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ON INVERSE LIMITS AND TYCHONOFF PRODUCTS OF ALMOST EXPANDABLE CLASS

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In this paper, we mainly prove the following: (1) Let X be the inverse limit of an inverse system $\left\{X_\alpha, \pi_\beta^\alpha, \Lambda\right\}$ and let the projection π_α be an open and onto map for each $\alpha \in \Lambda$, if X is $|\Lambda|$-paracompact and each X_α has the property $\mathcal{P}$, then X has also the property $\mathcal{P}$. (2) Let $X = \prod_{\sigma \in \Sigma} X_\sigma$ be $|\Sigma|$-paracompact, X has the property $\mathcal{P}$ iff $\prod_{\sigma \in F} X_\sigma$ has the property $\mathcal{P}$ for each $F \in [\Sigma]^{<\omega}$. Where $\mathcal{P}$ denotes one of six properties: almost expanability, almost θ-expandability, almost σ-expandability, almost discrete expandability, almost discrete θ-expandability, almost discrete σ-expandability.

Key Words : Almost Expandable; Almost θ-Expandable; Almost σ-Expandable; $|\Sigma|$-Paracompact; Countable Paracompact

In 1990, K. Chiba proved the following: Let X be the inverse limit of an inverse system $\left\{X_\alpha, \pi_\beta^\alpha, \Lambda\right\}$ and let the projection π_α be an open and onto map for every $\alpha \in \Lambda$, if X is $|\Lambda|$-paracompact and each X_α is normal (paracompact, collectionwise normal, metacompact, subparacompact, submetacompact, paralindelof, metalindelof, σ-paralindelof, σ-metacompact, shrinking, property $\mathcal{B}$), then X is normal (paracompact, collectionwise normal, metacompact, subparacompact, submetacompact, paralindelof, metalindelof, σ-paralindelof, σ-metacompact, shrinking, property $\mathcal{B}$). On the basis of this, various people ask:

Question — Are there similar results about two classes of both almost expandable spaces and almost discrete expandable spaces?

In this paper, we positively answer the above question. Using this, two groups of characterizations of infinite Tychonoff products of almost expandable spaces and almost discrete expandable spaces under the condition of $|\Sigma|$-paracompactness. And we also show that almost θ-expandable spaces, almost σ-expandable spaces, almost discrete θ-expandable spaces and almost discrete σ-expandable spaces have respectively similar results.

We use that $N_Y(x)$ denotes the neighborhood system of a point x of a subspace Y of a space X. Expectly, $N(x)$ denotes $N_Y(x)$ when $Y = X$; $|A|$, cl$A$, Int$A$ and A^c denote respectively the cardinality, the closure, the interior and the complementary set of a set A; $(\mathcal{U})_x$ and $(\mathcal{U})|_A$ denote respectively $\left\{U \in \mathcal{U} : x \in U\right\}$ and $\left\{U \cap A : U \in \mathcal{U}\right\}$; ω and $[\Sigma]^{<\omega}$ denote, respectively, the first infinite ordinal number and the collection of all non-empty finite subsets of a non-empty set Σ. And assume that all spaces are Hausdorff spaces throughout this paper.

Definition 1 — Let κ be a cardinal number, *A* space is κ-paracompact iff its every open cover $\mathcal{U}$ of cardinal $|\mathcal{U}| \leq \kappa$ has a locally finite open refinement; A space is $|\Sigma|$-paracompact iff it is κ-paracompact, where $\kappa = |\Sigma|$.

Definition 2 — A space X is said to be almost expandable if its every locally finite closed family $\{F_\xi : \xi \in \Xi\}$ has a point finite open family $\{U_\xi : \xi \in \Xi\}$ such that $F_\xi \subset U_\xi$ for every $\xi \in \Xi$; A space X is said to be almost θ-expandable (respectively almost σ-expandable) if its every locally finite closed family $\{F_\xi : \xi \in \Xi\}$ has a sequence $\langle \{U_{n\xi} : \xi \in \Xi\} \rangle_{n \in \omega}$ of open families of X such that $F_\xi \subset U_{n\xi}$ for every $\xi \in \Xi$ and every $n \in \omega$ (respectively $F_\xi \subset \bigcup_{n \in \omega} U_{n\xi}$ for every $\xi \in \Xi\}$.

Definition 3 — A space X is said to be almost discrete expandable if its every discrete closed family $\{F_\xi : \xi \in \Xi\}$ has a point finite open family $\{U_\xi : \xi \in \Xi\}$ such that $F_\xi \subset U_\xi$ for every $\xi \in \Xi$; A space X is said to be almost discrete θ-expandable (respectively almost discrete σ-expandable) if its every discrete closed family $\{F_\xi : \xi \in \Xi\}$ has a sequence $\langle \{U_{n\xi} : \xi \in \Xi\} \rangle_{n \in \omega}$ open families of X such that $F_\xi \subset U_{n\xi}$ for every $\xi \in \Xi$ and every $n \in \omega$ (respectively $F_\xi \subset \bigcup_{n \in \omega} U_{n\xi}$ for every $\xi \in \Xi$)

The following are main results and the proofs of this paper.

Theorem 1 — *Let X be the inverse limit of an inverse system $\{X_\alpha, \pi_\beta^\alpha, \Lambda\}$ and let the projection π_α be an open and onto map for each $\alpha \in \Lambda$. Suppose X is $|\Lambda|$-paracompact. If each X_α has one of the following properties, then X has also the corresponding property.*

(1) almost expandability.

(2) almost θ-expandability

(3) almost σ-expandability.

PROOF : Let $\{F_\xi : \xi \in \Xi\}$ be a family of locally finite closed sets of X. For every $\alpha \in \Lambda$, put $U_\alpha = \bigcup \{U : U$ is open in X_α and $\left|\left\{\xi \in \Xi : \pi_\alpha^{-1}(U) \cap F_\xi \neq \phi\right\}\right| < \omega\}$, then

(*a*) $\left\{\pi_\alpha^{-1}(U_\alpha) : \alpha \in \Lambda\right\}$ is an open cover of X and $\pi_\alpha^{-1}(U_\alpha) \subset \pi_\beta^{-1}(U_\beta)$ for every $\alpha \leq \beta$.

In fact, for every $x \in X$, there is some $W \in N(x)$ such that $\left\{\xi \in \Xi : W \cap F_\xi \neq \phi\right\}$ is a finite set. By [2, 2.5.5 Proposition], there are both a $\alpha \in \Lambda$ and an open set U of X_α such that $x \in \pi_\alpha^{-1}(U) \subset W$, i.e., $\left|\left\{\xi \in \Xi : \pi_\alpha^{-1}(U) \cap F_\xi \neq \phi\right\}\right| < \omega$, then $x \in \pi_\alpha^{-1}(U) \subset \pi_\alpha^{-1}(U_\alpha)$. So, $\bigcup_{\alpha \in \Lambda} \pi_\alpha^{-1}(U_\alpha) = X$. Next, for every $x \in \pi_\alpha^{-1}(U_\alpha)$, there is some open set U of X_α such that $x \in \pi_\alpha^{-1}(U)$ and $\left|\left\{\xi \in \Xi : \pi_\alpha^{-1}(U) \cap F_\xi \neq \phi\right\}\right| < \omega$. i.e., $\left|\left\{\xi \in \Xi : (\pi_\alpha^\beta)^{-1}(U) \cap F_\xi \neq \phi\right\}\right| < \omega$

since $x \in \pi_\alpha^{-1}(U) = \pi_\beta^{-1}(\pi_\alpha^\beta)^{-1}(U)$, hence $x \in \pi_\beta^{-1}(U_\beta)$.

By [1, Lemma 2], there is an open cover $\{W_\alpha : \alpha \in \Lambda\}$ of X such that

(*b*) cl $W_\alpha \subset \pi_\alpha^{-1}(U_\alpha)$ if every $\alpha \in \Lambda$, and $W_\alpha \in W_\beta$ for every $\alpha \leq \beta$.

Pick $T_\alpha = X_\alpha - \pi_\alpha(X - cl\, W_\alpha)$ for every $\alpha \in \Lambda$, then T_α is a closed set of X_α and $T_\alpha \subset U_\alpha$. Again pick $C_\alpha = Int\, \pi_\alpha^{-1}(T_\alpha)$, now we prove:

(*c*) $\{C_\alpha : \alpha \in \Lambda\}$ is an open cover of X.

For every $x \in X$, there is some $\alpha \in \Lambda$ such that $x \in W_\alpha$ since $\{W_\alpha : \alpha \in \Lambda\}$ covers X. There exist $\beta \in \Lambda$ and an open subset V of X_β such that $x \in \pi_\beta^{-1}(V) \subset W_\alpha$. Pick $\gamma \in \Lambda$ satisfying $\gamma \geq \alpha$ and $\gamma \geq \beta$, then $x \in C_\gamma$. To show this, we only assert that $\pi_\beta^{-1}(V) \subset \pi_\gamma^{-1}(T_\gamma)$. In fact, if there is some $y = (y_\delta)_{\delta \in \Lambda} \in \pi_\beta^{-1}(V) - \pi_\gamma^{-1}(T_\gamma)$, then $y_\beta \in V$ and $y_\gamma \in \pi_\gamma(X - cl\, W_\gamma)$. There is some $z = (z_\delta)_{\delta \in \Lambda} \in X - cl\, W_\gamma$ such that $y_\gamma = \pi_\gamma(z) = z_\gamma$, i.e., $y_\beta = \pi_\beta^\gamma(z_\gamma)$. So, $z \in \pi_\gamma^{-1}(\pi_\beta^\gamma)^{-1}(V) \subset W_\alpha \subset W_\gamma$ since $y_\beta \in V$ and $z_\gamma \in (\pi_\beta^\gamma)^{-1}(V)$. This contradicts to $z \in X - cl\, W_\gamma$. Thus $x \in \pi_\beta^{-1}(V) \subset \pi_\gamma^{-1}(T_\gamma)$, then $x \in C_\gamma$.

By $|\Lambda|$-paracompactness of X, there is a locally finite open cover $\{O_\alpha : \alpha \in \Lambda\}$ of X such that

(*d*) $O_\alpha \subset C_\alpha$ for every $\alpha \in \Lambda$

Define $\mathcal{F}_\alpha = \left\{ T_\alpha \cap cl\, \pi_\alpha(F_\xi) : \xi \in \Xi \right\}$ for every $\alpha \in \Lambda$, then

(*f*) F_α is a locally finite closed family of T_α.

In fact, for every $y \in T_\alpha \subset U_\alpha$, there is some open set U of X_α such that $y \in U$ and $\left| \left\{ \xi \in \Xi : \pi_\alpha^{-1}(U) \cap F_\xi \neq \phi \right\} \right| < \omega$, then

$$\left\{ \xi \in \Xi : \pi_\alpha^{-1}(U) \cap F_\xi \neq \phi \right\} = \left\{ \xi \in \Xi : U \cap \pi_\alpha(F_\xi) \neq \phi \right\}$$

since $\quad \pi_\alpha[\pi_\alpha^{-1}(U) \cap F_\xi] = U \cap \pi_\alpha(F_\xi)$ for $U \in N_{X_\alpha}(y)$. So,

$$\left| \left\{ \xi \in \Xi : U \cap \pi_\alpha(F_\xi) \neq \phi \right\} \right| < \omega.$$

Again since $\quad \left\{ \xi \in \Xi : U \cap T_\alpha \cap cl\, \pi_\alpha(F_\xi) \neq \phi \right\} \subset \left\{ \xi \in \Xi : U \cap \pi_\alpha(F_\xi) \neq \phi \right\}$

then $\mathcal{F}_\alpha$ is a locally finite family of closed subsets of T_α.

PROOF OF (1) — Assume that each X_α is almost expandable, then so is T_α since any closed subset of almost expandable spaces is almost expandable.There is a point finite open family $\left\{ V_{\alpha\xi} : \xi \in \Xi \right\}$ such that

(1f) $T_\alpha \cap cl\, \pi_\alpha F_\xi \subset v_{\alpha\xi}$ for every $\xi \in \Xi$.

We put

$$W_\xi = \bigcup_{\alpha \in \Lambda} [O_\alpha \cap \pi_\alpha^{-1}(V_{\alpha\xi})], \text{ then}$$

(1g) $F_\xi \subset W_\xi$ for every $\xi \in \Xi$.

In fact, for every $x \in F_\xi$, there is some $\alpha \in \Lambda$ such that $x \in O_\alpha \subset C_\alpha \subset \pi_\alpha^{-1}(T_\alpha)$, then

$$x_\alpha = \pi_\alpha(x) \in T_\alpha \cap \pi_\alpha(F_\xi) \subset V_{\alpha\xi}.$$

So,
$$x \in \pi_\alpha^{-1}(V_{\alpha\xi}) \cap O_\alpha \subset W_\xi.$$

(1*h*) $\{W_\xi : \xi \in \Xi\}$ is a point finite open family.

In fact, for every $x \in X$, we may let $\{\alpha \in \Lambda : x \in O_\alpha\} = \{\alpha_0, \alpha_1, \ldots, \alpha_m\}$, then

$$x \in O_{\alpha_i} \subset \pi_{\alpha_i}^{-1}(T_{\alpha_i}) \text{ for every } i \le m$$

i.e., $x_{\alpha_i} = \pi_{\alpha_i}(x) \in T_{\alpha_i}$ and $\{V_{\alpha_i\xi} : \xi \in \Lambda\}$ is point finite in T_{α_i}. Thus

$\Delta_i = \left\{ \xi \in \Xi : x \in \pi_{\alpha_i}^{-1}(V_{\alpha_i\xi}) \right\}$ is a finite set. It is easy to prove :

$$\left\{ \xi \in \Xi : x \in W_\xi \right\} \subset \bigcup_{i \le m} \Delta_i .$$

In fact, if $x \in W_\xi$, there is $\alpha \in \Lambda$ such that $x \in O_\alpha \cap \pi_\alpha^{-1}(V_{\alpha\xi})$, then there is $i \le m$ such that $\alpha = \alpha_i$ and $x \in \pi_{\alpha_i}^{-1}(V_{\alpha_i\xi})$, i.e., $x \in \Delta_i$.

Therefore, X is a almost expandable space.

PROOF OF (2) : Let each X_α be almost θ-expandable, since T_α is closed in X_α, then the locally finite closed family $\mathcal{F}_\alpha = \left\{ T_\alpha \cap cl\, \pi_\alpha(F_\xi) : \xi \in \Xi \right\}$ of T_α has a sequence $V_{\alpha n} = \left\{ V_{\xi\alpha n} : \xi \in \Xi \right\}\rangle_{n \in \omega}$ of open families of T_α such that

(2*f*) $T_\alpha \cap cl\, \pi_\alpha(F_\xi) \subset V_{\xi\alpha n+1} \subset V_{\xi\alpha n}$ for every $\xi \in \Xi$ and for every $n \in \omega$, and

(2*g*) For every $x \in T_\alpha$, there is some $n \in \omega$ such that $1 \le ord(x, V_{\alpha n}) < \omega$.

Put $W_{\xi n} = \bigcup_{\alpha \in \Lambda} [O_\alpha \cap \pi_\alpha^{-1}(V_{\xi\alpha n})]$ and let $\mathcal{W}_n = \left\{ W_{\xi n} : \xi \in \Xi \right\}$. Then

(2*h*) $F_\xi \subset W_{\xi n+1} \subset W_{\xi n}$ for every $\xi \in \Xi$ and for every $n \in \omega$.

Finally, we prove :

(2*i*) For every $x \in X$, there is $n \in \omega$ such that $1 \le$ ord $(x, \mathcal{W}_n) < \omega$.

Pick $x \in X$, let $\{\alpha \in \Lambda : x \in O_\alpha\} = \{\alpha_0, \alpha_1, \ldots, \alpha_k\}$ where $k \in \omega$. For every $i \le k$, since $x \in O_{\alpha_i} \subset C_{\alpha_i} \subset \pi_{\alpha_i}^{-1}(T_{\alpha_i})$, there is $n_i \in \omega$ such that $1 \le$ ord $(x_{\alpha_i}, V_{\alpha_i n}) < \omega$. Let $n' = \max\left\{ n_i : i \le k \right\}$, then

$$\left\{ \xi \in \Xi : x \in W_{\xi n'} \right\} \subset \left\{ \xi \in \Xi : x_{\alpha_i} \in V_{\xi \alpha_i n_i}, i \le k \right\}.$$

In fact, if $x \in W\xi n' = \bigcup_{\alpha \in \Lambda} [O\alpha \cap \pi_\alpha^{-1}(V\xi \alpha_i n')]$, then there is $\alpha \in \Lambda$ such that $x \in O\alpha \cap \pi_\alpha^{-1}(V\xi \alpha n')$. We have $i \le k$ such that $\alpha = \alpha_i$, i.e., $x \in O\alpha_i \cap \pi_{\alpha_i}^{-1}(V\xi \alpha_i n') \subset O\alpha_i \cap \pi_{\alpha_i}^{-1}(V\xi \alpha_i n_i)$. Thus, $1 \le ord(x, \mathcal{W}_{n'}) < \omega$, i.e., x is almost θ-expandable.

PROOF OF (3) : If every $X\alpha$ is almost σ-expandable, then $T\alpha$ is almost σ-expandable. For $\alpha \in \Lambda$, the locally finite closed family $\mathcal{F}\alpha = \left\{ T\alpha \cap cl\, \pi_\alpha(F\xi) : \xi \in \Xi \right\}$ of $T\alpha$ has a sequence $\langle \mathcal{V}\alpha n = \{ V\xi \alpha n : \xi \in \Xi \} \rangle_{n \in \omega}$ of point finite open families of $T\alpha$ satisfying :

(3*f*) $T\alpha \cap cl\, \pi_\alpha(F\xi) \subset \bigcup_{n \in \omega} V\xi \alpha n$ for every $\xi \in \Xi$.

We put $W_{\xi n} = \bigcup_{\alpha \in \Lambda} [O_\alpha \cap \pi_\alpha^{-1}(V_{\xi \alpha n})]$ for every $\xi \in \Xi$..

By using the way of (1i) and (1j), it is easy to prove the following :

(3*g*) $F\xi \subset \bigcup_{n \in \omega} W\xi_n$ for every $\xi \in \Xi$., and

(3*h*) $\{W\xi_n : \xi \in \Xi\}$ is point finite for every $n \in \omega$.

So, X is almost σ-expandable. □

***Theorem* 2** — *Let X be the inverse limit of an inverse system* $\{X\alpha, \pi_\beta^\alpha, \Lambda\}$ *and let the projection* π_α *be an open and onto map for each* $\alpha \in \Lambda$. *Suppose X is* $|\Lambda|$*-paracompact. If each* $X\alpha$ *has one of the following properties, then X has also the corresponding property.*

(1) almost discrete expandability.

(2) almost discrete θ-expandability.

(3) almost discrete σ-expandability.

PROOF : Let $\{F\xi : \xi \in \Xi\}$ be a family of discrete closed sets of X. For every $\alpha \in \Lambda$, we put $U\alpha = \bigcup \{U : U$ is open in $X\alpha$ and $\pi_\alpha^{-1}(U) \cap F\xi \ne \phi$ for at most one $\xi\}$.

then it is easy that we prove (1), (2) and (3) by the way of Theorem 1. □

Now we use that $\mathcal{P}$ denotes one of six properties: almost expanability, almost θ-expandability and almost σ-expandability, almost discrete expandability, almost discrete θ-expandability, almost discrete σ-expandability, then the following hold :

***Theorem* 3** — *Let* $X = \prod_{\sigma \in \Sigma} X\sigma$ *be* $|\Sigma|$*-paracompact, X has the property* $\mathcal{P}$ *iff* $\prod_{\sigma \in F} X\sigma$ *has the property* $\mathcal{P}$ *for each* $F \in [\Sigma]^{<\omega}$.

PROOF : ($\Leftarrow$) When $|\Sigma| < \omega$, it is obvious that $X = \prod_{\sigma \in \Sigma} X\sigma$ has the property $\mathcal{P}$ since $F = \Sigma \in [\Sigma]^{<\omega}$. Without loss of generality, we suppose $|\Sigma| \ge \omega$. Define the relation $\le : F \le E$ if and only if $F \subset E$ for any $(F, E) \in [\Sigma]^{<\omega} \times [\Sigma]^{<\omega}$. Then $[\Sigma]^{<\omega}$ is a directed set on the relation $\leq$. Let us put $X_F = \prod_{\sigma \in F} X_\sigma$ for every $F \in [\Sigma]^{<\omega}$ and define the projection :

$$\pi_F^E : X_E \to X_F \text{ when } F \leq E, \text{ where } \pi_F^E(x) = (x_\sigma)_{\sigma \in F} \in X_F$$

for any $x = (x_\sigma)_{\sigma \in E} \in X_E$.

It is easy to prove that π_F^E is an open and onto map, then $\{X_E, \pi_F^E, [\Sigma]^{<\omega}\}$ is an inverse system of spaces X_E with bounding maps $\pi_F^E : X_E \to X_F \ (E \geq F)$.

Let X' be the inverse limit of the inverse system $\{X_E, \pi_F^E, [\Sigma]^{<\omega}\}$ by [2, 2.5.3 Example], X' is homeomorphic to $X = \prod_{\sigma \in \Sigma} X_\sigma$.

Next, since $X_F = \prod_{\sigma \in F} X_\sigma$ has the property $\mathcal{P}$ for every $F \in [\Sigma]^{<\omega}$, X' has the property $\mathcal{P}$ by Theorem 1 and Theorem 2. Hence, $X = \prod_{\sigma \in \Sigma} X_\sigma$ has the property $\mathcal{P}$.

($\Leftarrow$) Assume that the product $X = \prod_{\sigma \in \Sigma} X_\sigma$ has the property $\mathcal{P}$. For every $F \in [\Sigma]^{<\omega}$, pick a point $x_\sigma \in X_\sigma$ for every $\sigma \in \Sigma - F$, then the closed subspace $Y_F = \prod_{\sigma \in F} X_\sigma \times \prod_{\sigma \in \Sigma - F} \{x_\sigma\}$ of X has the property $\mathcal{P}$. Therefore, $X_F = \prod_{\sigma \in F} X_\sigma$ has the property $\mathcal{P}$. $\square$

Theorem 4 — *Let $X = \prod_{i \in \omega} X_i$ is countable paracompact, then the following are equivalent:*

(1) X has the property $\mathcal{P}$.

(2) $\prod_{i \in F} X_i$ has the property $\mathcal{P}$ for each $F \in [\Sigma]^{<\omega}$.

(3) $\prod_{i \leq n} X_i$ has the property $\mathcal{P}$ for each $n \in \omega$.

PROOF : The equivalence of both (1) and (2) is direct by Theorem 3. (2) $\Rightarrow$ (3) hold trivially. Now we prove (3) $\Rightarrow$ (2)

In fact, for every $F \in [\Sigma]^{<\omega}$, we may assume $m = \max F$ since $F \neq \phi$. We pick some $x_\sigma \in X_\sigma$ when $\sigma \in \{0, 1, ..., m\} - F$, then $\prod_{\sigma \in F} X_\sigma \times \prod_{\sigma \in (0, 1, ..., m) - F} \{x_\sigma\}$ is a closed set of $\prod_{i \leq m} X_i$. So, $\prod_{i \in F} X_i$ has the property $\mathcal{P}$. $\square$

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