g, 0) = \overline{E}(g', a)$. This implies that $g''(\xi) \not\equiv 0$.

Finally, suppose that $\lim_{n \to \infty} z_n = z_1$. Since $g''(\xi) \not\equiv 0$, we may choose a point ξ_0 such that $g(\xi_0) \ne \infty, g''(\xi_0) \ne 0$.

Since $g(\xi_0) = \lim_{n \to \infty} \dfrac{f(z_n + z_n + \rho_n \xi_0) - a}{\rho_n}$, we have $\lim_{n \to \infty} f(z_n + z_n + \rho_n \xi_0) = a$, so that

$f(z_1+z_0^*)=a$. On the other hand, $g''(\xi_0)=\lim_{n\to\infty}\rho_n f''(z_n+z_n+\rho_n\xi_0)$, therefore, we have $\lim_{n\to\infty} f''(z_n+z_n'+\rho_n\xi_0)=\infty$, thus $f''(z_1+z_0^*)=\infty$, which is a contradiction. Thus, $\lim_{n\to\infty} z_n'=\infty$.

For a given finite value b (possibly $b\neq 0$ or $b\neq\frac{a}{k+1}$, where k is a positive integer), then from Lemma 7, there exists ξ_0 such that $g'(\xi_0)=b$. As $g''(\xi)\not\equiv 0$, by Hurwitz's theorem, there exist a sequence $\xi_n\to\xi_0$ such that

$$g_n'(\xi_n)=f'(z_n+z_n'+\rho_n\xi_n)=b.$$

Noting that $\lim_{n\to\infty}(z_n+z_n'+\rho_n\xi_n)=\infty$, we deduce that $f'(z)$ assumes b infinitely often. This completes the proof of Theorem 1.

PROOF OF THEOREM 2

Suppose that F is not normal on Δ. Define $F_1=\{f-a: f\in F\}$, then F_1 is not normal on Δ. By Lemma 8, we can find that there exist a sequence of complex numbers z_n, a sequence of positive numbers ρ_n, $\rho_n\to 0$, and a sequence of functions $f_n\in F$ such that

$$g_n(\xi)=\frac{f_n(z_n+\rho_n\xi)-a}{\rho_n}\to g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on C such that $g^{\#}(\xi)\le g^{\#}(0)=(|a|+1)+1=|a|+2$.

In the same way as in the proof of Theorem 1, we may also obtain that $\overline{E}(g,0)=\overline{E}(g',a)$, $g''(\xi)\not\equiv 0$ and g is of order at most 2.

We now claim that $g'(\xi)\neq b$. In fact, suppose there exists ξ_0 such that $g'(\xi_0)=b$. Since $g''\not\equiv 0$, there exist $\xi_n\to\xi_0$, such that

$$g_n'(\xi_n)=f_n'(z_n+\rho_n\xi_n)=b.$$

Noting that $|f_n(z_n+\rho_n\xi_n)-a|\ge\varepsilon$ when $f_n'(z_n+\rho_n\xi_n)=b$, we have

$$g(\xi_0)=\lim_{n\to\infty} g_n(\xi_n)=\infty,$$

which contradicts $g'(\xi_0)=b$. Thus $g'(\xi)\neq b$.

By Lemma 4, we get

$$g(\xi)=b(\xi-c)+\frac{A}{(\xi-c)^k},$$

and $a=(k+1)b$, where k is a positive integer. This gives a contradiction to the condition that $\frac{a}{b}$ is not any positive integer. Therefore, F_1 is normal on Δ, that is, F is normal on Δ. This completes the proof of Theorem 2.

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REFERENCES

1. W. K. Hayman, *Research Problems in Function Theory*, Athlone Press University of London, 1967.
2. W. Bergweiler and A. Eremenko, *Rev. Mat. Iberoam,* **11** (1995), 355-73.
3. H. H. Chen and M. L. Fang, *Science in China*, **25** (1995), 121-27.
4. L. Zalcman, *On some problems of Hayman*, Preprint (Bar-Ilan University), 1995.
5. Y. F. Wang and M. L. Sang, *Acta Math. Sinica*, New Series, 41 (1998), 743-48.
6. W. Schwick, *Arch. Math.* (Basel), **59** (1992), 52-54.
7. X. Pang and L. Zalcman, *Ark. Math.*, **38** (2000), 171-82.
8. X. Pang and L. Zalcman, *Bull. London Math. Soc.*, **32** (2000), 325-31.
9. P. Rippon and G. Stallard, *Proc. Amer. Math. Soc.*, **115** (1999), No.2, 355-62.
10. W. Bergweiler, *Arch. Math.* (Basel), **64** (1995), 199-202.
11. M. Fang, *Analysis*, **20** (2000), 45-49.
12. L. Yang, *Value Distribution Theory*, Springer-Verlag, Berlin-Heidelberg, 1993.