Indian J. pure appl. Math., **34**(4) : 551-568, April 2003

EXISTENCE OF PERIODIC SOLUTIONS FOR $\dddot{x} + \omega^2 \dot{x} - \mu F(x, \dot{x}, \ddot{x}) = 0$

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(*Received* 23 *August* 2001; *accepted* 9 *May* 2002)

In this paper existence of periodic solutions for $\dddot{x} + \omega^2 \dot{x} = \mu F(x, \dot{x}, \ddot{x})$ is studied. This equation may appear in some problems of energy and acceleration. We consider conditions for the third order nonlinear system in order to reduce it to a second order nonlinear system. Then, studying the fixed points of the reduced system, we will impose some conditions on the map F in order to have periodic solutions for the third order system. For example we will show that if the map F is the sum of partial derivations of a map $f(x, \dot{x})$, then under some conditions, the third order equation has periodic solutions. Also we will prove that we do not need to have the parameter μ small enough. At the end we will show in some examples and theorems that how we can have infinity many periodic solutions and homoclinic orbits for the third equation if it is truncated at some quadratic terms.

Key Words : Periodic Solution; Homoclinic Orbit

1. INTRODUCTION

The oscillator systems are very important in physics and engineering, because these kind of systems occur in many mechanical systems such as vibration systems. The differential equation

$$\dddot{x} + \omega^2 \dot{x} - \mu F(x, \dot{x}, \ddot{x}) = 0 \qquad \text{... (1)}$$

where μ is a real parameter, may occur in the energy and acceleration problems. Also we know that periodic orbits play quite important role in analysing a system. Existence of periodic solutions for the system can provide the situation for it, the system has stable physical behaviour i.e. the physical behaviour of the system saves frequently similar along the time. In fact if we provide the initial conditions for the system such that the corresponding solution is periodic or starts near to a periodic solution, then we can approximate the physical behaviour of the system in a finite time. Also the homoclinic orbits give global information about the behaviour of the orbits of eq. (1). So we consider some conditions for eq. (1) in order to have periodic orbits, in fact we will impose some conditions on the map $F(x, \dot{x}, \ddot{x})$ in order to have periodic solutions. We will show that if the map F is the sum of partial derivations of a map $f(x, \dot{x})$, then under some conditions, the eq. (1) can be reduced to a second order differential equation which has periodic solutions. For example, we will see that the two differential equations

$$\dddot{x} + \omega^2 \dot{x} - \mu (\dot{x}^2 + x\ddot{x} + \ddot{x}\, Df(\dot{x})) = 0$$

and

$$\dddot{x} + \omega^2 \dot{x} - \mu (\dot{x}^2 + x\ddot{x} + \dot{x}\, Df(x)) = 0$$

have infinity many periodic solutions. In section 1, we give definitions and theorems in order to use them in the other sections. Section 2 will give conditions for reducing the order of eq. (1). Section 3 is devoted to existence of fixed points for the reduced system. Section 4 is devoted to existence of periodic solutions for eq. (1) under some conditions. In section 5 by giving some examples, we give a few systems and some theorems about them to show the effect of the some quadratic terms on the system. The theorems are concluded by applications of our talk in section 4. In addition we will show that in some of the examples, there exist infinity many homoclinic orbits.

2. PRELIMINARY

In this section we give some definitions and theorems in order to use them in the other sections. Our main tool in this paper is the Poincare-Andronove-Hopf's theorem which is usually called Hopf bifurcation theorem.

Definition 1 — Let $\alpha : R \to R$ be a C^1 map such that $\alpha(0) = \alpha_0$. We say that α passes transversally through α_0 if

$$\frac{d\alpha}{dt}(0) \neq 0.$$

Also "transversal passing through α_0" means that we pass through α_0 by a map α which passes transversally through α_0.

Definition 2 — Let $\dot{X} = F(X)$ be a vector field such that $F(X_0) = 0$. A solution $\gamma = \gamma(t)$ of $\dot{X} = F(X)$ is a homoclinic solution (orbit) if $\gamma \neq X_0$ and $\lim_{t \to \pm\infty} \gamma(t) = X_0$. Also γ is a periodic solution if there exists $T > 0$ such that for each $t \in R$, $\gamma(T+t) = \gamma(t)$. A periodic solution γ is nontrivial if $\gamma \neq X_0$.

There are several versions of the Poincare-Andronove-Hopd's theorem, but we give the version which is in [H], page 344. The interested reader can find a various version of this theorem in [*G/H*], page 151 or [W1], page 270.

Theorem 1 — *(Poincare-Andronove-Hopf) Let $\dot{X} = A(\lambda)X + F(X, \lambda)$ be a C^k, with $k \geq 3$, planar vector field depending on a scalar parameter λ such that $F(0, \lambda) = 0$ and $D_X F(0, \lambda) = 0$ for all sufficiently small $|\lambda|$. Assume that the linear part $A(\lambda)$ at the origin has the eigenvalues $\alpha(\lambda) \pm i\beta(\lambda)$ with $\alpha(0) = 0$ and $\beta(0) \neq 0$. Furthermore, suppose that the eigenvalues cross the imaginary axis with nonzero speed, that is,*

$$\frac{d\alpha}{d\lambda}(0) \neq 0.$$

Then, in any neighbourhood U of the origin in R^2 and any given $\lambda_0 > 0$ there is a $\bar{\lambda}$ with $\bar{\lambda} < \lambda_0$ such that the differential equation $\dot{X} = A(\bar{\lambda})X + F(X, \bar{\lambda})$ has a nontrivial periodic orbit in U.

For proof see [H].

In order to use this theorem in our work, we prove the following deformation of it.

Proposition 1 — Let

$$\dot{X} = F(X, \lambda) \qquad \text{... (2)}$$

be a C^r, with $r \geq 3$, planar vector field depending on a scalar parameter λ such that $F(X_0, \lambda_0) = 0$. Moreover let $x : I \to R$, with I is an open interval containing λ_0, be a C^1 map such that $x(\lambda_0) = X_0$ and for all $\lambda \in I, x(\lambda)$ is a fixed point for (2). Also assume that the linear part of (2) at $x(\lambda)$ has the eigenvalues $\alpha(\lambda) \pm i\beta(\lambda)$ with $\alpha(\lambda_0) = 0$ and $\beta(\lambda_0) \neq 0$. Furthermore, suppose that the eigenvalues cross the imaginary axis with nonzero speed, that is,

$$\frac{d\alpha}{d\lambda}(\lambda_0) \neq 0.$$

Then for each $\lambda_1 > \lambda_0$ and each $\varepsilon > 0$, we can find $\bar{\lambda}$ with $|\bar{\lambda} - \lambda_0| < \lambda_1 - \lambda_0$ such that the differential equation $\dot{X} = F(X, \bar{\lambda})$ has a nontrivial periodic solution which lies in the open disk $D(x(\bar{\lambda}), \varepsilon)$ in R^2.

PROOF : We can assume that $X_0 = 0$ and $\lambda_0 = 0$. Otherwise, with simple transformation we transfer them to 0. Now let $\lambda \in I$, Taylor expansion of F around $x(\lambda)$ gives

$$X = D_X F(x(\lambda), \lambda)(X - x(\lambda)) + G(X, \lambda)$$

which is a C^r vector field depending on λ. Furthermore for each $\lambda \in I$, $G(x(\lambda), \lambda) = 0$ and $D_X G(x(\lambda), \lambda) = 0$. Putting $D_X F(x(\lambda), \lambda) = A(\lambda)$ and $Y = X - x(\lambda)$ and $H(Y, \lambda) = G(Y + x(\lambda), \lambda)$, we get

$$\dot{Y} = A(\lambda) Y + H(Y, \lambda) \qquad \text{... (3)}$$

which is a C^r vector field such that for each $\lambda \in I$, $H(0, \lambda) = 0$ and $D_X H(0, \lambda) = 0$. Furthermore the linear part of (3) at the origin has the eigenvalues $\alpha(\lambda) \pm i\beta(\lambda)$ with $\alpha(0) = 0$ and $\beta(0) \neq 0$. Moreover

$$\frac{d\alpha}{d\lambda}(0) \neq 0.$$

So Theorem 1 holds for (3). Now let $\lambda_1 > 0$ and U be the open disk $D(0, \varepsilon)$ in R^2 and let $\bar{\lambda}$, with $|\bar{\lambda}| < \lambda_1$ be such that $\dot{Y} = A(\bar{\lambda}) Y + H(Y, \bar{\lambda})$ has a nontrivial periodic solution in U. This implies that

$$\dot{X} = F(X, \bar{\lambda}) = D_X F(x(\bar{\lambda}), \bar{\lambda})(X - x(\bar{\lambda})) + G(X, \bar{\lambda})$$

has a nontrivial periodic solution in $U + x(\bar{\lambda})$ where $U + x(\bar{\lambda}) = \{x + x(\bar{\lambda}) : x \in U\}$. □

We end this section with the definition of Hopf bifurcation

Definition 3 — Consider the system (2) and suppose that proposition 1 holds for (2). Then we say that transversal passing through λ_0, a Hopf bifurcation occurs at X_0 for (2). (See [W1], page 270)

2. REDUCTION

In this section we show that under some conditions, we can reduce the third order differential equation

$$\dddot{x} + \omega^2 \dot{x} - \mu F(x, \dot{x}, \ddot{x}) = 0 \qquad \text{... (4)}$$

to a second order differential equation, such that if the reduced differential equation has periodic solutions (or homoclinic orbits) then (4) has too. We can write (4) in the following form

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -\omega^2 y + \mu F(x, y, z) \end{cases} \qquad \text{... (5)}$$

Suppose that $F : R^3 \to R$ is a C^r $(r \geq 2)$ map with $F(0) = 0$ and $DF(0) = 0$ such that satisfies the following condition

$$F(x, y, z) = \frac{\partial f(x, y)}{\partial x} y + \frac{\partial f(x, y)}{\partial y} z \qquad \text{... (6)}$$

where $f(x, y)$ is a C^{r+1} map. Substituting (6) in (4) where $y = \dot{x}$ and $z = \ddot{x}$, we get

$$\frac{d}{dt}(\ddot{x} + \omega^2 x - \mu f(x, \dot{x})) = \dddot{x} + \omega^2 \dot{x} - \mu \Bigg(\underbrace{\frac{\partial f(x, \dot{x})}{\partial x} \dot{x} + \frac{\partial f(x, \dot{x})}{\partial y} \ddot{x}}_{F(x,\, \dot{x},\, \ddot{x})} \Bigg) = 0. \qquad \text{... (7)}$$

This implies that

$$\ddot{x} + \omega^2 x - \mu f(x, \dot{x}) = k$$

where k is a real constant. Let us put $k = \omega^2 \lambda$, then we get

$$\ddot{x} + \omega^2 (x - \lambda) - \mu f(x, \dot{x}) = 0. \qquad \text{... (8)}$$

Now changing the variable x to $x = \bar{x} + \lambda$ in (8), we obtain

$$\ddot{\bar{x}} + \omega^2 \bar{x} - \mu f(\bar{x} + \lambda, \dot{\bar{x}}) = 0. \qquad \text{... (9)}$$

Dropping the bars, (9) becomes

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + \mu f(x + \lambda, y) \end{cases} \qquad \text{... (10)}$$

which is a two parameters family of perturbations of the oscillator system $\ddot{x} + \omega^2 x = 0$. Now in the following lemma we prove that if (10) has periodic (or homoclinic) solutions then (5) has too.

Lemma 1 — Suppose that $(x(t), \dot{x}(t))$ is a solution of (10). Then $(x(t) + \lambda, \dot{x}(t), \ddot{x}(t))$ is a solution of (5). Moreover, if (10) has periodic (or homoclinic) solutions then (5) has periodic (or homoclinic) solutions.

PROOF : It is obvious that if $(x(t), \dot{x}(t))$ is a solution of (10) then $(x(t) + \lambda, \dot{x}(t), \ddot{x}(t))$ is a solution of (5). Now let $(x(t), \dot{x}(t))$ be a periodic solution of (10) with period T. Then for each $t \in R$ we have

$$\ddot{x}(t + T) = \lim_{h \to 0} \frac{\dot{x}(t + T + h) - \dot{x}(t + T)}{h} = \lim_{h \to 0} \frac{\dot{x}(t + h) - \dot{x}(t)}{h} = \ddot{x}(t)$$

which shows that $(x(t) + \lambda, \dot{x}(t), \ddot{x}(t))$ is a periodic solution of (5).

Now let $(x(t), \dot{x}(t))$ be a homoclinic orbit of (10) such that $\lim_{t \to \infty} (x(t), \dot{x}(t)) = \lim_{t \to -\infty} (x(t), \dot{x}(t)) = (x_0, 0)$ where $(x_0, 0)$ is a fixed point for (10). From (10) it can be easily checked that

$$\lim_{t \to \infty} \ddot{x}(t) = \lim_{t \to \infty} \ddot{x}(t) = -\omega^2 x_0 + \mu f(x_0 + \lambda, 0) = 0$$

and so $$\lim_{t \to \pm\infty} (x(t) + \lambda, \dot{x}(t), \ddot{x}(t)) = (x_0 + \lambda, 0, 0).$$

Since $(x_0, 0)$ is a solution of (10) so $(x_0 + \lambda, 0, 0)$ is a solution of (5), hence it is a fixed point for (5). The above relations shows that $(x(t) + \lambda, \dot{x}(t), \ddot{x}(t))$ is a homoclinic orbit for (5). □

Remark 1 : Since $F(0) = 0$ and $DF(0) = 0$, so $Df(0, 0) = 0$. This is because of the fact that

$$DF = (f_{xx} \cdot y + f_{xy} \cdot z, f_{xy}\, y + f_x + f_{yy}\, z, f_y)$$

where f_x, f_y, f_{xx}, f_{xy} and f_{yy} are the partial derivations of f. Therefore $DF\ (0) = (0, f_x\,(0, 0), f_y\ (0, 0))$ $= 0$ which implies that $f_x\,(0, 0) = f_y\,(0, 0) = 0$. Hence we have $Df = (0,\ 0) = 0$. In addition if $f(0, 0) \neq 0$, putting $g\,(x, y) = f(x, y) - f(0, 0)$, then g is a C^{r+1} map which satisfies (6). Moreover $g\,(0, 0) = D_g\,(0, 0) = 0$. So in (10) we can assume that $f\ (0,\ 0) = 0$.

3. Existence of Fixed Points

We know that for $\lambda = 0$, the origin is a fixed point for (10). If $\lambda \neq 0$, it is important to know about fixed points of (10). In the following theorem we consider the fixed points of (10) with respect to the parameters λ and μ.

***Theorem* 2** — *For each $0 < M \in R$, there exist an open rectangular $\mathcal{R}$ in R^2 containing the set $[-M, M] \times \{0\}$ and a C^1 map $x : \mathcal{R} \to R$ such that $(x, (\mu, \lambda), 0)$ is a fixed point for (10). In addition the real part of the eigenvalues corresponding to $(x\,(\mu, \lambda), 0)$ is $\mu f_y\,(x\,(\mu, \lambda) + \lambda, 0)$. Also for each $(\mu, 0), (0, \lambda) \in \mathcal{R}$ we have $x\,(\mu, 0) = x\,(0, \lambda) = 0$.*

PROOF : Consider the map

$$B : R^3 \to R$$

$$B : (x, \mu, \lambda) \mapsto -\omega^2 x + \mu f(x + \lambda, 0).$$

Since for each $\mu \in R, B\,(0, \mu, 0) = 0$ and $B_x\,(0, \mu, 0) = -\omega^2 \neq 0$, so by the implicit function theorem, there exist an open neighbourhood U_μ containing $(\mu, 0)$ and a C^1 map $x_\mu : U_\mu \to R$ such that for each $(v, \lambda) \in U, B\,(x_\mu\,(v, \lambda)) = 0$. Hence $(x_\mu\,(v, \lambda), 0)$ is a fixed point for (10). A finite number of such U_μ covers $[-M, M] \times \{0\}$ in R^2. Let $U_{\mu_0}, \ldots, U_{\mu_k}$ be a such cover. We can consider $\mathcal{R}$ an open rectangular contained in the union of U_{μ_i} $(i = 0, 1, \ldots, k)$ such that it contains $[-M, M] \times \{0\}$. If (v, λ) belongs to U_{μ_i} and U_{μ_j}, $(0 \leq i, j \leq k)$ then by the uniqueness of the solution in the implicit function theorem, $x_{\mu_i}\,(v, \lambda) = x_{\mu_j}\,(v, \lambda)$. This implies that we can define the C^1 map

$$x : \mathcal{R} \to R$$

$$x : (\mu, \lambda) \mapsto x_{\mu_i}\,(\mu, \lambda)\ (\mu, \lambda) \in U_{\mu_i}$$

now computing the linear part of (10) at $(x\,(\mu, \lambda), 0)$ $((\mu, \lambda) \in \mathcal{R})$, we obtain the following matrix

$$A\,(\mu, \lambda) = \begin{pmatrix} 0 & 1 \\ -\omega^2 + \mu f_x & \mu f_y \end{pmatrix}$$

where all partial derivations are computed at $(x\,(\mu, \lambda) + \lambda, 0)$. The eigenvalues of $A\,(\mu, \lambda)$ are

$$\Gamma_{1,2}\,(\mu, \lambda) = \frac{\mu f_y \pm \sqrt{(\mu f_y)^2 + 4\,(\mu f_x - \omega^2)}}{2}. \qquad \ldots (11)$$

If $(\mu f_y)^2 + 4(\mu f_x - \omega^2) < 0$ then the real part of $\Gamma_{1,2}(\mu, \lambda)$ is μf_y. On the other hand by Remark 1, $f_x(0, 0) = f_y(0, 0) = 0$, hence for μ fixed we have

$$\lim_{\lambda \to 0} (\mu f_y(x(\mu, \lambda) + \lambda, 0))^2 + 4 \mu f_x(x(\mu, \lambda) + \lambda, 0) = 0.$$

Since $[-M, M] \times \{0\}$ is compact and contained in $\mathcal{R}$ so we can shrink $\mathcal{R}$ such that $[-M, M] \times \{0\} \subset \mathcal{R}$ and for each $(\mu, \lambda) \in \mathcal{R}$ $(\mu f_y)^2 = 4(\mu f_x - \omega^2) < 0$. The proof is now complete. □

Corollary 1 — Let $\mathcal{R}$ be as it in Theorem 2 and $\mu \in R$ be fixed. Since the constant M in theorem 2 can be arbitrary large, so we can have $\mathcal{R}$ such that $(\mu, 0) \in \mathcal{R}$. Hence in eq. (10) we can consider $|\mu|$ arbitrary large and suppose that $(\mu, 0) \in \mathcal{R}$.

Corollary 2 — Suppose that $\mathcal{R}$ be as it in corollary 1. The fixed point $(x(\mu, \lambda), 0)$, $((\mu, \lambda) \in \mathcal{R})$, is isolated.

PROOF : Let us consider the map

$$\Omega : R \to R$$
$$x \mapsto -\omega^2 x + \mu f(x + \lambda, 0).$$

We know that $\Omega(x(\mu, \lambda), 0) = 0$ and $\Omega_x(x(\mu, \lambda)) = -\omega^2 + \mu f_x(x(\mu, \lambda) + \lambda, 0)$. But in $\mathcal{R}$

$$(\mu f_y(x + \lambda, 0))^2/4 + \mu f_x(x + \lambda, 0) - \omega^2 < 0.$$

Hence $-\omega^2 + \mu f_x(x(\mu, \lambda) + \lambda, 0) < 0$. So Ω is decreasing in a neighbourhood of $x(\mu, \lambda)$. This shows that $(x(\mu, \lambda), 0)$ is isolated. □

Remark 2 : Suppose that we have a system such

$$\begin{cases} \dot{x} = y \\ \dot{y} = g(x, y) \end{cases}$$

where g is a C^1 function. Let $\gamma = (x(t), y(t))$ be a solution of this system and $I = (a, b)$ be the interval of $y(t) > 0$ (res. $y(t) < 0$). Then for each $t \in I$, $x(t)$ is increasing (res. decreasing). Hence we can consider $y(t)$ as a C^1 function of $x(t)$. Indeed if we define the map $Y : (x(a), x(b)) \to R$ such that $Y(x(t)) = y(t)$ $(t \in I)$ then Y is a well defined C^1 map such that $(x, Y(x)) \in \gamma$.

4. EXISTENCE OF PERIODIC SOLUTIONS

In this setion we impose some onditions on (10) in order to have periodic solutions. Indeed we look for Hopf bifuration for it. Throughout this setion we assume that the set $\mathcal{R}$ and the map $x : \mathcal{R} \to R$ are as them in Corollary 1.

Lemma 3 — Let $(\mu, \lambda) \in \mathcal{R}$. Then $(\partial x / \partial \lambda)(\mu, \lambda) \neq -1$.

PROOF : If $(\mu, \lambda) \in \mathcal{R}$ then

$$-\omega^2 x(\mu, \lambda) + \mu f(x(\mu, \lambda) + \lambda, 0) = 0.$$

Computing the derivation of this equation with respect to λ, we get

$$-\omega^2 \frac{\partial x}{\partial \lambda}(\mu, \lambda) + \mu \frac{\partial f}{\partial x}(x(\mu, \lambda) + \lambda, 0)\left(\frac{\partial x}{\partial \lambda}(\mu, \lambda) + 1\right) = 0.$$

If $(\partial x/\partial \lambda)(\mu, \lambda) = -1$ then $-\omega^2 (\partial x/\partial \lambda)(\mu, \lambda) = 0$ which implies that $(\partial x/\partial \lambda)(\mu, \lambda)$ 0. This is a contradiction, so $(\partial x/\partial \lambda)(\mu, \lambda) \neq -1$. □

***Theorem* 3** — *Let* $(\mu_0, \lambda_0) \in \mathcal{R}$ *and* $\mu_0 \neq 0$ *and consider the two hypothesis*

$H_1 : f_y(x(\mu_0, \lambda_0) + \lambda_0, 0) = 0$

$H_2 : f_{xy}(x(\mu_0, \lambda_0) + \lambda_0, 0) \neq 0.$

Then (*i*) For $\mu = \mu_0$, transversal passing through λ_0, a Hopf bifurcation occurs at $(x(\mu_0, \lambda_0), 0)$ for (10).

(*ii*) There exist an interval I containing μ_0 and a C^1 map $\lambda : I \to R$ such that $0 \notin I$ and $\lambda(\mu_0) = \lambda_0$ and for each $\mu \in I$, H_1 and H_2 hold for $(\mu\, \lambda(\mu))$.

(*iii*) There exist an interval I containing μ_0 such that $0 \notin I$ and for each $\mu \in I$ the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -\omega^2 y + \mu(f_x(x, y)\, y + f_y(x, y), z) \end{cases} \qquad \ldots (12)$$

has infinity many periodic solutions.

PROOF : (*i*) We show later that the eigenvalues corresponding to $(x(\mu, \lambda), 0)$ are obtained by (11). By H_1 we have $Re\ (\Gamma_{1,2}(\mu_0, \lambda_0)) = 0$. On the other hand

$$\frac{d\,\mathrm{Re}\,(\Gamma_{1,2})}{d\lambda}(\mu_0, \lambda_0) = \mu_0 f_{xy}(x(\mu_0, \lambda_0) + \lambda_0, 0)\left(\frac{\partial x}{\partial \lambda}(\mu_0, \lambda_0) + 1\right).$$

Hence by H_2 and Lemma 3 we have $(d\ Real\ (\Gamma_{1,2}))/(d\lambda)(\mu_0, \lambda_0) \neq 0$. Hence from Definition 3, for $\mu = \mu_0$ transversal passing from λ_0 a Hopf bifurcation occurs at $(x(\mu_0, \lambda_0), 0)$ for (10).

(*ii*) Let $(\mu_0, \lambda_0) \in \mathcal{R}$ with $\mu_0 \neq 0$, H_1 and H_2 hold. Since $x(\mu, \lambda)$ is a C^1 map and H_2 holds at (μ_0, λ_0) so there exists a neighbourhood W of (μ_0, λ_0), with $W \subset \mathcal{R}$ such that for each $(\mu, \lambda) \in W$,

$$f_{xy}(x(\mu, \lambda) + \lambda, 0)\left(\frac{\partial x}{\partial \lambda}(\mu, \lambda) + 1\right) \neq 0.$$

Now we consider the map

$\Psi : W \to R$

$(\mu, \lambda) \mapsto f_y(x(\mu, \lambda) + \lambda, 0).$

By H_1, $\Psi(\mu_0, \lambda_0) = 0$. On the other hand

$$\frac{\partial \Psi}{\partial \lambda}(\mu_0, \lambda_0) = f_{xy}(x(\mu_0, \lambda_0) + \lambda_0, 0)\left(\frac{\partial x}{\partial \lambda}(\mu_0, \lambda_0) + 1\right) \neq 0.$$

Hence by the implicit function theorem there exist an interval I containing μ_0 and a C^1 map $\lambda : I \to R$ such that $\lambda(\mu_0) = \lambda_0$ and for each $\mu \in I$, $(\mu, \lambda(\mu)) \in W$ and $\Psi(\mu, \lambda(\mu)) = 0$. We can shring I such that $0 \notin I$ and so for each $\mu \in I$, H_1 and H_2 hold for $(\mu, \lambda(\mu))$.

(*iii*) Suppose that the interval I and the map $\lambda : I \to R$ are as them in (*ii*) and let $\mu_1 \in I$. By (*i*), transversal passing through $\lambda(\mu_1)$ a Hopf bifurcation occurs at $(x(\mu_1, \lambda(\mu_1)), 0)$ for (10). Let $\varepsilon_1 > 0$ be given and $\lambda_1 > \lambda(\mu_1)$, then, by Proposition 1, we can find $\bar{\lambda}_1$, with $|\bar{\lambda}_1 - \lambda(\mu_1)| < \lambda_1 - \lambda(\mu_1)$, such that

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + \mu_1 f(x + \bar{\lambda}_1, y) \end{cases}$$

has a nontrivial periodic solution γ_1 in the open disk $D(x(\mu_1, \bar{\lambda}_1), \varepsilon_1)$ in R^2. Now let $\varepsilon_2 > 0$ be such that $\gamma_1 \cap D(x(\mu_1, \bar{\lambda}_1), \varepsilon_2) = \phi$. We can find $\bar{\lambda}_2$, with $|\bar{\lambda}_2 - \lambda(\mu_1)| < \lambda_1 - \lambda(\mu_1)$, such that

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + \mu_1 f(x + \bar{\lambda}_2, y) \end{cases}$$

has a nontrivial periodic solution γ_2 in the open disk $D(x(\mu_1, \bar{\lambda}_2), \varepsilon_2)$ in R^2. Continuing this method, we can find infinity many nontrivial periodic solutions (depending on λ) for (10). Hence by Theorem 1 we conclude that (12) has infinity many nontrivial periodic solutions. □

Remark 3 : We note that part (*i*) of Theorem 3, if $\lambda_0 = 0$ then the condition H_1 holds automatically. Indeed the fixed point $x(\mu_0, \lambda_0) = x(\mu_0, 0) = 0$ hence $f_y(x(\mu_0, 0) + 0, 0) = f_y(0, 0) = 0$. Therefore H_1 holds. In this case if

$$f_{xy}(0, 0) = f_{xy}(x(\mu_0, 0) + 0, 0) \neq 0$$

then H_2 holds too. Hence by (1) of Theorem 3, transversal passing through 0, a Hopf bifurcation occurs at the origin for (10). □

Proposition 2 — Let $f : R^2 \to R$ be a C^3 function such that $f(0,0) = 0$ and $Df(0, 0) = 0$. If $f_{xy}(0) \neq 0$ then for each $\mu \in R$ with $(\mu, 0) \in \mathcal{R}$ the following system has infinity many periodic solutions,

$$\ddot{x} + \omega^2 \dot{x} - \mu(\dot{x} f_x(x, \dot{x}) + \ddot{x} f_y(x, \dot{x})) = 0$$

PROOF : By Remark 3 transversal passing through $\lambda = 0$ a Hopf bifurcation occurs at the origin for

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + \mu f(x + \lambda, y) \end{cases}$$

Using Theorem 3 the proof is complete. □

Proposition 3 — Let $g : R^2 \to R$ be C^3 map such that $g(0, 0) = 0$ and $Dg(0, 0) = 0$. In addition let $g_{xy}(0, 0 \neq -a$ where $a \in R$ is constant. Then for each $\mu \in R$, the following system has infinity many periodic solutions.

$$\ddot{x} + \omega^2 \dot{x} - \mu(a\dot{x} + g_x(x, \dot{x}))\dot{x} - \mu(ax + g_y(x, \dot{x}))\ddot{x} = 0. \qquad \text{... (13)}$$

PROOF : Consider the system (10) with $f(x, y) = axy + g(x, y)$. We get

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + \mu a(x + \lambda) y + g(x + \lambda, y)) \end{cases} \qquad \text{... (14)}$$

For $\lambda = 0$ and each $\mu \in R$ fixed, (0, 0) is a fixed point for (14). Hence by Theorem 2, we can assume that $(\mu, 0) \in \mathcal{R}$. Now we have

$$\frac{\partial^2 (axy + g(x, y))}{\partial x \partial y}(0, 0) = a + g_{xy}(0, 0) \neq 0$$

By (*i*) in theorem 3 and remark 3, transversal passing through $\lambda = 0$, a Hopf bifurcation occurs at the origin for (14). Hence by Theorem 3, (13) has infinity many periodic solutions. □

Remark 4 : Let $f: R \to R$ be a C^3 function and $f(0) = Df(0) = 0$. For each $\mu \in R$ fixed, by Theorem 2, we can assume that $(\mu, 0) \in \mathcal{R}$. Then by (*i*) of Theorem 3 and Remark 3, for each $\mu \in R$ fixed, transversal passing through $\lambda = 0$, a Hopf bifurcation occurs at the origin for the following systems

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + \mu((x + \lambda)y + f(y)) \end{cases}, \quad \begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + \mu((x + \lambda)y + f(x + \lambda)) \end{cases}$$

Hence by Theorem 3,

$$\ddot{x} + \omega^2 \dot{x} - \mu(\dot{x}^2 + x\ddot{x} + \ddot{x}\, Df(\dot{x})) = 0$$

and
$$\ddot{x} + \omega^2 \dot{x} - \mu(\dot{x}^2 + \dot{x}\, Df(x) + x\ddot{x}) = 0$$

have infinity many periodic solutions.

We end this section by a theorem which will be used in section 5

Theorem 4 — *Let $f: R^2 \to R$ be a C^r map which satisfies the condition H3: For all $x, y \in R, f(x, y) = f(x, -y)$. Also let $\gamma(t) = (x(t), y(t))$ be a solution of the system*

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + f(x, y) \end{cases}$$

and suppose that γ intersects x-axis at distinct points $(x_1, 0)$ and $(x_2, 0)$. Then

(i) If both $(x_1, 0)$ and $(x_2, 0)$ are regular points, then γ is a periodic solution.

(ii) If $(x_1, 0)$ is a fixed point and $(x_2, 0)$ is a regular point, then γ is a homoclinic.

(iii) If both $(x_1, 0)$ and $(x_2, 0)$ are fixed point, then γ is a part of a hetroclinic.

PROOF : (*i*) We can consider $(x_1, 0)$ and $(x_2, 0)$ such that γ has no intersection with x-axis between $(x_1, 0)$ and $(x_2, 0)$. Let us define those part of γ which lies between $(x_1, 0)$ and $(x_2, 0)$ by

$$\begin{cases} \alpha : [0, T] \to R^2 \\ t \mapsto (x(t), y(t)) \end{cases}$$

where $T > 0$ and $\alpha(0) = (x_1, 0)$ and $\alpha(T) = (x_2, 0)$. If we define

$$\begin{cases} \beta_1 : [-T, 0] \to R^2 \\ t \mapsto (x(-t), -y(-t)) \end{cases}$$

then β_1 is a solution of the system such that its graph is symmetric **with** α with respect to y-axis. Furthermore $\beta_1(0) = (x_1, 0)$ and $\beta_1(-T) = (x_2, 0)$. So β_1 lies along α and hence γ is periodic.

(*iii*) Without loss of genericity we can suppose that $\gamma(0) = (x_2, 0)$ and $\lim_{t \to -\infty} \gamma(t) = (x_1, 0)$. Then the curve

$$\begin{cases} \beta_2 : [0, \infty] \to R^2 \\ t \mapsto (x(-t), -y(-t)) \end{cases}$$

is a solution of the system such that its graph is symmetric with γ with respect to y-axis. Furthermore, $\lim_{t \to \infty} \beta_2(t) = (x_1, 0)$ and $\beta_2(0) = (x_2, 0)$. Hence β_2 lies along γ and so γ is a homoclinic.

(*iii*) It is similar to (*ii*). □

5. Applications

Throughout this section the set $\mathcal{R}$ is as it in Corollary 1. In this section we give some examples in order to show the applications of the Theorem 3 and results of this paper. We are going to study the effects of the second order terms on the system. For this purpose, we put $g(x, y, \lambda) = f(x + \lambda, y)$. Then (10) can be written as

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + \mu g(x, y, \lambda) \end{cases}.$$

We know that $g(0, 0, 0) = 0$ and $Dg(0, 0, 0) = 0$. The Taylor expansion of g at (0, 0,0) is

$$g(x, y, \lambda) = \frac{1}{2}(g_{xx}(0, 0, 0)x^2 + g_{yy}(0, 0, 0)y^2 + g_{\lambda\lambda}(0, 0, 0)$$

$$\lambda^2 + 2g_{xy}(0, 0, 0)xy + 2g_{x\lambda}(0, 0, 0)x + (....)$$

From definition of g we have $g_{xx}(0, 0, 0) = g_{x\lambda}(0, 0, 0) = g_{\lambda\lambda}(0, 0, 0) = f_{xx}(0, 0)$ and $g_{xy}(0, 0, 0) = g_{y\lambda}(0, 0, 0) = f_{xy}(0, 0)$ and $g_{yy}(0, 0, 0) = f_{yy}(0, 0)$. Putting $f_{xx}(0, 0) = a, f_{xy}(0, 0) = b, f_{yy}(0, 0) = c$ and truncating the Taylor expansion at $o(2)$, we get

$$\begin{cases} \dot{x} = y \\ \dot{y} = -a\mu\lambda^2 + (2a\mu\lambda - \omega^2)x + 2b\mu\lambda y + \mu(ax^2 + 2bxy + cy^2) \end{cases} \quad ... (15)$$

Hence the (15) can be considered as

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + \mu f(x + \lambda, y) \end{cases}$$

with $f(x, y) = ax^2 + 2bxy + cy^2$. If $b \neq 0$ then $f_{xy}(0, 0) = b \neq 0$. Hence by Remark 3, (15) has infinity many periodic solutions. Hence we have the following theorem.

***Theorem* 5** — *The following differential equation with $b \neq 0$ has infinity many periodic solutions*

$$\ddot{x} + \omega^2 \dot{x} = \mu(ax + b\dot{x})\dot{x} + \mu(bx + c\dot{x})\ddot{x}$$

PROOF : By Theorem 3 and the above discussion it is obvious. □

Now we give some examples

Example I — We consider (15) with $a = b = 0$ and $c \neq 0$. In this case (15) has the following form

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + \mu c y^2 \end{cases}$$

Putting $v = \mu c$, we get

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + v y^2 \end{cases} \qquad \text{... (16)}$$

which has (0, 0) as the only fixed point. We study (16) where $v > 0$, the case $v < 0$ is similar. On the curve $x = (v/\omega^2)\, y^2$ we have $\dot{y} = 0$. In addition the sign of $\dot{y}$ is shown in the Figure 1.

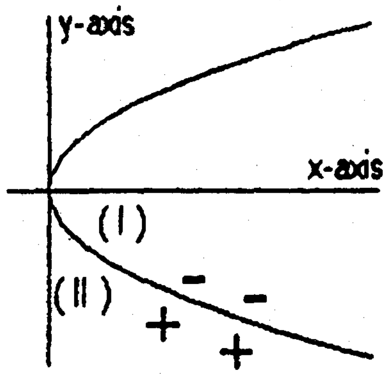

FIG. 1. The sign of y

Hence, with respect to figure 1, we can conclude that the local behaviour of a solution $\gamma = (x(t), y(t))$ of (16), which intersects the positive x-axis is as the figure. 2 If $y(t) < 0$ (res. $y(t) > 0$), then $x(t)$ is decreasing (res. increasing). On the other hand in region (*I*) of the figure 1, $\dot{y} < 0$, hence $y(t)$ is decreasing. This implies that γ intersects $x = (v/\omega^2)\, y^2$. Furthermore, because of the fact that in the intersection point $\dot{y} = 0$, so the intersection must be transversal.

After passing through $x = (v/\omega^2)\, y^2$, $y(t)$ increases. Hence γ intersects x-axis or convergents to (0, 0). In first, by Theorem 4, γ is a periodic solution. In the second, since $y(t)$ in region (*II*) of the figure 1 is increasing, so γ can not intersect the negative y-axis. On the other hand the slop of $x = (v/\omega^2)\, y^2$ where $x \to 0^+$ is unbounded. Hence we have

$$\lim_{t \to \infty} \left| \frac{\dot{y}}{\dot{x}} \right| = \infty.$$

After passing through $x = (v/\omega^2)\, y^2, \dot{y} > 0$, so if t is sufficiently large, we have $|y(t)| < 1$. Hence

$$x(t) < \frac{v}{\omega^2}\, y^2(t) < \frac{v}{\omega^2}\, |y(t)|.$$

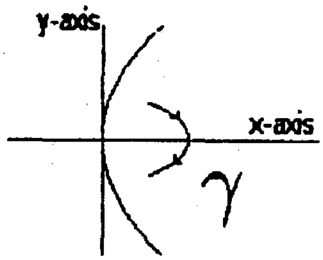

FIG. 2. The local behaviour of γ

So
$$\left|\frac{\dot{y}}{\dot{x}}\right| < \frac{|-\omega^2 x + \nu y^2|}{|y|} < \nu + \nu\,|y(t)|$$

which shows that

$$\lim_{t \to \infty} \left|\frac{\dot{y}}{\dot{x}}\right| \leq \nu.$$

This is a contradiction. Therefore γ is a periodic solution. Since, at first, we assume that γ intersects the positive x-axis at an arbitrary point, so (0, 0) is a centre for (16).

***Theorem* 6** — *For each $\mu \in R$, the system*

$$\dddot{x} + \omega^2 \dot{x} - \mu c\,(x\ddot{x})^{\cdot} = 0 \quad (c \in R)$$

has infinity many periodic solutions.

PROOF : By Theorem 3 and the above discussion it is obvious. □

Example II — We consider (15) with $b = c = 0$ and $a \neq 0$. In this case (15) is

$$\begin{cases} \dot{x} = y \\ \dot{y} = a\mu\lambda^2 + (2a\mu\lambda - \omega^2)x + a\mu x^2. \end{cases}$$

Putting $\nu = a\mu$, we get

$$\begin{cases} \dot{x} = y \\ \dot{y} = \nu\lambda^2 + (2\nu\lambda - \omega^2)x + \nu x^2. \end{cases} \qquad \text{... (17)}$$

If $\nu \neq 0$ and $(\nu, \lambda) \in \{(\nu, \lambda) : \nu\lambda < \omega^2/4\}$, then (17) has two different fixed pints $(\alpha(\nu, \lambda) \pm \beta(\nu, \lambda), 0)$ where $\alpha(\nu, \lambda) = (\omega^2 - 2\nu\lambda)/2\nu$ and $\beta(\nu, \lambda) = \sqrt{\omega^4 - 4\nu\lambda\omega^2}/2\nu$. For convenience, we put $\alpha = \alpha(\nu, \lambda)$ and $\beta = \beta(\nu, \lambda)$. $(\alpha + \beta, 0)$ is a saddle and the eigenvalues corresponding to $(\alpha - \beta, 0)$ are purely imaginary. Also (17) is a Hamiltonian system with Hamiltonian

$$H(x, y) = \frac{y^2}{2} - \nu\lambda^2 x + (\omega^2 - 2\nu\lambda)\frac{x^2}{2} - \nu\frac{x^3}{3}.$$

The trajectories of (17) are obtained by the level curves of $H(x, y)$ i.e. $H(x, y) = h$ where $h \in R$. Let us put $f(x) = 2\nu\lambda^2 x + (2\nu\lambda - \omega^2)x^2 + (2\nu x^3)/3$ and $g_h(x) = f(x) + h$. Then the trajectories of (17) are obtained by

$$y^2 = g_h(x) \quad (h \in R). \qquad \ldots (18)$$

The orbit of (17) can be obtained by (18) provided $g_h(x) \geq 0$. Let $\nu \neq 0$ and $(\nu, \lambda) \in \mathcal{R}$, and put $h_p = -f(\alpha+\beta)$. Then $g_{h_p}(\alpha+\beta) = 0$ and $g_{h_p}'(\alpha+\beta) = g_{h_p}'(\alpha-\beta) = 0$. Also $\alpha - \beta$ is the maximal point and $\alpha+\beta$ is the minimal point for $g_{h_p}(x)$. Hence considering the graph of $g_{h_p}(x)$, (17) exhibit a homoclinic for the saddle $(\alpha+\beta, 0)$ (see Figure 3).

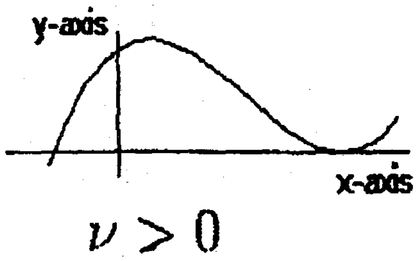

FIG. 3. The graph of $g_h(x)$

Now let $\bar{h} = -f(\alpha-\beta)$ and for each $\delta \geq 0$, we define $h_\delta = \bar{h} + \delta$. Then $g_{h_\delta}(\alpha-\beta) = \delta$. On the other hand $g_{h_\delta}'(\alpha-\beta) = g_{h_\delta}'(\alpha+\beta) = 0$. Furthermore, $\alpha-\beta$ is the maximal point and $\alpha+\beta$ is the minimal point for $g_{h_\delta}(x)$. Since $g_{h_0}(\alpha-\beta) = 0$ and the roots of $g_{h_\delta}(x)$ varies continuously with respect to δ, so if δ is sufficiently small, then $y^2 = g_{h_\delta}(x)$ indicates a non trivial periodic solution. Hence $(\alpha \S - \beta, , 0)$ is a center and the homoclinic must be filled by these periodic solutions. (See the Figure 4). □

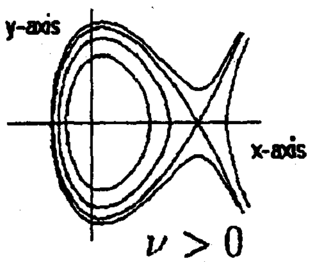

FIG. 4. The phase space of (20)

***Theorem* 7** — *For each $\mu \in R$ the system*

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -\omega^2 x + a\mu xy \end{cases}$$

has infinity many homoclinic orbits and periodic solutions. The homoclinics make a 2-dimensional C^1 surface. In addition each homoclinic lines on a C^1 2-dimensional orientable manifold and in it filled by periodic solutions (See figure 5).

PROOF : Existence of infinity many periodic solutions and homoclinics is obvious by Lemma 1 and the above discussion. Also it can be easily checked that, the graph of g_{h_p} varies C^1 with

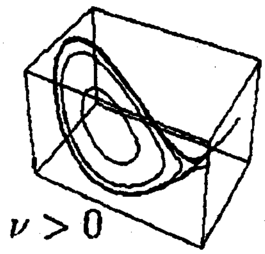

FIG. 5. The 2-dimentional manifold cased by (20)

respect to the (ν, λ). So by lemma 1 we conclude that The homoclinics make a 2-dimensional C^1 surface. Finally, since the solutions of (18) are distinct and varies C^1 with respect to initial conditions, so for each $(\nu, \lambda) \in \mathcal{R}$ fixed, the solutions of (18) make a C^1 2-dimensional orientable manifold in the phase space of the 3-dimensional system. □

Example III — Let us consider (15) with $b = 0$ and $a, c \neq 0$. In this case (15) is

$$\begin{cases} \dot{x} = y \\ \dot{y} = a\mu\lambda^2 + (2a\mu\lambda - \omega^2)x + a\mu x^2 + c\mu y^2 \end{cases}.$$

Putting $\nu = \mu a$ and $d = c / a$, we get

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x + \nu f(x + \lambda, y) \end{cases} \quad \text{... (19)}$$

with $f(x, y) = x^2 + dy^2$. For each $\nu \neq 0$ and $(\nu, \lambda) \in \{(\nu, \lambda) : \nu\lambda < \omega^2/4\}$, (19 has two fixed point $(\alpha \pm \beta, 0)$ with α and β is as themin example II. $(\alpha + \beta, 0)$ is a hyperbolic saddle and the corresponding eigenvalues of $(\alpha - \beta, 0)$ are purely imaginary. Putting $\bar{x} = x - (\alpha + \beta, 0)$ and $\bar{y} = y$, we get

$$\begin{cases} \dot{\bar{x}} = \bar{y} \\ \dot{\bar{y}} = \sqrt{\omega^4 - 4\nu\lambda\omega^2}\,\bar{x} + \nu\bar{x}^2 + d\nu\bar{y}^2 \end{cases}$$

For convenience we drop at the bars and obtain

$$\begin{cases} \dot{x} = y \\ \dot{y} = \sqrt{\omega^4 - 4\nu\lambda\omega^2}\,x + \nu x^2 + d\nu y^2 \end{cases}. \quad \text{... (20)}$$

which has two fixed points (0, 0) and $(x_0, 0) = (-\sqrt{\omega^4 - 4\nu\lambda\omega^2}/\nu, 0)$. Now we consider two cases

Case 1 — $d > 0$. In this case the local phase space of (20) has shown in Figure 6.

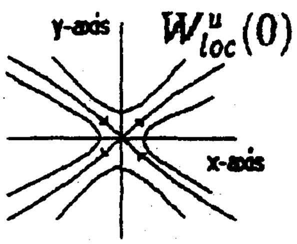

FIG. 6. The local phase space of (23)

We consider $v<0$. The case $v>0$ is similar. Let $W^u(0)$ be the trajectory of a solution of (20) $\gamma(t)=(x(t), y(t))$. If γ is unbounded in the first region of the plan, then $\dot{y}(t)\to-\infty$. Hence for $t>0$ sufficiently large $y(t)$ decreases. Hence γ crosses the positive x-axis or convergents to $(x_0, 0)$. In the first, by theorem 4, γ is a homocliic. The second show that γ is a part of a heteroclinic.

If the second case occurs then each solution of (20) which lies in heteroclinic is bounded, hence by the Poincare-Bendixon theorem about two dimensional flows the solution (when $t\to+\infty$ and $t\to-\infty$) must be convergents to a cycle or a fixed point. Because of the facts that each closed orbit in the plan has a fixed point in it and there is no fixed point in the linear of the heteroclinic, so this heteroclinic must be filled by homoclinics cased from $(x_0, 0)$ (see Figure 7).

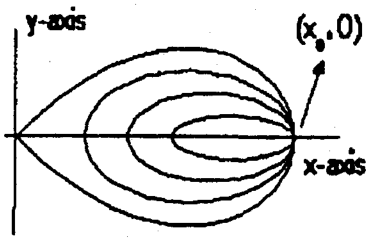

FIG. 7. The heteroclinic orbit filled by homoclinic orbits

We claim that it is a contradiction. If γ is a heteroclinic then there exists $T>0$ such that for each $t>T$, $\dot{y}<0$ and so $\sqrt{\omega^4-4v\lambda\omega^2}\,x+vx^2<-dvy^2$ or equivalent

$$h(x)=\sqrt{\frac{-1}{vd}(\sqrt{\omega^4-4v\lambda\omega^2}\,x+vx^2)}<y.$$

By Remark 2, we can write $y(t)$ as a function of $x(t)$. For $x(T)<x<x_0$ we have $h(x)<y(x)$ and $\lim_{x\to-x_0} y(x)=\lim_{x\to-x_0} h(x)=0$. Furthermore $y(x)$ and $h(x)$ are decreasing so

$$\frac{y(x)-y(x_0)}{x-x_0}<\frac{h(x)-h(x_0)}{x-x_0}\ (x(T)<x<x_0).$$

Hence by the mean value theorem

$$\frac{dy}{dx}(\xi_x)<\frac{dh}{dx}(\eta_x)\quad (x<\xi_x, \eta_x<x_0).$$

Now if $x\to-x_0$ then $\xi_x, \eta_x\to x_0$. But $\lim_{x\to-x_0} dh/dx=-\infty$ and so $\lim_{x\to-x_0} dy/dx=-\infty$. On the other hand from (20)

$$\frac{dy}{dx}=\frac{\sqrt{\omega sp4-4v\lambda\omega^2}\,x+vx^2}{y}+dvy.$$

For $x(T)<x<x_0$, $(\sqrt{\omega^4-4v\lambda\omega^2}\,x+vx^2)/y>0$ and $\lim_{x\to-x_0} dvy(x)=0$. Hence dy/dx can not convergent to $-\infty$. This is a contradiction. Hence γ is a homoclinic. A solution of (20) which lies in the homoclinic by similar proof can not convergent to $(x_0, 0)$, hence it must cross x-axis in two different points. By theorem 4, we shows that this solution is a periodic solution. hence the homoclinic filled by periodic solutions.

Case 2 — $d < 0$. In this case the phase space of (20) is as Figure 6.

We consider $v < 0$. The case $v > 0$ is similar. For $p \in R$ let $\gamma_p(t) = (x_p(t), y_p(t))$ be a solution of (20) such that $\gamma_p(0) = (p, 0)$. If $0 < p < x_0$ then $\dot{y}_p(0) > 0$ and if $p > x_0$ then $\dot{y}(0) < 0$. Now let $0 < p < x_0 < q$ then $\gamma_p(t)$ and $\gamma_q(t)$ are as the Figure 8.

If $y_p(t) > 0$ $(y_q(t) > 0)$ then $x_p(t)$ $(x_q(t))$ is increasing and if $y_p(t) < 0$ $(y(t)_q < 0)$ then $x_p(t)$ $(x_q(t))$ is decreasing so phase space of γ_p and γ_q must be as one of the figures 8(a), 8(b) or 8(c).

Fgure 8(a) shows that γ_p is a periodic solution. If Figure 8(b) occurs (Figure 8(c) is similar) then $\gamma_p(t)$ must cross x-axis in two different point or convergents to $(x_0, 0)$. We can show as case $d > 0$ that $\gamma_p(t)$ can not convergent to $(x_0, 0)$, hence by theorem 4, $\gamma_p(t)$ is a periodic solution. Also for each $p < \xi < x_0$, $\gamma_\xi(t)$ must be a periodic solution and so $(x_0, 0)$ is a center.

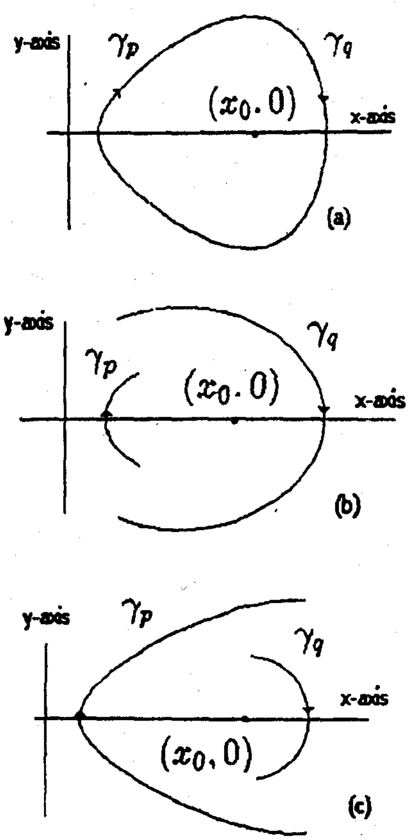

FIG. 8a,b,c. The local behaviour of γ_q and γ_p

***Theorem* 8** — *For each $\mu \in R$ the system*

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -\omega^2 y + \mu (ax + cz) y \end{cases} \qquad (a, c \in R)$$

has infinitely many periodic solutions. In addition if $ac > 0$ then the system has infinity many homoclinic orbits which make a C^1 2-dimensional surface. In this case each homoclinic lies on a C^1 orientable 2-dimensional manifold and in it filled by periodic solutions.

PROOF : By Lemma 1 and similar proof as Theorem 6 It is obvious. □

In the above examples we apply Theorem 3 when the Hoph bifurcation occurs for $\lambda = 0$. Now, as the final example of this paper, we give an example of application of Theorem 3 with $\lambda \neq 0$.

Example IV — Consider (10) with $\mu = 1, \omega = 3$ and $f(x, y) = x^2 + x^3 y - \frac{7 - 3\sqrt{5}}{2} x^2 y$. Then we have the following system.

$$\begin{cases} \dot{x} = y \\ \dot{y} = \lambda^2 - (9 - 2\lambda) x + x^2 + y\left((x + \lambda)^3 - \frac{9 - 3\sqrt{5}}{2}(x + \lambda)^2\right) \end{cases} \qquad \ldots (21)$$

Since $f_{xy}(0, 0) = 0$ so we can not apply remark 3. But we can see that for each λ close to 1 we have $(x_\lambda, 0) = \left(\frac{9 - 2\lambda - \sqrt{81 - 36\lambda}}{2}, 0\right)$ as a fix point for (21). For $\lambda = 1$, $x_1 = \frac{7 - 3\sqrt{5}}{2}$ with corresponding eigenvalues $\pm i\sqrt{3\sqrt{5}}$. Furthermore, $f_y(x_1 \; 1, 0) = 0$ and $f_{xy}(x_1 + 1, 0) = \frac{9 - 3\sqrt{5}}{2} \neq 0$. Hence transversal passing from $\lambda = 1$, a Hoph bifurcation occurs for (21) at $(x_1, 0)$. Hence we have

Corollary 4 — The following system has infinity many periodic solutions

$$\dddot{x} + \omega^2 \dot{x} = 2x\dot{x}(1 + \dot{x}) + x^2 (x\ddot{x} + 3\dot{x}^2 + \ddot{x}).$$

CONCLUSION

In this paper, we show that if the map $F(x, \dot{x}, \ddot{x})$ is sum of partial derivations of a map $f(x, \dot{x})$, so we have many periodic solutions as the parameter λ varies. We hope to extent the problem and put weak restriction on map $F(x, \dot{x}, \ddot{x})$. Also we guess that the chaotic behavior, bifurcation and Poincare map of this third equation can be obtained by our results of this paper.

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