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STABILITY OF TRIANGULAR POINTS IN THE GENERALISED PHOTOGRAVITATIONAL ROBE'S RESTRICTED THREE-BODY PROBLEM

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We study the effect of oblateness and radiation pressure forces of the primaries on the location and the stability of the triangular points in the Robe's restricted three-body problem. It is observed that the equations of motion and location of the triangular points are affected by the radiation pressure force and oblateness of the primaries. We further note that these points are stable for $0 \le \mu \le \mu_{crit}$ and unstable for $\mu_{crit} \le \mu \le 1/2$. It is also seen that for these points the range of stability increase or decreases according as radiating and oblateness coefficients increase or decrease.

Key Words : Stability; Triangular Points; Generalised Photogravitational Robe's RTBP

1. INTRODUCTION

Robe (1977) introduced a new kind of restricted three-body problem that incorporates the effect of buoyancy force. He regards one of the two principal bodies as a rigid spherical shell of mass m_1, filled with homogeneous incompressible fluid of density ρ_1. The other one is a point mass m_2, located outside the shell. The third body i.e. the particle of negligible mass has density ρ_3 and moves inside the shell under the influences of the gravitational attraction of the principal bodies and the buoyancy force of the fluid ρ_1. The Robe model may provide some insight into the problemof small oscillations of the earth's core in the gravitational field of the earth moon system.

Two situations were considered by Robe

(*i*) That in which m_2 describes a circular orbit around the shell and (*ii*) the case of elliptical orbits for m_2, assuming the shell empty (i.e. $\rho_1 = 0$) or the densities $\rho_1 = \rho_3$. The centre of the shell is an equilibrium point for the third body in both the instances, which led him to study the conditions for its linear stability.

Shrivastava and Gorain (1991) studied the effect of a small perturbation in the coriolis and centrifugal forces on the location of the equilibrium point. They considered the circular case with equal densities ($\rho_1 = \rho_3$) and evaluated the resulting shift in the location of the equilibrium point.

Robe's RTBP was further reanalysed by Plastino & Plastino (1995) and a new assumption was incorporated by them namely the configuration of the fluid body was that described by an hydrostatic equilibrium figure i.e. Roche's ellipsoid.

In our present model of Robe's RTBP, we have considered two additional parametric influences on the third body with the assumption that the first principal body is an oblate spheroid and the second body is radiating; and then we have examined the location and stability of triangular points.

2. Equations of Motion

m_1 and m_2 are supposed to be the masses of the two principal bodies, ρ_1 is the density of the fluid inside the shell having mass m_1; while ρ_3 is the density of the third infinitesimal mass point m_3, moving inside it :

A_1 is used for oblateness coefficient of the first primary such that $0 < A_1 \ll 1$ and $A_1 = (AE_1^2 - AP_1^2)/5R^2$ where AE_1 is the equatorial radius and AP_1 is the polar radius of the primary.

Further radiation repulsive force is denoted by q which is given by the equation $F_p = F_g(1-q)$: F_g being the gravitational attractive force and $q \sim 1$ i.e. $0 < 1 - q \ll 1$.

Perturbation in the potential between the two primaries due to the radiation pressure is being neglected because mass of the first is supposed to be sufficiently large.

The following forces are acting on m_3 :

(*i*) Gravitational attraction of m_2; (*ii*) Oblateness effect of m_1, (*iii*) Effect of radiation of m_2; (*iv*) The gravitational force, exerted by the fluid density ρ_1 i.e. $F_A = -(4/3)\pi G \rho_1 m_3 M_1 M_3$; (*v*) The buoyancy force, $F_B = (4/3)\pi G \rho_1^2 m_3 M_1 M_3/\rho_3$.

We suppose that $M_1 M_3 = r_1$ and $M_2 M_3 = r_2$; M_1, M_2 & M_3 being the centres of m_1, m_2 and m_3 respectively and G is the gravitational constant (x, y, z) are the coordinates of the infinitesimal mass m_3; and the line joining m_1 and m_2 is the x-axis.

The total potential acting at m_3 is

$$-\frac{Gm_2 q}{r_2} + \frac{4}{3}\pi G \rho_1 \left(1 - \frac{\rho_1}{\rho_3}\right) r_1^2 - \frac{Gm_1}{r_1} - \frac{Gm_1 A_1}{2r_1^3}$$

The equations of motion in this problem are

$$\left.\begin{aligned} \ddot{x} - 2n\dot{y} &= \frac{\partial \Omega}{\partial x} \\ \ddot{y} + 2n\dot{x} &= \frac{\partial \Omega}{\partial y} \end{aligned}\right\} \qquad \text{... (1)}$$

with

$$\Omega = \frac{n^2}{2}(x^2 + y^2) - k r_1^2 + \frac{\mu q}{r_2} + \frac{1-\mu}{r_1} + \frac{(1-\mu)A_1}{2r_1^3} \qquad \text{... (2)}$$

where

$$k = \frac{4}{3}\pi \rho_1 \left(1 - \frac{\rho_1}{\rho_3}\right), \mu = \frac{m_2}{m_1 + m_2}, 0 < \mu < 1$$

$$n^2 = 1 + \frac{3}{2}A_1; x_1 = -\mu, x_2 = 1 - \mu, 0 < \mu < 1$$

and

$$\left.\begin{aligned} r_1^2 &= (x+\mu)^2 + y^2 \\ r_2^2 &= (x - 1 + \mu)^2 + y^2 \end{aligned}\right\}. \qquad \text{... (3)}$$

We have assumed $\rho_1 \neq \rho_3$ i.e. $k \neq 0$ in eq. (2).

3. Location of Triangular Points

The solutions of the equations $\Omega_x = 0, \Omega_y = 0, y \neq 0$ give the locations of triangular points.

i.e.
$$x\left[(n^2-2k)-\frac{\mu q}{r_2^3}-\frac{(1-\mu)}{r_1^3}-\frac{3}{2}\frac{(1-\mu)A_1}{r_1^5}\right]$$

$$-2k\mu-\frac{(\mu-1)\mu q}{r_2^3}-\frac{(1-\mu)\mu}{r_1^3}-\frac{3}{2}\frac{(1-\mu)A_1}{r_1^5}\mu=0 \qquad \dots (4)$$

and
$$y\left[(n^2-2k)-\frac{\mu q}{r_2^3}-\frac{(1-\mu)}{r_1^3}-\frac{3}{2}\frac{(1-\mu)A_1}{r_1^5}\right]=0. \qquad \dots (5)$$

Eqs. (4) and (5) disclose that

$$r_2^3 = q/n^2, \qquad \dots (6)$$

$$\frac{1}{r_1^3}+\frac{3}{2}\frac{A_1}{r_1^5}=\left(1-\frac{2k}{1-\mu}\right)+\frac{3}{2}A_1 \qquad \dots (7)$$

i.e. $r_1 = 1$ equating the coefficients of A_1 in (7) Knowing r_2 and r_1 from the eqs. (6) and (7) the co-ordinates of the triangular points are found by solving eq. (3) for x and y.

The co-ordinates of the triangular points corresponding to L_4 and L_5 are given exactly by

$$\left.\begin{aligned} x &= -\mu+\frac{1}{2}+\frac{r_1^2-r_2^2}{2} \\ y &= \pm\left[\frac{(r_1^2+r_2^2)}{2}-\frac{1}{4}-\frac{(r_1^2-r_2^2)^2}{4}\right]^{1/2} \end{aligned}\right\}. \qquad \dots (8)$$

When the respective primaries are neither oblate spheroid nor radiating i.e. $A_1 = 0$ and $q = 1$, the solutions of the eq. (6) and (7) are $r_i = 1\ (i = 1, 2)$.

Therefore, we can assume that solutions of (6) and (7) are given by $r_i = 1 + \varepsilon_i$ where ε_i's are very small. Restricting only linear terms in ε_i, A_1 and q and coupling terms $A_1 q$, we have

$$\varepsilon_1 = \frac{2}{3}k+\frac{2}{3}k\mu-\frac{5}{3}kA_1$$

$$\varepsilon_2 = \frac{q}{3}-\frac{1}{3}-\frac{A_1 q}{2} \qquad \dots (9)$$

$$r_1 = 1+\frac{2}{3}k+\frac{2}{3}k\mu-\frac{5}{3}kA_1$$

$$r_2 = \frac{2}{3}+\frac{q}{3}-\frac{A_1 q}{2}$$

putting the values of $r_i\ (i = 1, 2)$ in (8) we get

$$x = -\mu+\frac{7}{9}+\frac{2}{3}k+\frac{2}{3}k\mu-\frac{5}{3}kA_1-\frac{2}{9}q+\frac{A_1 q}{3}$$

$$y = \pm \sqrt{2}\left[\frac{4}{9}+\frac{k}{6}+\frac{k\mu}{6}-\frac{5}{12}kA_1-\frac{7}{36}q-\frac{A_1 q}{12}\right]. \quad \text{... (10)}$$

4. Stability of the Triangular Points

The characteristics equation of the variational equation corresponding to (1) is

$$\lambda^4 + (4n^2 - \Omega^0_{xx} - \Omega^0_{yy})\,\lambda^2 + \Omega^0_{xx}\cdot\Omega^0_{yy} - (\Omega^0_{xy})^2 = 0 \quad \text{... (11)}$$

where the second partial derivatives of Ω are denoted by subscripts and the superscript 0 indicates that those derivatives are to be evaluated at the triangular point (x_0, y_0), co-ordinates of $L_{4,5}$.

At the triangular points L_4 and L_5, we have

$$\Omega^0_{xx} = f(x-1+\mu)^2 + g(x+\mu)^2 - 2k,$$

$$\Omega^0_{xy} = \pm y\,[f(x-1+\mu)+g(x+\mu)],$$

$$\Omega^0_{yy} = y^2\,[f+g]-2k, > 0,$$

where $$f = \frac{3\mu n^2}{r_2^2} > 0 \text{ and } g = 3(1-\mu)\left(1+\frac{5}{2}A_1\right) > 0. \quad \text{... (12)}$$

Substituting the values of Ω^0_{xx}, Ω^0_{yy} and Ω^0_{xy} in eq. (11) we have

$$\lambda^4 + [4n^2 - f r_2^2 - g + 4k]\,\lambda^2 + y^2 fg - 2k\,[f r_2^2 + g - 2k] = 0. \quad \text{... (13)}$$

Let $\lambda^2 = \Lambda$

Then the characteristic eq. (13) becomes

$$\Lambda^2 + [4n^2 - f r_2^2 - g + 4k]\,\Lambda + y^2 fg - 2k\,[f r_2^2 + g - 2k] = 0 \quad \text{... (14)}$$

We can easily evaluate the following expressions as

$$4n^2 - f r_2^2 - g + 4k = n^2 - 3(1-\mu)A_1 + 4k;$$

$$y^2 fg = 9\mu(1-\mu)n^2\left(1+\frac{5}{2}A_1\right)\left(1-\frac{r_2^2}{4}\right);$$

and $$2k\,[f r_2^2 + g - 2k] = 15A_1 k - 3\mu A_1 k - 4k^2$$

Then the characteristic eq. (14) becomes

$$\Lambda^2 + [n^2 - 3(1-\mu)A_1 + 4k]\,\Lambda + 9\mu(1-\mu)n^2\left(1+\frac{5}{2}A_1\right)\left(1-\frac{r_2^2}{4}\right)$$
$$-\ 15A_1 k + 3\mu A_1 k + 4k^2 = 0. \quad \text{... (15)}$$

Then $$\Lambda_{1,2} = \frac{1}{2}[\{3(1-\mu)A_1 - n^2 - 4k\}$$
$$\pm\left\{(3(1-\mu)A_1 - n^2 - 4k)^2 - 36\mu(1-\mu)n^2\left(1+\frac{5}{2}A_1\right)\left(1-\frac{r_2^2}{4}\right)\right.$$

$$-\ 60 A_1 k + 12 \mu A_1 k + 16 k^2\}^{1/2}].$$

We observe that the roots of the eq (13)

$$\lambda_1 = \Lambda_1^{1/2},\ \lambda_2 = -\Lambda_1^{1/2},\ \lambda_3 = \Lambda_2^{1/2},\ \lambda_4 = -\Lambda_2^{1/2} \qquad \dots (16)$$

are functions of μ, q and A_1 and their nature determines the nature of the discriminant,

$$\Delta = [n^2 + 4k - 3(1-\mu) A_1]^2 - 36 \mu (1-\mu) n^2 \left(1 + \frac{5}{2} A_1\right)\left(1 - \frac{r_2^2}{4}\right)$$

$$-60 A_1 k + 12 \mu A_1 k + 16 k^2.$$

However, the following three cases can be discussed here :

(*i*) When Δ is positive, $\Lambda_{1,2}$ are negative and the roots (16) as

$$\lambda_{1,2} = \pm i(-\Lambda_2)^{1/2} = \pm i s_4,\ \lambda_{3,4} = \pm i(-\Lambda_2)^{1/2} = \pm i s_5$$

show that the triangular points are linearly stable, s_5 and s_4 are the angularfrequencies of short and long period terms, which the solution of variational equations consists.

(*ii*) When Δ is negative, the real parts of two of the four roots (16) are positive and equal and, hence, the equilibria are unstable.

(*iii*) When $\Delta = 0$, both the values of the four roots (15) in pairs are equal. So the solution of the variational equations contain secular terms and consequently the triangular point is unstable.

We are, however, considering only the first two cases in this paper.

5. Critical Mass

The discriminant of quadratic eq. (15) is zero

i.e.
$$[(n^2 - 3(1-\mu) A_1 + 4k]^2 \left[36 \mu (1-\mu) n^2 \left(1 + \frac{5}{2} A_1\right)\left(1 - \frac{r_2^2}{4}\right)\right.$$

$$\left. - 60 A_1 k + 12 \mu A_1 k + 16 k^2\right] = 0$$

when
$$\left[9 A_1^2 + 36 n^2 \left(1 + \frac{5}{2} A_1\right)\left(1 - \frac{r_2^2}{4}\right)\right] \mu^2 - \left[18 A_1^2 - 6n^2 A_1\right.$$

$$\left. - 12 A_1 k + 36 n^2 \left(1 + \frac{5}{2} A_1\right)\left(1 - \frac{r_2^2}{4}\right)\right] \mu + n^4 + 9A_1^2$$

$$-6n^2 A_1 + 8kn^2 + 36 A_1 k = 0 \qquad \dots (17)$$

or,
$$\left[36 n^2 \left(1 + \frac{5}{2} A_1\right)\left(1 - \frac{r_2^2}{4}\right)\right] \mu^2$$

$$-\left[36 n^2 \left(1 + \frac{5}{2} A_1\right)\left(1 - \frac{r_2^2}{4}\right) - 6n^2 A_1 - 12 A_1 k\right] \mu$$

$$+\ n^4 - 6n^2 A_1 + 8kn^2 + 36 A_1 k = 0 \qquad \dots (18)$$

ignoring the terms having higher powers of A_1. When $A_1 = 0$, eq. (18) coincides with that of Chernikov (1970)

Solution of eq. (18) for $0 \le \mu \le 1/2$ is

$$\mu_{crit} = \frac{\alpha - \beta^{1/2}}{\gamma} \qquad \text{... (19)}$$

where

$$\alpha = \left(1 - \frac{q^{2/3}}{4}\right)(1 + 3A_1) - \frac{A_1 k}{3}$$

$$\beta = \left[\left(1 - \frac{q^{2/3}}{4}\right)(1 + 3A_1) - \frac{A_1 k}{3}\right]^2$$

$$-4\left[\left(1 - \frac{q^{2/3}}{4}\right)(1 + 3A_1) + A_1\right] \cdot \left[\frac{1}{36} - \frac{A_1}{12} + \frac{2}{9}k + \frac{4}{3}A_1 k\right] = 0$$

$$\Upsilon = 2\left[\left(1 - \frac{q^{2/3}}{4}\right)(1 + 3A_1) + A_1\right]$$

Let $q = 1 - \varepsilon$, where ε is small. Restricting computation with linear terms in A_1 upto quadratic terms in ε and retaining terms upto $\varepsilon^2 A_1$, we obtain

$$\alpha = \left[\frac{3}{4} + \frac{1}{6}\varepsilon + \frac{1}{36}\varepsilon^2\right] + \left[\left(\frac{9}{4} - \frac{k}{3}\right) + \frac{1}{2}\varepsilon + \frac{1}{12}\varepsilon^2\right] A_1$$

$$\beta = \left[\left(\frac{23}{48} - \frac{2}{3}k\right) + \left(\frac{25}{108} - \frac{4}{27}k\right)\varepsilon + \left(\frac{43}{648} - \frac{2}{81}\right)\varepsilon^2\right]$$

$$+ \left[\left(\frac{235}{72} - \frac{133}{18}k\right) + \left(\frac{3}{2} - \frac{13}{9}k\right)\varepsilon + \left(\frac{5}{12} - \frac{13}{54}k\right)\varepsilon^2\right] A_1$$

$$\Upsilon = \left[\frac{3}{2} + \frac{1}{3}\varepsilon + \frac{1}{18}\varepsilon^2\right] + \left[\frac{13}{2} + \varepsilon + \frac{\varepsilon^2}{6}\right] A_1$$

and the expression of μ_{crit} in (19) becomes

$$\mu_{crit} = \left[\frac{2}{3}\left\{\frac{3}{4} - \left(\frac{23}{48} - \frac{2}{3}k\right)^{1/2}\right\} + 3\left\{\frac{1}{6}\left(\frac{25}{108} - \frac{4}{27}k\right)^{1/2}\right\}\varepsilon\right.$$

$$\left. + 18\left\{\frac{1}{36} - \left(\frac{43}{648} - \frac{2}{81}k\right)^{1/2}\right\}\varepsilon^2\right]$$

$$+ \left[\frac{2}{13}\left\{\left(\frac{9}{4} - \frac{k}{3}\right) - \left(\frac{235}{72} - \frac{133}{18}k\right)^{1/2}\right\} + \left\{\frac{1}{2} - \left(\frac{3}{2} - \frac{13}{9}k\right)^{1/2}\right\}\varepsilon\right.$$

$$+ 6\left\{\frac{1}{12}-\left(\frac{5}{12}-\frac{13}{54}k\right)^{1/2}\right\}\varepsilon^2\Bigg]A_1 \qquad \dots (20)$$

i.e.

$$\mu_{crit} = [(0.0385211 + 0.3210289\,k)$$
$$- (0.9433756 - 0.4618802\,k)\,\varepsilon$$
$$- (4.1368092 - 0.8626621\,k)\,\varepsilon^2]$$
$$+ [(0.0682121+0.2633248\,k)$$
$$- (0.7247448 - 0.5896918\,k)\,\varepsilon$$
$$- (3.3729833) - 1.1188614\,k)\,\varepsilon^2]\,A_1 \qquad \dots (21)$$

We can easily observe in eq. (21) that the value of critical mass parameter μ_{crit} decreases with the increase in ε and A_1; that is the range of stability of the points increases or decreases according as radiating and oblateness coefficients increase or decrease.

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