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TRANSVERSE VIBRATIONS OF ORTHOTROPIC NONUNIFORM RECTANGULAR PLATES WITH CONTINUOUSLY VARYING DENSITY

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An analysis and numerical results are presented for free transverse vibrations of orthotropic nonuniform rectangular plates with continuously varying density on the basis of classical plate theory. The variation in density is assumed as exponential along one direction of the plate which gives rise nonhomogeneity of the plate material. Following Lévy approach i.e. the two parallel edges are simply supported, the fourth order differential equatoon governing the motion of such plates of exponentially varying thickness in one direction, has been solved using Chebyshev polynomials for three different combinations of clamped, simply supported and free boundary conditions at the other two edges. The effect of the nonhomogenity parameter together with the thickness variation and aspect ratio on the natural frequencies has been illustrated for the first three modes of vibration. Normalized transverse displacements are presented for two different values of taper parameter, keeping other plate parameters fixed, for all the three boundary conditions. A comparison of results with those available in the literature has been presented.

Key Words : Orthotropic Nonuniform Rectangular Plates; Variable Thickness

INTRODUCTION

Plates of variable thickness are often encountered in engineering applications and their use in machine design, nuclear reactor technology, naval structures and acoustical components is quite common. The consideration of anisotropy of the plate material together with the variation in thickness of the structural components not only ensures the reduction in the weight and size but also meets the desirability of high strength in various technological situations of aerospace industry, ocean engineering and electronic and optical equipments. An extensive review on linear vibration of plates has been given by Leissa in his monograph[1] and a series of review articles[2-5].

Many applications of plates demand that the nonhomogeneity of materials should be taken into account for the analysis of plate vibration. Nonhomogeneity can be natural or it can be developed artificially. Plywood, deltawood, timber and fiber-reinforced plastics etc. form an important class of nonhomogeneous materials which are used in engineering design and technology to strengthen the construction. There are some artificially nonhomogeneous materials such as glass epoxy and boron epoxy in steel alloys for making rods in nuclear reactors. A number of papers dealing with nonhomogeneous anisotropic plates of uniform thickness nonhomogeneous isotropic plates of nonuniform thickness of various geometries has been reported by Leissa[1-5]. In a recent survey of the literature the author has found no work dealing with vibration of nonhomogeneous orthotropic rectangular plates of variable thickness. However few good papers on vibrations of nonhomogeneous membranes are Masad[6], Laura *et al.*[7] and Gutierrez *et al.*[8].

The present work deals with the transverse vibrations of nonhomogeneous orthotropic rectangular plates of exponentially varying thickness in one direction on the basis of classical plate theory. The nonhomogeneity is assumed to arise due to the variation in the density of the plate material in exponential manner along the direction of thickness variation. The governing differential equation has been solved using Chebyshev polynomials. This method is preferred because Chebyshev polynomials have minimax property (i.e., of all the monic polynomials, the maximum error is minimum : Fox and Parker[9], p.5). A five-ply maple plywood has been taken as an example of a rectangular orthotropic material.

MATHEMATICAL FORMULATION

Consider a rectangular plate of length a, breadth b and thickness $h = h(x, y)$. The plate is referred to a system of rectangular cartesian coordinates (x, y, z), the middle surface being $z = 0$ and origin at one of the corners of the plate. The x and y axes are taken along the principal directions of orthotropy and the axis of z is perpendicular to the xy-plane. The differential equation which governs the transverse vibrations is given by

$$D_x \frac{\partial^4 w}{\partial x^4} + D_y \frac{\partial^4 w}{\partial y^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + 2 \frac{\partial H}{\partial x} \frac{\partial^3 w}{\partial x \partial y^2}$$

$$+ 2 \frac{\partial H}{\partial y} \frac{\partial^3 w}{\partial y \partial x^2} + 2 \frac{\partial D_x}{\partial x} \frac{\partial^3 w}{\partial x^3}$$

$$+ 2 \frac{\partial D_y}{\partial y} \frac{\partial^3 w}{\partial y^3} + \frac{\partial^2 D_x}{\partial x^2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 D_y}{\partial y^2} \frac{\partial^2 w}{\partial y^2}$$

$$+ \frac{\partial^2 D_1}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 D_1}{\partial y^2} \frac{\partial^2 w}{\partial x^2}$$

$$+ 4 \frac{\partial^2 D_{xy}}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \rho h \frac{\partial^2 w}{\partial t^2} = 0, \quad \text{... (1)}$$

where $(D_x, D_y) = (E_x^*, E_y^*)\, h^3/12,\ D_{xy} = G_{xy}\, h^3/12,\ D_1 = E^*\, h^3/12,$

$$H = D_1 + 2D_{xy},\ (E_x^*, E_y^*) = (E_x, E_y)/(1 - \nu_x \nu_y),\ E^* = \nu_y E_x^* = \nu_x E_y^*,$$

$w(x, y, t)$ is the transverse deflection, t is the time, ρ is the density and E_x, E_y, ν_x, ν_y and G_{xy} are material constants in proper directions defined by an orthotropic stress-strain law.

Let us assume that the two opposite edges of the plate $y = 0$ and $y = b$ are simply supported and that the thickness h is independent of y i.e., $h = h(x)$.For harmonic vibrations, the deflection w (Lévy approach) is assumed to be

$$w(x, y, t) = \bar{w}(x) \sin(p \pi y/b)\, e^{i \omega t} \quad \text{... (2)}$$

where p is a positive integer and ω is the radian frequency.

Introducing the non-dimensional variables

$$X = x/a,\ Y = y/b,\ \bar{h} = h/a,\ W = \bar{w}/a \quad \text{... (3)}$$

eq. (1) reduces to

$$\bar{h}^3 W^{iv} + 6 \bar{h}^2 \bar{h}' w'' + [3 \{\bar{h}^2 \bar{h}'' + 2 \bar{h} \bar{h}'^2\} - 2 (\eta^*/E_x^*) \bar{h}^3 \lambda^2] W''$$

$$- 6 (\eta^*/E_x^*) \bar{h}^2 \bar{h}' \lambda^2 W' + [\{E_y^*/E_x^*\} \bar{h}^3 \lambda^4 - 3 (E^*/E_x^*) \{\bar{h}^2 \bar{h}''$$

$$+ 2 \bar{h} \bar{h}'^2\} \lambda^2 - 12 (\rho a^2 \omega^2/E_x^*) \bar{h}] W = 0, \quad \text{... (4)}$$

where $\lambda^2 = p^2 \pi^2 a^2/b^2$, $\eta^* = E^* + 2G_{xy}$ and primes denote differentiation with respect to X.

It is assumed that the thickness of the plate and density of the plate material vary according to the functional relations

$$\bar{h} = h_0 e^{\alpha X}, \rho = \rho_0 e^{\beta X}, \qquad \text{... (5)}$$

i.e. an exponential variation along x-direction, where h_0, ρ_0 are thickness and density of the plate at $X = 0$ and α, β are the taper and density parameters, respectively. This type of variation is of interest since it provides reasonable approximation to linear variation. Eq. (4) now reduces to

$$A_0 W^{iv} + A_1 W''' + A_2 W'' + A_3 W' + A_4 W = 0, \qquad \text{... (6)}$$

where $A_0 = 1, A_1 = 6\alpha, A_2 = 9\alpha^2 - 2\lambda^2(\eta^*/E_x^*), A_3 = -6\alpha\lambda^2(\eta^*/E_x^*),$

$$A_4 = (E_y^*/E_x^*)\lambda^4 - 9(E^*/E_x^*)\alpha^2\lambda^2 - \Omega^2 e^{(\beta - 2\alpha)X}, \Omega^2 = 12\rho_0 a^2 \omega^2/E_x^* h_0^2.$$

The solution of eq. (6) together with the boundary conditions at the edges $X = 0$ and $X = 1$ constitutes a two point boundary value problem. Due to the presence of variable coefficients in eq. (6), its closed form solution is not possible. Keeping this in view an approximate solution is obtained by applying the Chebyshev Collocation technique.

METHOD OF SOLUTION

By taking a new independent variable ξ defined by

$$\xi = 2X - 1, \qquad \text{... (7)}$$

the range $0 \le X < x \le 1$ is transformed to $-1 \le \xi \le 1$, the applicability range of the present technique. Eq. (6) now reduces to

$$V_0 W^{iv} + V_1 W''' + V_2 W'' + V_3 W' + V_4 W = 0, \qquad \text{...(8)}$$

where $V_i = 2^{4-i} A_i$, $i = 0, 1, 2, 3, 4.$

According to Chebyshev collocation technique[9-12], we assume the solution as

$$W = c_1 + c_2 T_1 + c_3 \overset{1}{T_1} + c_4 \overset{2}{T_1} + \sum_{k=0}^{m-5} c_{k+5} \overset{4}{T_k}, \qquad \text{... (9)}$$

where $c_j\ (j = 1, 2, ..., m)$ are unknown constants, $T_k\ (k = 0, 1, 2, ..., m-5)$ are Chebyshev polynomials and $\overset{j}{T_k}$ represents the jth integral of T_k with respect to ξ.

Substitution of W and its derivatives in eq. (8) gives an equation in terms of the unknown constants c's and Chebyshev polynomials T_j. The satisfaction of this resultant equation at $(m-4)$ collocation points given by

$$\xi_i = \cos\left(\frac{2i+1}{m-4}\frac{\pi}{2}\right), \quad i = 0, 1, 2, ..., m-5, \qquad \text{... (10)}$$

provides a set of $(m-4)$ equations in terms of unknowns $c_j\ (j = 1, 2, ..., m)$, which can be denoted by the matrix equation,

$$[B]\ \{C^*\} = \{0\}, \qquad \text{... (11)}$$

where B is a matrix of order $(m-4) \times m$ and, $\{C^*\}$ and $\{0\}$ are column vectors of order $m \times 1$ and $(m-4) \times 1$, respectively.

BOUNDARY CONDITIONS AND FREQUENCY EQUATIONS

The three sets of boundary conditions, namely, $C - C$, $C - S$, $C - F$, have been considered in which the first symbol represents the condition at the edge $X = 0$ and the second at the edge $X = 1$ and C, S, F, respectively, stand for clamped, simply supported and free edge conditions. The relations which should be satisfied at C, S and F edge are

$$W = dW/dX = 0;\ W = (d^2 W/dX^2) - \nu_y \lambda^2 W = 0$$

and

$$(d^2 W/dX^2) - \nu_y \lambda^2 W = (d^3 W/dX^3) \quad - (\nu_y + 4G_{xy}/E_x^*)\ \lambda^2 (dW/dX) = 0, \text{ respectively.}$$

Applying the $C - C$ boundary conditions to the displacement function (9), one obtains a set of four homogeneous equations in terms of unknowns c_j, $j = 1, 2, ..., m$. These equations together with the field eqs. (11) give a complete set of m equations in m unknowns, which can be written in matrix form as

$$[B/B_{CC}]\ \{C^*\} = \{0\}, \quad ... (12)$$

where B_{CC} is a matrix of order $4 \times m$ and $\{0\}$ is a column vector of order $m \times 1$.

For a non-trivial solution of eq. (12), the frequency determinant must vanish and hence,

$$|\, B/B_{CC} \,| = 0. \quad ... (13)$$

Similarly for $C - S$ and $C - F$ plates, the frequency determinants can be written as

$$|\, B/B_{CS} \,| = 0,\ |\, B/B_{CF} \,| = 0, \quad .... (14,\ 15)$$

respectively.

NUMERICAL RESULTS AND DISCUSSION

The characteristic eqns. (13-15) have been solved numerically for various values of density parameter β = (– 0.5 (0.2) 0.5, 0.0, 1.0) taper constant α = (– 0.5 (0.2) 0.5, 0.0) and the aspect ratio a/b for the first three modes of vibration. This is achieved by writing $p = 1$ in these equations. In all computations, we have fixed $m = 15$, because a further increase in m does not improve the results except in the fourth place of decimal (Fig. 1). The values of elastic constants used for the plate material are (E_x, E_y, G_{xy}) = (1.3147, 0.4218, 0.1118) × 10^5 kg/cm^2 and $\nu_x = 0.12$ (five-ply maple plywood[13]). The thickness h_0 has been taken as 0.1.

From the results it is observed that the frequencies for a C-S plate are higher than that for a C-F plate but less than that for a C-C plate for the same set of values of plate parameters. Fig. (2) shows the behaviour of density parameter β on the frequency parameter Ω for the first mode for $\alpha = -0.5$, 0.5 and aspect ratio a/b = 0.5, 1.0. The frequency parameter is found to decrease continuously with the increase in density parameter β for all other plate parameters. However the rate of decrease is less in C-F case as compared to C-S and C-C plates. The effect of taper parameter is more pronounced for negative values of β as compared to those for positive values for all the boundary conditions. The increase in the value of aspect ratio increases the value of frequency parameter for a set of the values of all other plate parameters for all the three boundary conditions.

A similar inference can be drawn from Figs. (3-4) showing the plots for frequency parameter versus density parameter except that the rate of decrease of frequency parameter Ω with density parameter β increases with the increase in number of modes.

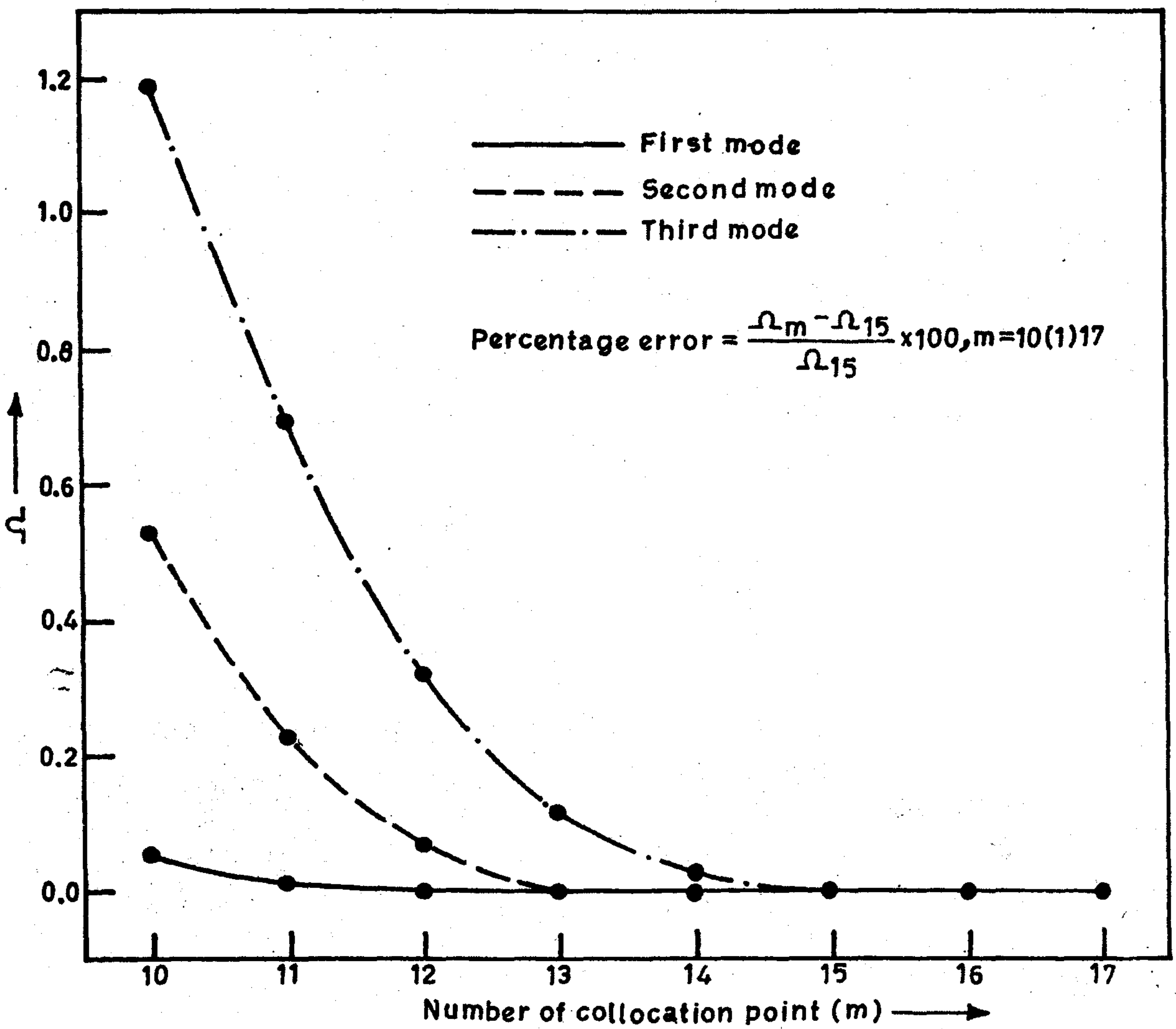

FIG. 1. Percentage error in Ω for C-C plate for a/b = 1.0, α = 0.5 and β = –0.5

Figs. (5-6) show the effects of density parameter for the varying values of aspect ratio a/b on the natural frequencies for the first two modes of vibration for taper constant α = – 0.5, 0.5 for all the three boundary conditions.

Mode shapes have been computed for $\alpha = \pm 0.5, \beta = 0.1$ and a/b = 1.0 for all the boundry conditions. Normalized displacements $W_{norm}\,(= W/W_{\max})$ are shown in Figs. (7-9) for first three modes of vibration. The nodal lines are seen to shift towards the edge X = 0 as the edge X = 1 increases in thickness. No appreciable change was noticed in the mode shapes due to change in the values of density parameter and aspect ratio. In fact the numerical values of normalized displacements were changing only at the second place of decimal.

Table 1 shows a comparison of results for uniform isotropic plates obtained by taking $\alpha = 0, (E^* + 2G_{xy})/E_x^* = 1, E^*/E_x^* = \nu, E_y^*/E_x^* = 1$ and $E_x = E_y = E$ in eq. (4) for β = 0.0 and a/b = 0.5, 1.0. Our results show a good agreement with those obtained by various techniques. Table 2 shows a comparison of the fundamental frequencies for orthotropic plates of uniform thickness with those

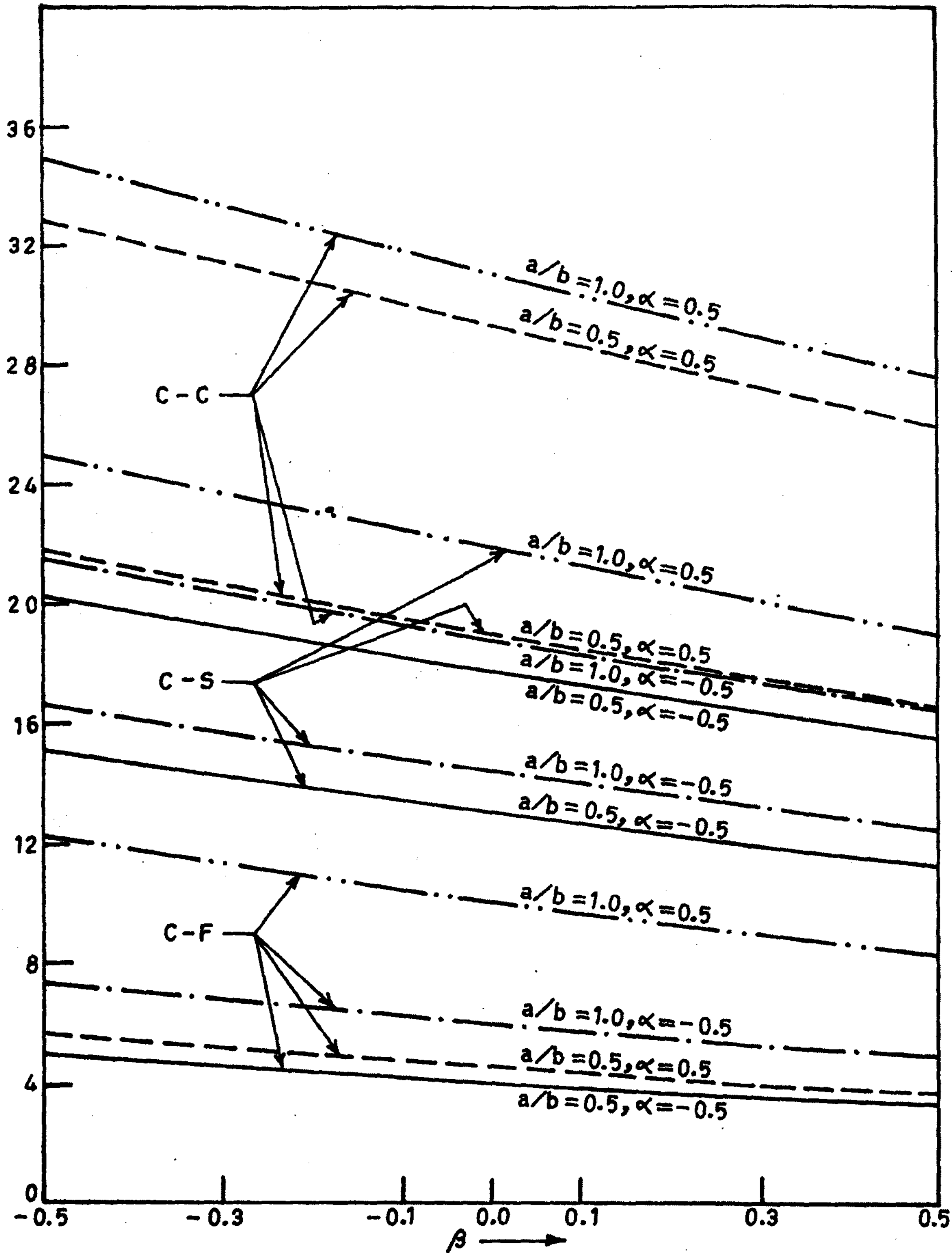

FIG. 2. Natural frequencies of C-C, C-S and C-F plates for first mode of vibration

of Hearmon[13] for all the six similar combinations of boundary conditions with aspect ratio a/b = 2.0. However, for higher modes and different values of p = 1, 3, 5 with a/b = 2.0 with a particular

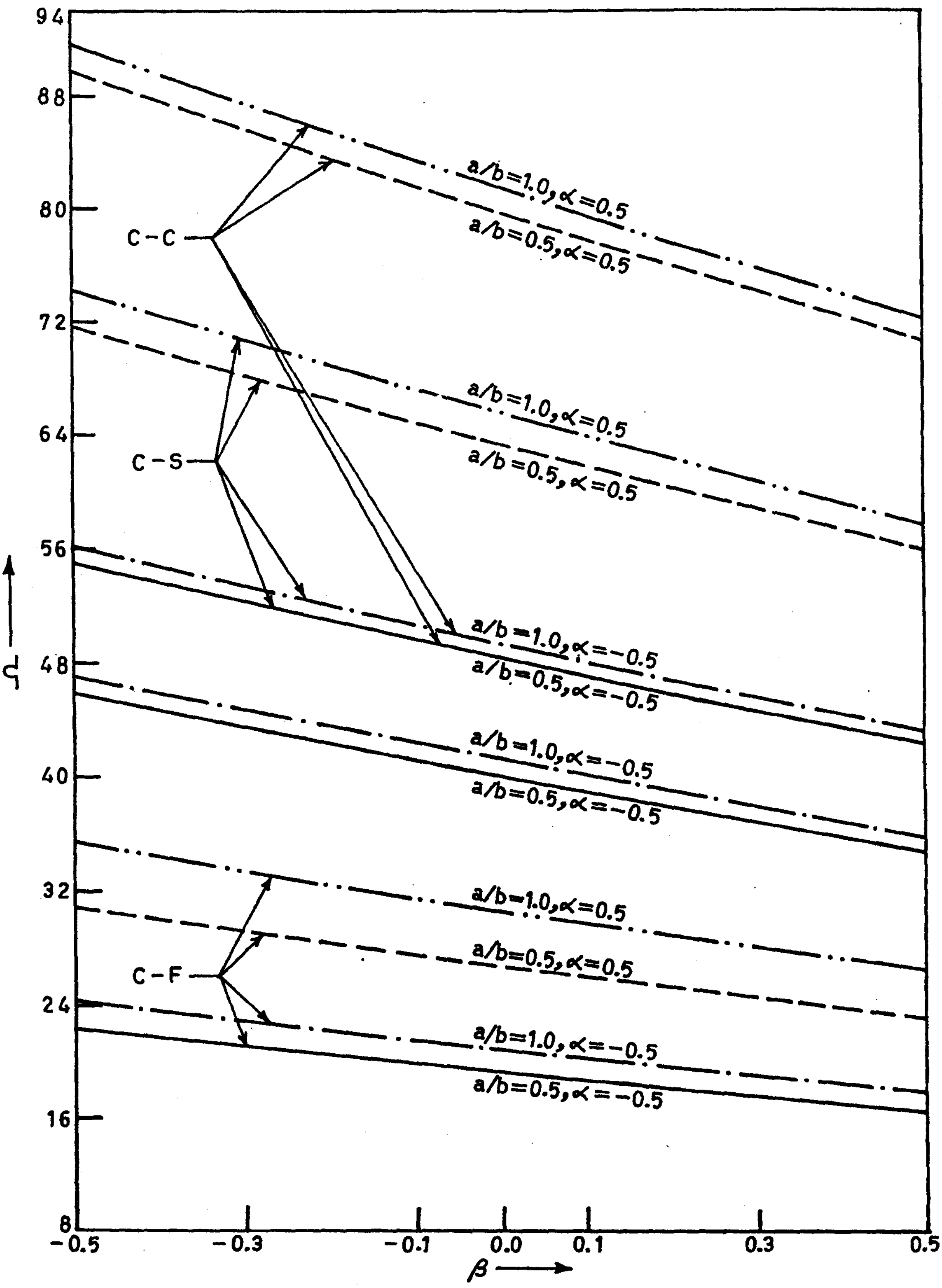

FIG. 3. Natural frequencies of C-C, C-S and C-F plates for second mode of vibration

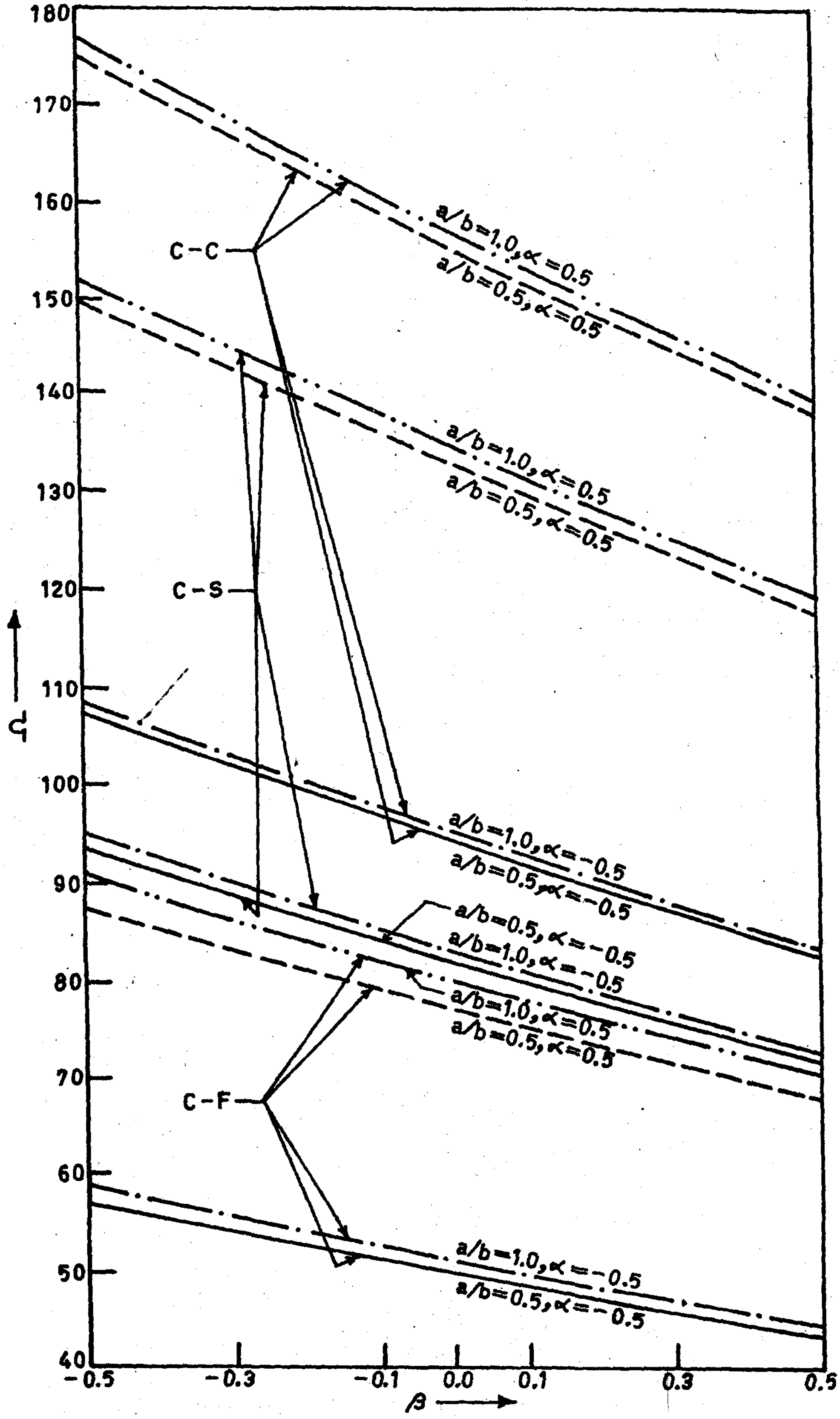

FIG. 4. Natural frequencies of C-C, C-S and C-F plates for third mode of vibration

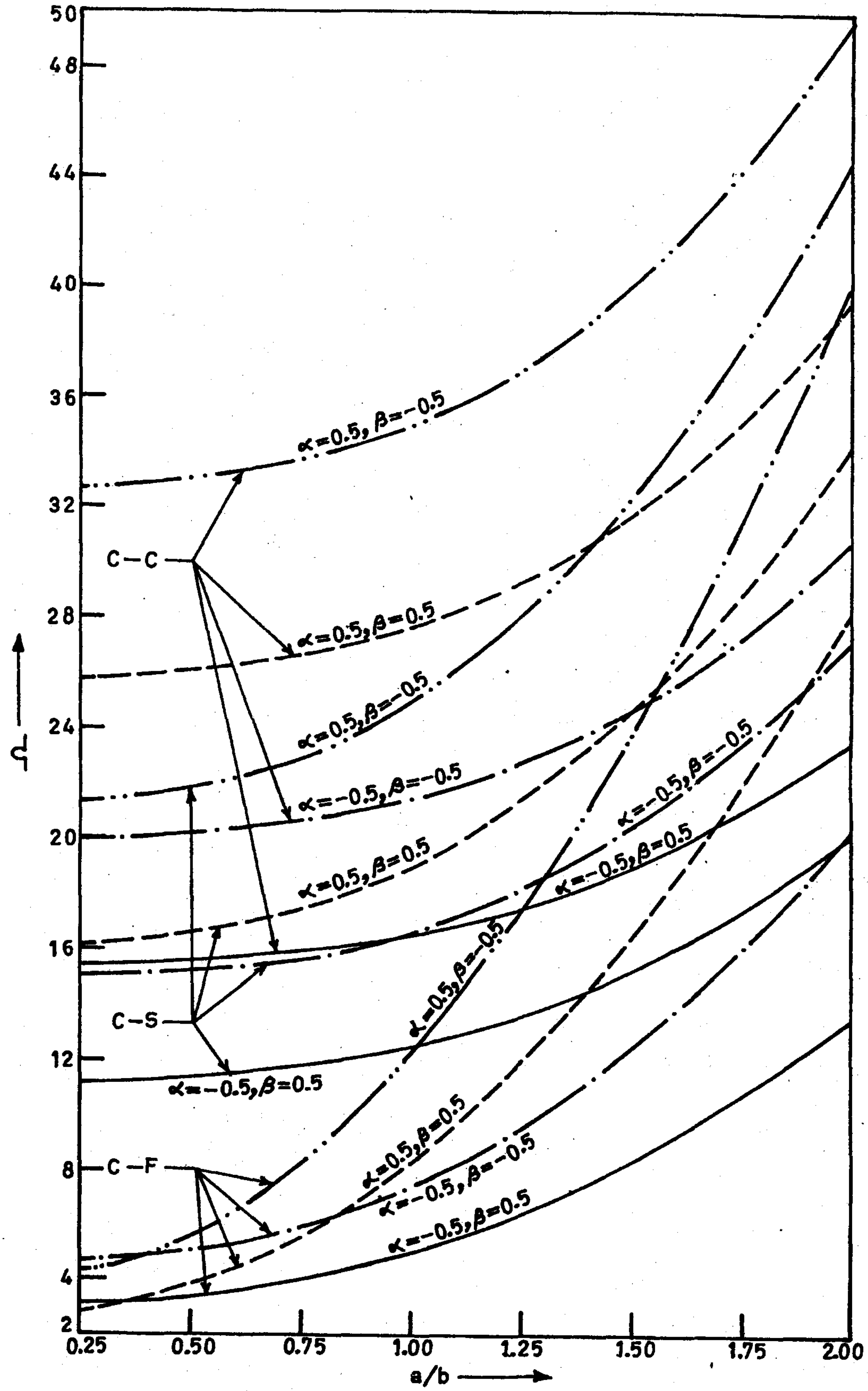

FIG. 5. Natural frequencies of C-C, C-S and C-F plates for first mode of vibration

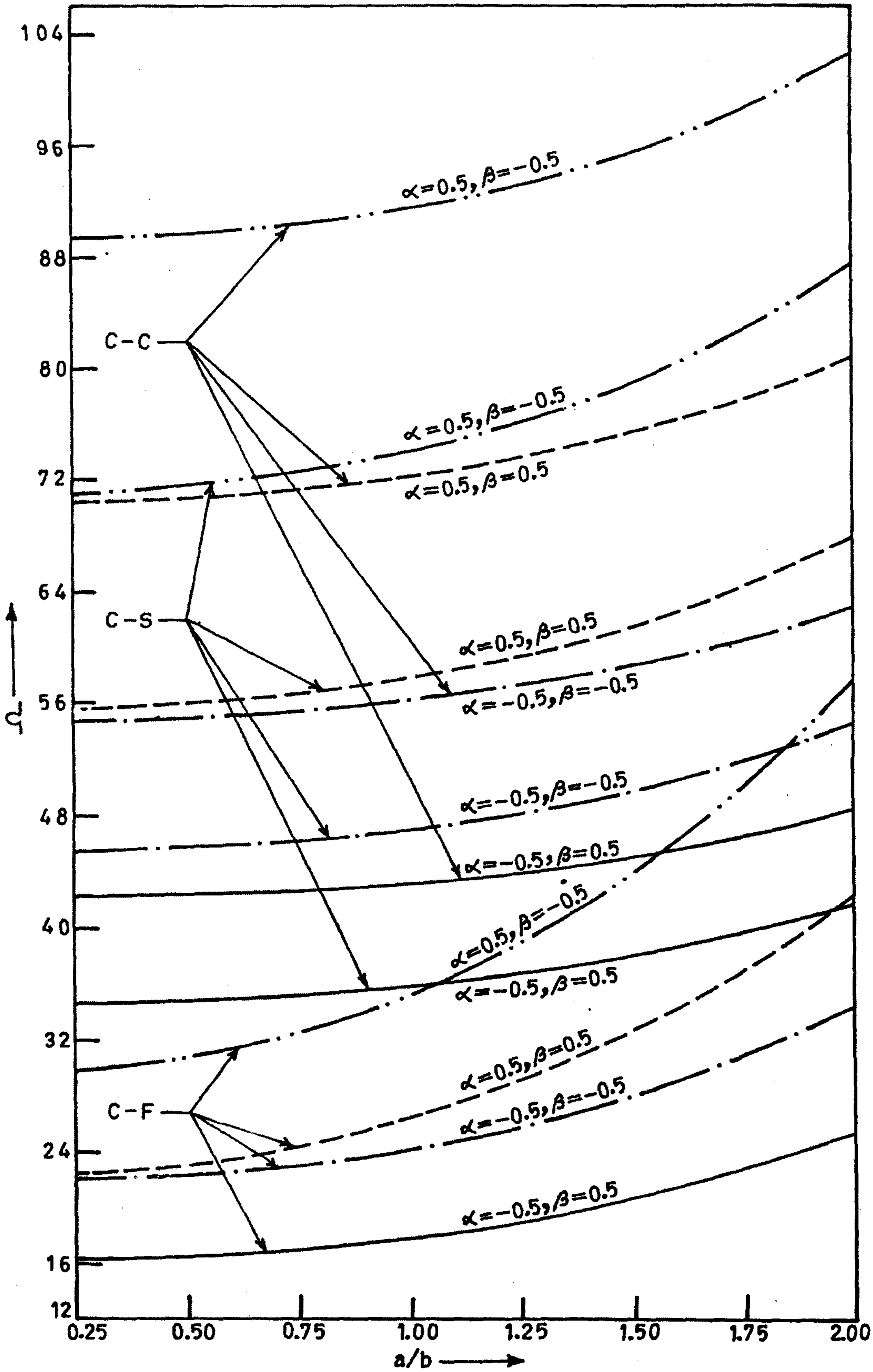

FIG. 6. Natural frequencies of C-C, C-S and C-F plates for second mode of vibration

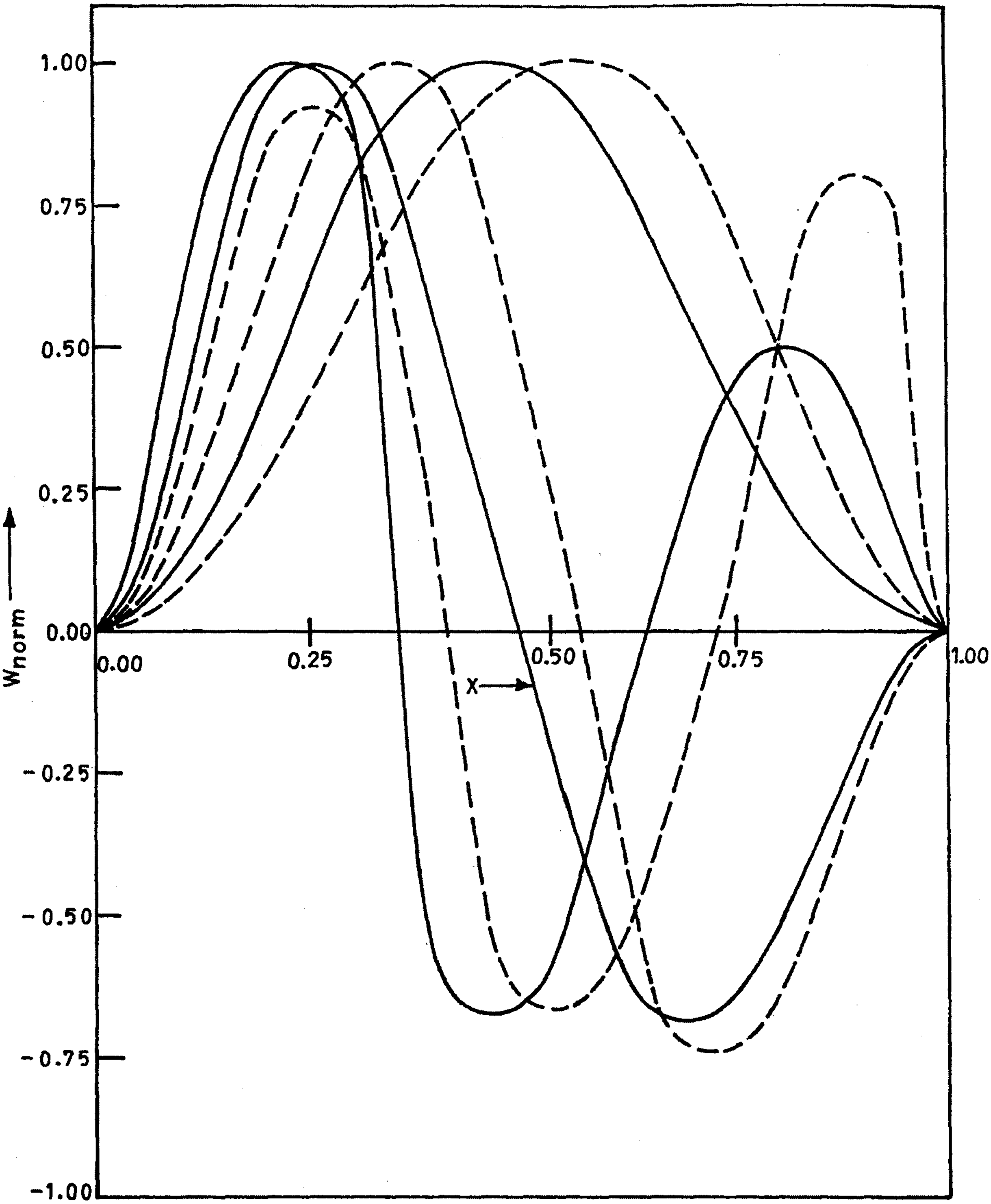

FIG. 7. Normalised displacements of C-C plate for the first three modes of vibration. a/b = 1.0, β = 1.0 : — — — — — — α = 0.5; - - - - - α = −0.5.

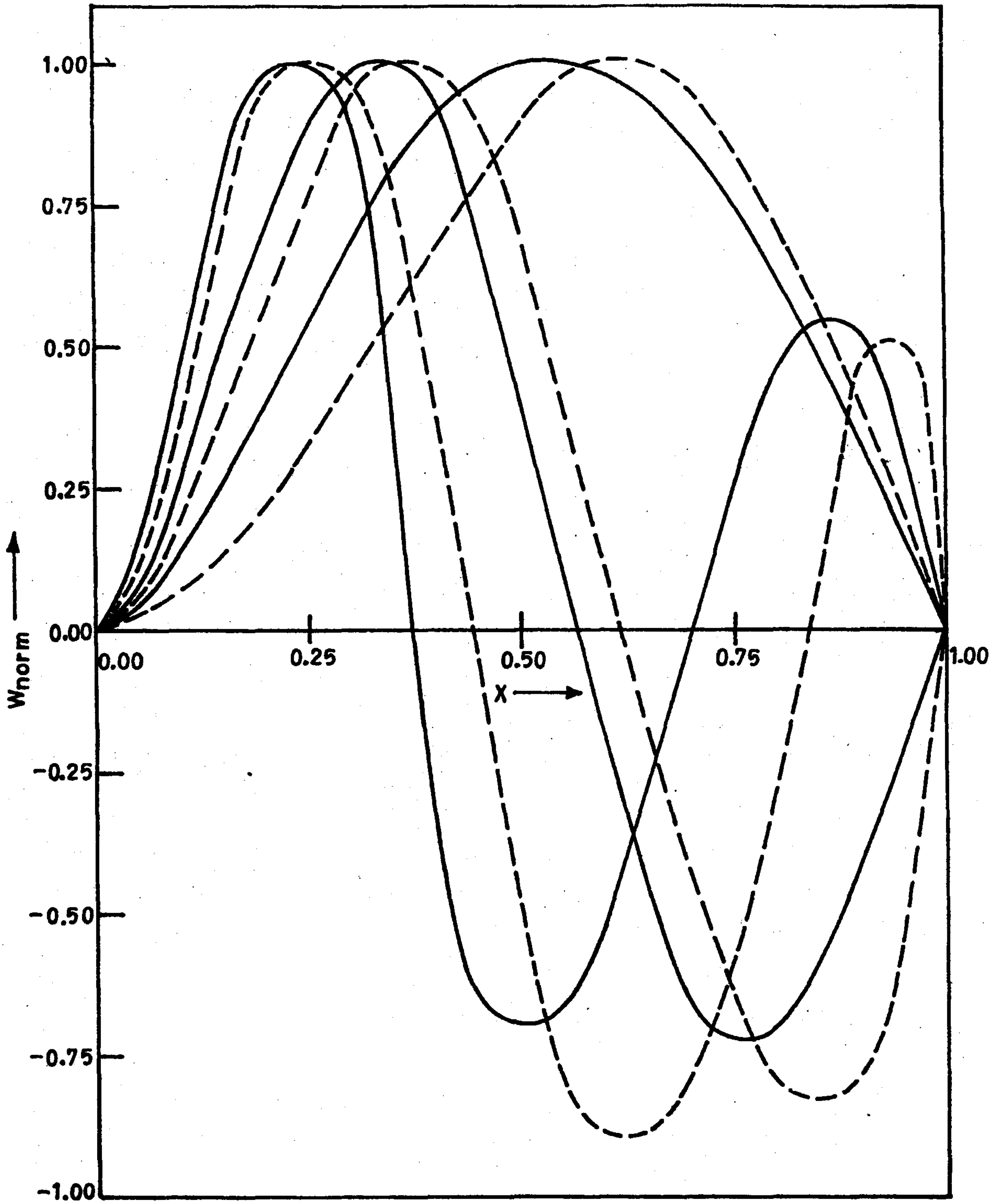

FIG. 8. Normalised displacements of C-S plate for the first three modes of vibration.
a/b = 1,0, β = 1.0 : — — — — α = 0.5; - - - - - α = −0.5.

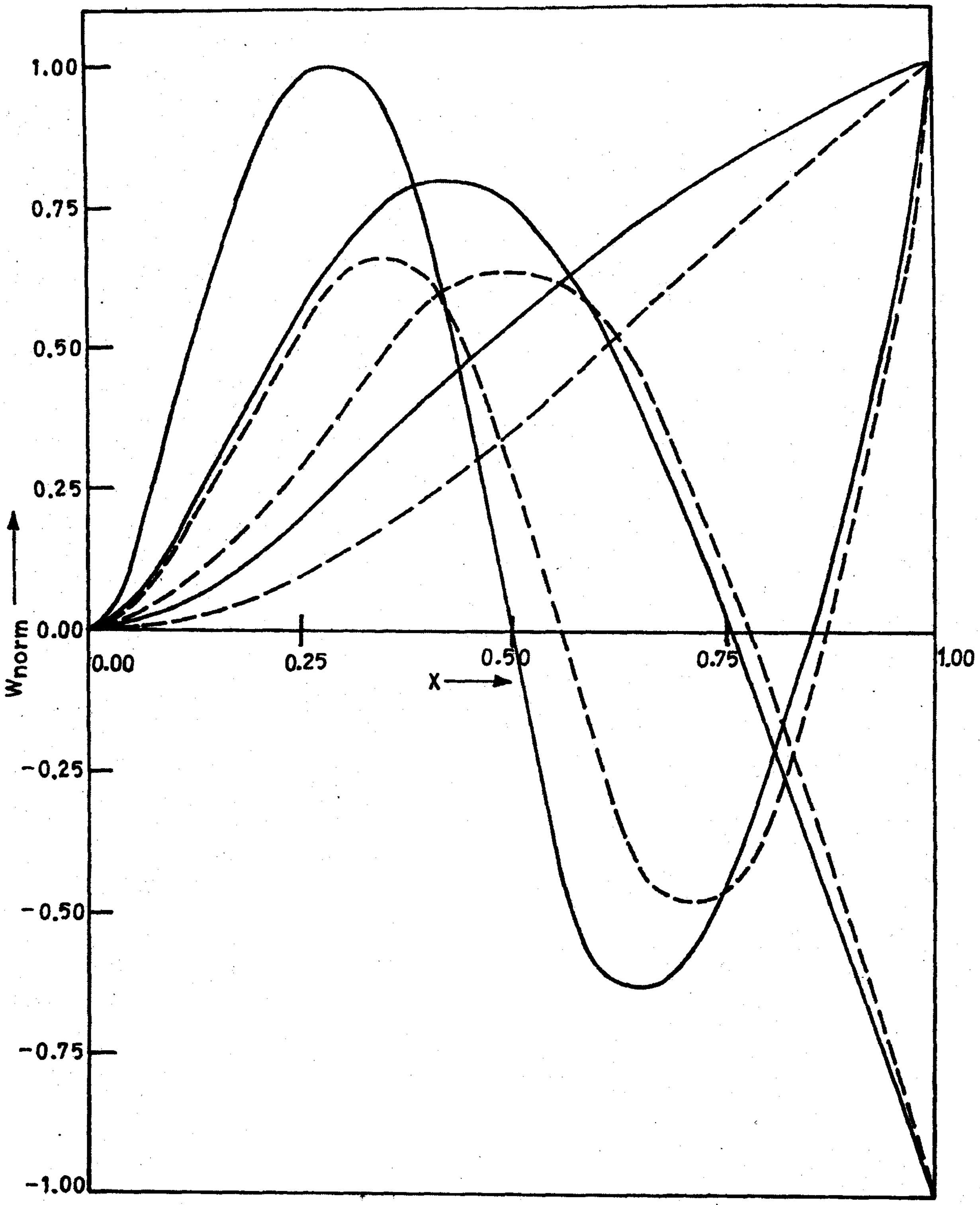

FIG. 9. Normalised displacements of C-S plate for the first three modes of vibration. a/b = 1.0, β = 1.0 : – – – – α = 0.5; - - - - - α = –0.5.

boundary condition, the results are reported in Table 3. An excellent agreement of the results show the computational accuracy of the Chebyshev collocation method.

TABLE I
Values of frequency parameter Ω for uniform isotropic plates $\alpha = 0, \beta = 0.0, \nu = 0.3$

Boundary conditions	C-C		C-S		C-F	
Mode a/b	0.5	1.0	0.5	1.0	0.5	1.0
I	23.8157	28.9509	17.3318	23.6464	5.703	12.679
	23.8156†	28.9508†	17.3316†	23.6463†	----	----
	23.828‡	28.949‡	17.341‡	23.648‡	5.702‡	12.679‡
						12.68°
						12.83•
II	63.5346	69.3271	52.0979	58.6464	24.958	33.062
	63.5345†	69.3270†	52.0966†	58.6463†	----	----
	63.709‡	69.462‡	52.231‡	58.753‡	24.959‡	33.063‡
III	122.9292	129.0952	106.4786	113.2281	64.578	72.541
	122.2295†	129.0956†	106.4785†	113.2281†	----	----
	123.702‡	129.793‡	107.115‡	113.808‡	64.579‡	72.540‡

† Values calculated by Frobenius' method [14]; ‡ Values calculated by spline technique method [15]
° Values calculated by finite element method [16]; • Values calculated by optimized Kantorovich [16].

TABLE II
Values of fundamental frequencies $\Omega\,(=(12\,\rho_0\,a^2\,\omega^2/(E_y^*\,h_0^2))^{1/2})$ for orthotropic plates of uniform thickness : $\beta = 0.0,\ a/b = 2.0\ (D_x/D_y = 3.117,\ D_1/D_y = \nu_x = 0.120,\ H/D_y = 0.648)$

Boundary Conditions	Present Study	Exact*	Rayleigh Method*
F SFSF:S S F	17.393	17.39	17.42
S SFSS:S S F	20.651	20.65	20.70
C SFSC:S S F	26.058	26.06	26.22
S SSSS:S S S	48.652	48.65	48.65
S CSSS:S S C	68.518	68.52	68.53
C SCSC:S S C	94.559	94.56	94.57

*Values taken from [13]: Symbols: C, clamped; S, simply supported; F, free.

TABLE III

Values of frequency parameter $\Omega\,(=(12\,\rho_0\,a^2\,\omega^2/(E_y^*\,h_0^2))^{1/2})$ for a uniform orthotropic SFSC plate (Five-ply maple plywood) : $\beta = 0.0, a/b = 2.0$

p	Mode	Present Study	Exact*	Rayleigh Method*
1	1	26.059	26.06	26.22
	2	97.678	97.68	97.70
	3	254.679	254.68	254.65
	4	490.985	490.98	491.00
3	1	161.724	161.72	162.67
	2	212.036	212.04	213.67
5	1	439.737	439.74	441.14

*Value taken from Hearmon[13].

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