Indian J. pure appl. Math., 34(4) : 569-577, April 2003

ON SOME CRITERIA FOR UNIVALENCE AND STARLIKENESS

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(*Received* 4 *July* 2001; *accepted* 21 *May* 2002)

Let A be the class of analytic functions $f(x)$ satisfying $f(0) = f'(0) - 1 = 0$ and let S, $S^*(\alpha)$, and K be the well known subclasses of A, respectively, called univalent, starlike of order $\alpha, 0 < \alpha < 1$ and convex. Robertson[12] proved that if $\left|\frac{f(z)f''(z)}{f'(z)^2}\right| \le 2\frac{(1-\beta)}{\beta}, \frac{1}{2} \le \beta < 1$ then $f \in S^*(\beta)$. This result was proved again and extended by Mocanu[7]. In this paper the results of Mocanu[7] are generalized.

Key Words : Univalent Functions; Starlike of Order α; Convex Functions and Subordination

INTRODUCTION

Let A denote the class of functions $f(z)$ analytic $U = \{|z| : |< 1\}$ with $f(0) = f'(0) - 1 = 0$ and let $S, S^*(\alpha)$ $(0 \le \alpha < 1)$ and K, respectively, denote the well-known sub-classes of A that are univalent, starlike of order α and convex. We shall denote $S^*(0) = S^*$. We note that $f \in S^*(\alpha)$ if and only if

$$Re\left\{z\frac{f'(z)}{f(z)}\right\} > \alpha, \quad z \in U, \quad 0 \le \alpha < 1, \qquad \dots (1)$$

and $f \in K$ if and only if

$$Re\left\{1 + z\frac{f''(z)}{f'(z)}\right\} > 0, \quad z \in U. \qquad \dots (2)$$

It is obvious that $f \in K$ if and only if $zf'(z) \in S^*$.

Over the past few decades considerable attention [1, 2, 4, 6, 7, 8, 9, 10, 11, 12, 14, 15] has been devoted to obtaining criteria for univalence and starlikeness depending on bounds of functionals containing $z\frac{f'(z)}{f(z)}$ and/or $z\frac{f''(z)}{f'(z)}$. We state below some of these results that are relevant to us

Theorem A — *Let $f(z) \in A$ with $f(z)f'(z) \ne 0, 0 \le |z| < 1$.*

(*i*) *For a given $\alpha > 0$ and a given $q(r)$ such that $q'(r)$ is continuous for $0 \le r < 1$ and $0 < q(r) \le q(0) = 1$, let f satisfy the inequality*

$$Re\left[z\frac{f'(z)}{f(z)}\left\{(\alpha - 1)z\frac{f'(z)}{f(z)} + z\frac{f''(z)}{f'(z)}\right\}\right] \ge r^2\left\{\left(\frac{q(r)'}{r}\right) + \alpha\left(\frac{q(r)}{r}\right)^2\right\}. \qquad \dots (3)$$

Then $$Re\left\{z\frac{f'(z)}{f(z)}\right\} \ge q(r) > 0, |z| = r < 1.$$

(ii) For a given constant $\beta, \frac{1}{2} \le \beta < 1$, *let* f *satisfy*

$$\left| \frac{f''(z)}{f'(z)} \right| \le 2 \frac{(1-\beta)}{\beta} \left| z \frac{f'(z)}{f(z)} \right|, \ z \in U. \qquad \ldots (4)$$

Then, $f \in S^*(\beta)$ *and*

$$z \frac{f'(z)}{f(z)} \prec \frac{\beta}{\beta - (1-\beta) z}, \ z \in U. \qquad \ldots (5)$$

Furthermore, if $\frac{3}{4} \le \beta < 1$, *then* $f \in K$ *and* $\left| z \frac{f''(z)}{f'(z)} \right| < 1$. *The constant* $\frac{3}{4}$ *cannot be replaced by smaller one.*

Theorem B — *(i)* [8]. If $f \in A, f'(z) \neq 0$ in U and

$$\left| \frac{f''}{f'(z)} \right| \le r_1 = 2.8329, \quad r_1 = \sqrt{1 + x_1^2} \qquad \ldots (6)$$

where x_1 is the smallest root of $x \sin x + \cos x = \frac{1}{e}$, then $f \in S$.

(ii) [8]. If $f \in A, \frac{f(z)}{z} \neq 0$ in U and

$$\left| \frac{f''}{f'(z)} \right| \le M_0 = 1.5986, \qquad \ldots (7)$$

where M_0 is the positive root of $(2 - M) e^M = 2$, then $\left| z \frac{f'(z)}{f(z)} - 1 \right| < 1$.

(iii) [7] If $f \in A, \frac{f(z)}{z} \neq 0$, in U and if there exists a $k > 1$ such that

$$\left| 1 + z \frac{f''(z)}{f'(z)} \right| \le k \left| z \frac{f'(z)}{f(z)} \right|, \qquad \ldots (8)$$

then
$$\frac{f(z)}{z f'(z)} \prec 1 + (1 - k^2) \log \left(1 + \frac{z}{k} \right). \qquad \ldots (9)$$

and $f \in S^*$ for $k \le k_0 = 1.808$ where k_0 is the root of the equation $1 + (1 - x^2) \log \left(1 + \frac{1}{x} \right) = 0$. Further, if $f'(z) \frac{f(z)}{z} \neq 0$ and if

$$\left| 1 + z \frac{f''(z)}{f'(z)} \right| \le \sqrt{2} \left| 1 + z \frac{f'(z)}{f(z)} \right|. \qquad \ldots (10)$$

then $f \in S^*$.

We say that an analytic function $f(z)$ is subordinate to an analytic function $g(z)$, denoted as f g, if $f(0) = g(0)$ and there exists an analytic function $\omega(z), \omega(0) = 0, |\omega(z)| < 1$, such that $f(z) = g(\omega(z))$.

We shall need the following

Theorem C^3 — *If* $p(z)$ is analytic in U, $p(0) = 0$, $h(z)$ is analytic and convex in U,

$\gamma \neq 0$, $Re\ \gamma \geq 0$, and $p(z) + \frac{1}{\gamma} z p'(z)\ h(z)$, then $p(z)\ q(z) = \frac{\gamma}{z^{\gamma}} \int_0^z h(t)\, t^{\gamma-1}\, dt$. and this $q(z)$ is convex and the best dominant.

We state below the main results of this paper.

Theorem 1 — *Let* $f \in A, \frac{f(z)}{z} f'(z) \neq 0, z \in U, \alpha \geq 0$ *and* $0 < \lambda \leq 1 + \alpha$.

(*i*) If

$$\left| (1-\alpha) z \frac{f'(z)}{f(z)} - \left(1 + z\frac{f''(z)}{f'(z)}\right) + \alpha \right| \leq \lambda \left| z^2 \frac{f'(z)}{f(z)} \right| \quad \text{... (11)}$$

then

$$z\frac{f'(z)}{f(z)} \prec \frac{1}{1 + \frac{\lambda}{1+\alpha} z}. \quad \text{... (12)}$$

In particular, f is starlike of order $\frac{1}{1 + \frac{\lambda}{1+\alpha}}$. Further, if $0 < \lambda \leq \frac{1+\alpha}{3}$ then $f \in K$ and in fact

$$\left| z\frac{f''(z)}{f'(z)} \right| < 1. \quad \text{... (13)}$$

This bound on λ is the best possible.

(*ii*) If for real $\beta \neq 0$

$$\left| \beta z \frac{f''(z)}{f'(z)} + \alpha (1 - [f'(z)]^{-\beta}) \right| \leq \lambda \,|\, z [f'(z)]^{-\beta} \,| \quad \text{... (14)}$$

then

$$[f'(z)]^{-\beta}\ \frac{1}{1 + \frac{\lambda}{1 = \alpha} z}. \quad \text{... (15)}$$

In particular, if $\beta > 0, \lambda < \frac{\beta}{1 + \frac{\beta + \alpha}{1+\alpha}}$ then f is convex and in fact

$$\left| z\frac{f''(z)}{f'(z)} \right| < 1 \quad \text{... (16)}$$

Further, if $\beta > 0$ and $0 < \lambda \leq \frac{\gamma_1 \beta}{1 + \frac{\gamma_1 \beta}{1+\alpha}}$, where $\gamma_1 = \sqrt{1 + x_1^2}$ and x_1 is the smallest root of x $\sin x + \cos x = \frac{1}{e}$, then f is starlike

(*iii*) If for real $\beta \neq 0$

$$\left| \left(z\frac{f'(z)}{f(z)} \right)^{\beta} \left\{ \beta \left(1 + z\frac{f''(z)}{f'(z)} - z\frac{f'(z)}{f(z)} \right) + \alpha \right\} - \alpha \right| \leq \lambda \,|\, z \,| \quad \text{... (17)}$$

then

$$\left(z\frac{f'(z)}{f(z)} \right)^{\beta} \prec 1 + \frac{\lambda}{1+\alpha} z. \quad \text{... (18)}$$

In (11), $\alpha = 1$ and $\lambda = 2\frac{1-\beta}{\beta}$ corresponds to (4) of Theorem A, for $|\beta| > 1$, (17) is a condition for f to be strongly starlike. On putting $\alpha = 0$ in (11) and in (14) and (17) putting $\alpha = 0$ and $\beta = 1$ we get the following

Corollary 1 — Let $f \in A, \frac{f(z)}{z} f'(z) \neq 0, z \in U$ and $0 < \lambda < 1$.

(*i*) If

$$\left| 1 + \frac{f''(z)}{f'(z)} - z\frac{f'(z)}{f(z)} \right| \leq \lambda \left| z^2 \frac{f'(z)}{f(z)} \right| \qquad \text{... (19)}$$

then
$$z\frac{f'(z)}{f(z)} \prec \frac{1}{1+\lambda z}. \qquad \text{... (20)}$$

This class of functions has been considered in [13].

(*ii*) If

$$|zf''(z)| \leq \lambda |z| \qquad \text{... (21)}$$

then
$$[f'(z)]^{-1} \; \frac{1}{1+\lambda z}. \qquad \text{... (22)}$$

(*iii*) If

$$\left| z\frac{f'(z)}{f(z)} \left(1 + z\frac{f''(z)}{f(z)} - z\frac{f'(z)}{f(z)} \right) \right| \leq \lambda |z| \qquad \text{... (23)}$$

then
$$\left| z\frac{f'(z)}{f(z)} - 1 \right| \leq \lambda . \qquad \text{... (24)}$$

***Theorem* 2** — *Let* $f \in A, \frac{f(z)}{2} f'(z) \neq 0$ *in* \U\, $\alpha \geq 0, k > 1$ *and real* λ.

(*i*) If

$$\left| \frac{\lambda}{1-k^2} - 1 + \frac{f(z)}{zf'(z)} \left(1 + z\frac{f''(z)}{f'(z)} \right) - \alpha \left(\frac{f(z)}{zf'(z)} - 1 \right) \right| \leq \frac{k|\lambda|}{k^2-1}, \qquad \text{... (25)}$$

then
$$\frac{f(z)}{zf'(z)} \quad 1 + \frac{\lambda z}{k} \int_0^1 \frac{t^\alpha}{1+\frac{z}{k}t} dt. \qquad \text{... (26)}$$

(*ii*) If

$$\left| \frac{\lambda}{1-k^2} + z\frac{f''(z)}{f'(z)^2} - \alpha \left(\frac{1}{f'(z)} - 1 \right) \right| \leq \frac{k|\lambda|}{k^2-1}, \qquad \text{... (27)}$$

then
$$\frac{1}{f'(z)} \quad 1 + \frac{\lambda z}{k} \int_0^1 \frac{t^\alpha}{1+\frac{z}{k}t} dt. \qquad \text{... (28)}$$

In (25) putting $\alpha = 0, \lambda = 1 - k^2 < 0$ gives (8) of Theorem B.

On putting $\lambda = \alpha(k^2 - 1) > 0$ in (27) we get

Corollary 2 — If

$$\left| z\frac{f''(z)}{f'(z)} - \alpha \right|^2 \le k\,\alpha\,|f'(z)|, \qquad \dots (29)$$

then

$$Re\,\frac{1}{f'(z)} > 1 - \frac{k^2-1}{k}\,\alpha \int_0^1 \frac{t^\alpha}{1-\frac{t}{k}}\,dt \qquad \dots (30)$$

On putting $\lambda = (1-\alpha)(1-k^2)$, in (25) we get

Corollary 3 — If

$$\left| 1 + z\frac{f''(z)}{f'(z)} - \alpha \right| \le k\,|1-\alpha| \left| z\frac{f'(z)}{f(z)} \right| \qquad \dots (31)$$

then

$$Re\,\frac{f(z)}{zf'(z)} > 1 - \frac{(1-\alpha)(k^2-1)}{k} \int_0^1 \frac{t^\alpha}{1+\frac{t}{k}}, \text{ if } 0 \le \alpha < 1 \qquad \dots (32)$$

and

$$Re\,\frac{f(z)}{zf'(z)} > 1 - \frac{(\alpha-1)(k^2-1)}{k} \int_0^1 \frac{t^\alpha}{1+\frac{t}{k}}, \text{ if } \alpha > 1 \qquad \dots (33)$$

On putting $\alpha = 1$ and $\lambda = k^2 - 1$, we get

Corollary 4 — If

$$\left| \frac{f(z)f''(z)}{f'(z)^2} - 1 \right| \le k \qquad \dots (34)$$

then

$$Re\,\frac{f(z)}{zf'(z)} > 1 + (k^2-1)\log\left(1 - \frac{1}{k}\right). \qquad \dots (35)$$

Hence, $f \in S^*$ if

$$1 + (k^2-1)\log\left(1 - \frac{1}{k}\right) > 0. \qquad \dots (36)$$

On putting $\lambda = 1 - k^2 < 0$, we get

Corollary 5 — If

$$\left| 1 + z\frac{f''(z)}{f'(z)} \right| \le k \left| z\frac{f'(z)}{f(z)} \right| + \alpha \left| 1 - z\frac{f'(z)}{f(z)} \right| \qquad \dots (37)$$

then

$$Re\,\frac{f(z)}{zf'(z)} > 1 - \frac{k^2-1}{k} \int_0^1 \frac{t^\alpha}{1+\frac{t}{k}}\,dt. \qquad \dots (38)$$

For $\alpha = 1$, the right side of (38) is positive if $k \le \sqrt{2}$.

Theorem 3 — *Let* $f \in A, f'(z)\frac{f(z)}{z} \ne 0$ *in* U, $\alpha > 0, 0 < \lambda \le 1$ and a be real. If

$$\left(z\frac{f'(z)}{f(z)} \right)^a \left[a\,\alpha \left\{ 1 + z\frac{f''(z)}{f'(z)} - z\frac{f'(z)}{f(z)} \right\} + 1 \right] \prec \frac{1+\lambda z}{1-\lambda z} \qquad \dots (39)$$

then $$\left(z\frac{f'(z)}{f(z)}\right)^{a} \prec \int_0^1 \frac{1-\lambda z t^{\alpha}}{1+\lambda z t^{\alpha}}\,dt. \quad \text{... (40)}$$

For $a = 1$ this is similar to (3) and for $a > 1$ this gives (39) as sufficient condition for f to be strongly starlike.

***Theorem* 4** — *Let* $g \in A, g' \neq 0$ *in* U, real $\beta \neq 0, \alpha \geq 0, k > 1$. If

$$f(z) = \int_z^1 (g'(t))^{\beta}\,dt. \quad \text{... (41)}$$

then $f \in A$, and

$$Re\,f'(z) \geq 1 - \frac{\alpha(k^2-1)}{k}\int_0^1 \frac{t^{\alpha}}{1-\frac{t}{k}}, \quad \text{... (42)}$$

if $$\left|\alpha + \beta z\frac{g''(z)}{g'(z)}\right| \leq k\,\alpha\,|g'(z)|^{-\beta}, \quad \text{... (43)}$$

Hence f is close-to-convex univalent if

$$1 \geq \alpha\frac{k^2-1}{k}\int_0^1 \frac{t^{\alpha}}{1-\frac{t}{k}}\,dt. \quad \text{... (44)}$$

PROOF OF THEOREM 1

Lemma 1 — If q is analytic in U, $q(z) = 1 + b_n z^n + b_{n+1} z^{n+1} + \ldots, n \geq 1, \alpha > 0$ and $0 < \lambda \leq \alpha + n$, then

$$zq'(z) + \alpha q(z) \quad \alpha + \lambda z \quad \text{... (45)}$$

implies

$$q(z) \quad 1 + \frac{\lambda}{n+\alpha} z. \quad \text{... (46)}$$

PROOF : From (45) we get

$$zq'(z) + \alpha q(z) = \alpha + \lambda\,\omega(z),\ \omega(0) = 0,\ |\omega(z)| \leq |z|. \quad \text{... (47)}$$

This yields

$$q(z) = 1 + \lambda \int_0^1 t^{\alpha-1}\,\omega(tz)\,dt$$

and family by Schwarz Lemma

$$|q(z) - 1| \leq \frac{\lambda}{n+\alpha}|z|. \quad \text{... (48)}$$

This is equivalent to (46). □

In most applications here we take $n = 1$.

If f is analytic in U, $\frac{f(z)}{z}f'(z) \neq 0$ in U then on putting $q(z) = \frac{f(z)}{zf'(z)}$, we get (11) and (12).

For convexity part of (i), we write (ii) in the form

$$\left| z\frac{f''(z)}{f'(z)} \right| \leq \lambda \left| z^2 \frac{f'(z)}{f(z)} \right| + |1-\alpha| \left| z\frac{f'(z)}{f(z)} - 1 \right|, \quad \text{... (49)}$$

$$\leq \frac{3\lambda}{1+\alpha},$$

when we use (12) in the right hand side. To see that this is the best possible value of λ consider

$$f(z) = z\left(1 - \frac{\lambda}{1+\alpha}z^3\right)^{-\frac{1}{3}}, \quad \text{... (50)}$$

Then

$$z\frac{f'(z)}{f(z)} = \frac{1}{1 - \frac{\lambda}{1+\alpha}z^3},$$

and

$$1 + z\frac{f''(z)}{f'(z)} = \frac{1 + \frac{3\lambda}{1+\alpha}z^3}{1 - \frac{\lambda}{1+\alpha}z^3}.$$

This shows that f (z) may ease to be convex if $\frac{3\lambda}{1+\alpha} > 1$, for in that case $Re \left(1 + z\frac{f''(z)}{f'(z)}\right)$ can be negative.

If we put $p(z) = [f'(z)]^{\beta}$ in (45) we obtain (14) and (15). Again, (14) yields in view of (15)

$$\left| \beta z \frac{f''(z)}{f'(z)} \right| \leq \lambda |z[f'(z)]^{-\beta}| + \alpha |1 - [f''(z)]^{-\beta}| \quad \text{... (51)}$$

$$\leq \frac{\lambda\left(1 + \frac{\alpha}{1+\alpha}\right)}{1 - \frac{\lambda}{1+\alpha}}$$

which gives (16). For the rest we use Theorem $B(i)$ (6)

The last part of Theorem 1 follows by taking $p(z) = \left(z\frac{f'(z)}{f(z)}\right)^{\beta}$. Corollary 1 is easy to deduce.

PROOF OF THEOREM 2

Lemma 2 — Let $p(z)$ be analytic in U, $p(c) = 1$, $\alpha \geq 0$, real $\lambda \neq 0$ and $k > 1$. If

$$\alpha p(z) + zp'(z) \quad \alpha + \frac{\lambda z}{k+z} \quad \text{... (52)}$$

then $$p(z) \quad 1+\frac{\lambda z}{k}\int_0^1 \frac{t^\alpha}{1+\frac{z}{k}t}\,dt. \qquad \text{... (53)}$$

In particular, $$Re\ p(z) \geq 1-\frac{\lambda}{k}\int_0^1 \frac{t^\alpha}{1-\frac{1}{k}}\,dt,\ \lambda>0 \qquad \text{... (54)}$$

and $$Re\ p(z) \geq 1+\frac{\lambda}{k}\int_0^1 \frac{t^\alpha}{1-\frac{t}{k}}\,dt,\ \lambda<0 \qquad \text{... (55)}$$

Lemma 2 easily follows from Theorem C.

On putting $p(z)=\frac{f(z)}{2f'(z)}$ in (52) and (53), we get after some elementary calculations (25) and (26). Corollary 3 now follows from (54) and (55) on taking $\lambda=(1-\alpha)(1-k^2)$. Corollary 4 follows from (55) on taking $\alpha=1, \lambda=k^2-1>0$. Corollary 5 is obvious.

The choice $p(z)=\frac{1}{f'(z)}$ in (52) and (53) gives (27) and (28)/

Proof of Theorem 3

Lemma 3 — Let $p(z)$ be analytic in U, $p(0) = 1$, $\alpha\geq 0$ and $0<\lambda\leq 1$. If

$$\alpha z p'(z)+p(z) \quad \frac{1+\lambda z}{1-\lambda z}, \qquad \text{... (56)}$$

then $$p(z) \quad -1+\int_0^1 \frac{2}{1-\lambda z t^\alpha}\,dt. \qquad \text{... (57)}$$

In particular

$$Re\, p(z) \geq -1+2\int_0^1 \frac{dt}{1+\lambda t^\alpha}. \qquad \text{... (58)}$$

Lemma 3 follows easily from theorem C. On putting $p(z)=\left(\frac{zf'(z)}{f(z)}\right)^a$, we get the theorem.

Proof of Theorem 4

In (52) put $p(z)=(g'(z))^\beta$ and choose $\lambda=\alpha(1-k^2)<0$. We obtain, if

$$\left|\alpha+\beta z\frac{g''(z)}{g'(z)}\right| \leq \alpha k\,|g'(z)|^{-\beta} \qquad \text{... (59)}$$

then from (28)

$$(g'(z))^\beta \geq 1-\frac{\alpha(k^2-1)}{k}\int_0^1 \frac{t^\alpha}{1-\frac{t}{k}}\,dt. \qquad \text{... (60)}$$

Since $f'(z)=(g'(z))^\beta$ the theorem follows.

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