Indian J. pure appl. Math., **34**(4) : 603-607, April 2003

FIVE DIMENSIONAL PLANE GRAVITATIONAL WAVES IN BIMETRIC RELATIVITY (II)

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(*Received* 29 *October* 2001; accepted 2 *May* 2002)

The plane gravitational wave solutions of the field equations $N^j_i = 0$ in five dimensional space-time V_5 for bimetric relativty are given by g_{ij} which satisfied

$$Q\,\rho^j_i + P\,\sigma^j_i = 0$$

which further breaks into

$$\bar{\bar{w}}_2\,\rho^j_i + \bar{w}_2\,\sigma^j_i = 0 = \bar{\bar{\phi}}_2\,\rho^j_i + \bar{\phi}_2\,\sigma^j_i$$

$$\bar{\bar{w}}_3\,\rho^j_i + \bar{w}_3\,\sigma^j_i = 0 = \bar{\bar{\phi}}_3\,\rho^j_i + \bar{\phi}_3\,\sigma^j_i$$

$$\bar{\bar{w}}_4\,\rho^j_i + \bar{w}_4\,\sigma^j_i = 0 = \bar{\bar{\phi}}_4\,\rho^j_i + \bar{\phi}_4\,\sigma^j_i$$

where
$$\rho^j_i = [(\phi_2^2 + \phi_3^2 + \phi_4^2) - 1]\, g^{hj}\,\bar{g}_{hi}$$

$$\sigma^j_i = \frac{d}{dZ}\left\{[1 - (\phi_2^2 + \phi_3^2 + \phi_4^2)]\, g^{hj}\,\bar{g}_{hi}\right\}$$

$$\phi_2 = \frac{Z_{,2}}{Z_{,5}},\ \phi_3 = \frac{Z_{,3}}{Z_{,5}},\ \phi_4 = \frac{Z_{,4}}{Z_{,5}}$$

$$w_2 = t + \phi_2\,x^2,\quad w_3 = t + \phi_3\,x^3,\quad w_4 = t + \phi_4\,x^4.$$

If Z is independent of the variables x^2 i.e. x and x^3 i.e. y, then our earlier work regarding the plane wave solutions in five dimensional space-time demonstrated in the paper [1] and [2] can be deduced respectively.

1. INTRODUCTION

In the paper refer it to [1], for BR theory of Rosen (1973, 74), we have obtained the plane wave solutions g_{ij} of the field equations $N^j_i = 0$ in five dimensional space-time V_5 by redefining Karade's (1994) definition of plane wave in V_4 as follows :

Definition — A plane wave g_{ij} is a nonflat solution of the field equations

$$N^j_i = 0,\ (i,\, j\ =\ 1,\ 2,\ 3,\ 4,\ 5) \qquad \text{... (1.1)}$$

in an empty region of the space-time such that

$$g_{ij} = g_{ij}\,(Z),\quad Z = Z\,(x^i),\ x^i = u, x, y, z, t \qquad \text{... (1.2)}$$

in some suitable co-ordinate system such that

$$g^{ij} Z_{,i} Z_{,j} = 0, \quad Z_{,i} = \frac{\partial Z}{\partial x^i} \qquad \text{... (1.3)}$$

$$Z = Z(y, z, t), \quad Z_{,3} \neq 0, Z_{,4} \neq 0, Z_{,5} \neq 0 \qquad \text{... (1.4)}$$

where $N^j_i = \frac{1}{2} f^{\alpha\beta} (g^{hj} g_{hi\,|\,\alpha})_{|\,\beta}$

$$N = N^i_i, \; k = \sqrt{g/f}, \; g = \det(g_{ij}), f = \det(f_{ij})$$

and the bar (|) stands for f-covariant differentiation.

In this definition the signature convention adopted is

$$g_{aa} < 0, \; a = 1, 2, 3, 4$$

$$\begin{vmatrix} g_{aa} & g_{ab} \\ g_{ba} & g_{bb} \end{vmatrix} > 0, \quad \begin{vmatrix} g_{aa} & g_{ab} & g_{ac} \\ g_{ba} & g_{bb} & g_{bc} \\ g_{ca} & g_{cb} & g_{cc} \end{vmatrix} < 0, \quad \begin{vmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{vmatrix} > 0,$$

$$g_{55} > 0 \qquad \text{... (1.5)}$$

[not summed for $a, b, c = 1, 2, 3, 4$]

and accordingly $g = \det(g_{ij}) > 0$.

The field equations $N^j_i = 0$ then yield

$$Q \rho^j_i + P \sigma^j_i = 0$$

which further breaks into

$$\bar{\bar{w}}_3 \rho^j_i + \bar{w}_3 \sigma^j_i = 0 = \bar{\bar{\phi}}_3 \rho^j_i + \bar{\phi}_3 \sigma^j_i$$

$$\bar{\bar{w}}_4 \rho^j_i + \bar{w}_4 \sigma^j_i = 0 = \bar{\bar{\phi}}_4 \rho^j_i + \bar{\phi}_4 \sigma^j_i$$

where $\rho^j_i = [(\phi_3^2 + \phi_4^2) - 1] \, g^{hj} \bar{g}_{hi}$

$$\sigma^j_i = \frac{d}{dZ} \left\{ [1 - (\phi_3^2 + \phi_4^2)] \, g^{hj} \bar{g}_{hi} \right\}$$

$$w_3 = t + \phi_3 x^3, \; w_4 = t + \phi_4 x^4.$$

$$Z_{,3} = \frac{\phi_3}{M_3}, \; Z_{,4} = \frac{\phi_4}{M_4}, \; Z_{,5} = \frac{1}{P}$$

$$\phi_3 = \frac{Z_{,3}}{Z_{,5}}, \; \phi_4 = \frac{Z_{,4}}{Z_{,5}}$$

$$M_3 = \bar{w}_3 - \bar{\phi}_3 x^3, \; M_4 = \bar{w}_4 - \bar{\phi}_4 x^4$$

$$N_3 = \bar{\bar{w}}_3 - \bar{\bar{\phi}}_3 x^3, \; N_4 = \bar{\bar{w}}_4 - \bar{\bar{\phi}} x^4.$$

In the present paper, we confine ourselves to the same space-time V_5 but relax the conditions (1.2), (1.3) and (1.5) with assuming

$$Z = Z(x, y, z, t), \quad Z_{,2} \neq 0, Z_{,3} \neq 0, Z_{,4} \neq 0, Z_{,5} \neq 0. \qquad \text{... (1.6)}$$

We get some interesting results in BR theory. If Z is independent of variable x^2 i.e. x then our work regarding the plane wave solutions in five dimensional space-time studied in the paper [1] can be brought out.

2. Solutions of Field Equations

We have to react with eqs. (1.1) along with the conditions (1.2), (1.3) and (1.6) to obtain different forms of the plane wave solutions. Noting eqs. (1.3) and (1.6), we get

$$g^{22}\phi_2^2 + 2g^{23}\phi_2\phi_3 + 2g^{24}\phi_2\phi_4 + 2g^{25}\phi_2 + g^{33}\phi_3^2$$
$$+ 2g^{34}\phi_3\phi_4 + 2g^{35}\phi_3 + g^{44}\phi_4^2 + 2g^{45}\phi_4 + g^{55} = 0 \qquad \text{... (2.1)}$$

where
$$\phi_2 = \frac{Z_{,2}}{Z_{,5}}, \phi_3 = \frac{Z_{,3}}{Z_{,5}}, \phi_4 = \frac{Z_{,4}}{Z_{,5}} \qquad \text{... (2.2)}$$

which further yield

$$t + \phi_2 x^2 = w_2 = t + \phi_2 x \qquad \text{... (2.3)}$$

$$t + \phi_3 x^3 = w_3 = t + \phi_3 y \qquad \text{... (2.4)}$$

$$t + \phi_4 x^4 = w_4 = t + \phi_4 z \qquad \text{... (2.5)}$$

where w_2, w_3 and w_4 are arbitrary functions of Z.

Differentiating partially (2.3) with respect to x, t and (2.4) with respect to y, t and (2.5) with respect to z, t we obtain

$$Z_{,2} = \frac{\phi_2}{M_2}, \; Z_{,5} = \frac{1}{M_2} \qquad \text{... (2.6)}$$

where
$$M_2 = \bar{w}_2 - \bar{\phi}_2 x \qquad \text{... (2.7)}$$

$$Z_{,3} = \frac{\phi_3}{M_3}, \; Z_{,5} = \frac{1}{M_3} \qquad \text{... (2.8)}$$

where
$$M_3 = \bar{w}_3 - \bar{\phi}_3 y \qquad \text{... (2.9)}$$

$$Z_{,4} = \frac{\phi_4}{M_4}, \; Z_{,5} = \frac{1}{M_4} \qquad \text{... (2.10)}$$

where
$$M_4 = \bar{w}_4 - \bar{\phi}_4 z \qquad \text{... (2.11)}$$

Differentiating partially (2.7) with respect to x, t and (2.9) with respect to y, t and (2.11) with respect to z, t we get

$$M_{2,2} = \frac{N_2}{M_2}\phi_2 - \bar{\phi}_2, \; M_{2,5} = \frac{N_2}{M_2} \qquad \text{... (2.12)}$$

where
$$N_2 = \bar{\bar{w}}_2 - \bar{\bar{\phi}}_2 x \qquad \text{... (2.13)}$$

$$M_{3,3} = \frac{N_3}{M_3}\phi_3 - \bar{\phi}_3, \; M_{3,5} = \frac{N_3}{M_3} \qquad \text{... (2.14)}$$

where $$N_3 = \overset{=}{w}_3 - \overset{=}{\phi}_3 y \qquad \ldots (2.15)$$

$$M_{4,4} = \frac{N_4}{M_4}\phi_4 - \overline{\phi}_4, \; M_{4,5} = \frac{N_4}{M_4} \qquad \ldots (2.16)$$

where $$N_4 = \overset{=}{w}_4 - \overset{=}{\phi}_4 z \qquad \ldots (2.17)$$

The eqs. (2.6), (2.8) and (2.10) and (2.12), (2.14), (2.16) reveal that

$$M_2 = M_3 = M_4 = P \text{ (say) and } N_2 = N_3 = N_4 = Q \text{ (say)} \qquad \ldots (2.18)$$

Then the eqs (2.6), (2.8), (2.10) and (2.12), (2.14), (2.16) can be rewritten as

$$Z_{,2} = \frac{\phi_2}{P}, Z_{,3} = \frac{\phi_3}{P}, Z_{,4} = \frac{\phi_4}{P}, Z_{,5} = \frac{1}{P} \qquad \ldots (2.19)$$

and $$P_{,2} = \frac{Q}{P}\phi_2 - \overline{\phi}_2, P_{,3} = \frac{Q}{P}\phi_3 - \overline{\phi}_3, P_{,4} = \frac{Q}{P}\phi_4 - \overline{\phi}_4, P_{,5} = \frac{Q}{P} \qquad \ldots (2.20)$$

where a bar (–) over a letter means the derivative with respect to Z. It is to be noted that the expressions for various quantities in our paper [1] are retained here too.

Presuming f_{ij} as Lorentz metric (–1, –1, –1, –1, +1), the f-covariant derivative becomes the ordinary partial derivative and the field eqs. (1.1) assume the simple form

$$f^{\alpha\beta}[g^{hj} g_{hi,\alpha}]_{,\beta} = 0 \qquad \ldots (2.21)$$

which in view of (1.6) becomes

$$f^{22}[g^{hj} g_{hi,2}]_{,2} + f^{33}[g^{hj} g_{hi,3}]_{,3} + f^{44}[g^{hj} g_{hi,4}]_{,4} + f^{55}[g^{hj} g_{hi,5}]_{,5} = 0 \qquad \ldots (2.22)$$

which then yield

$$Q\left\{[(\phi_2^2 + \phi_3^2 + \phi_4^2) - 1]\, g^{hj}\overline{g}_{hi}\right\} + P\left\{1 - (\phi_2^2 + \phi_3^2 + \phi_4^2)]\, g^{hj}\overset{=}{g}_{hi}\right.$$
$$\left. + [1 - (\phi_2^2 + \phi_3^2 + \phi_4^2)]\, \overline{g}^{hi}\overline{g}_{hi} - [2(\phi_2\overline{\phi}_2 + \phi_3\overline{\phi}_3 + \phi_4\overline{\phi}_4)]\, g^{hi}\overline{g}_{hi}\right\} = 0. \qquad \ldots (2.23)$$

Eq. (2.23) can be put in the form analogous to that of Karade (1994) as

$$Q\rho_i^j + P\sigma_i^j = 0 \qquad \ldots (2.24)$$

where $$\rho_i^j = [(\phi_2^2 + \phi_3^3 + \phi_4^2) - 1]\, g^{hi}\overline{g}_{hi}$$

$$\sigma_i^j = \frac{d}{dZ}\left\{[1 - (\phi_2^2 + \phi_3^2 + \phi_4^2)]\, g^{hj}\overline{g}_{hi}\right\}$$

Substituting the values of P and Q, the eqs. (2.24) reduce to

$$\overset{=}{w}_2\rho_i^j + \overline{w}_2\sigma_i^j = 0 = \overset{=}{\phi}_2\rho_i^j + \overline{\phi}_2\sigma_i^j$$

$$\overset{=}{w}_3\rho_i^j + \overline{w}_3\sigma_i^j = 0 = \overset{=}{\phi}_3\rho_i^j + \overline{\phi}_3\sigma_i^j$$

$$\overset{=}{w}_4\rho_i^j + \overline{w}_4\sigma_i^j = 0 = \overset{=}{\phi}_4\rho_i^j + \overline{\phi}_4\sigma_i^j \qquad \ldots (2.25)$$

which are again in the format of Karade (1994). It is observed that, here we have obtained three eqs. (2.25) instead of two as compared to our earlier paper refer it to [1]. This is due to $Z_{,2} \neq 0, Z_{,3} \neq 0, Z_{,4} \neq 0$ and $Z_{,5} \neq 0$ instead of $Z_{,3} \neq 0, Z_{,4} \neq 0$ and $Z_{,5} \neq 0$.

CONCLUSION

We conclude that the plane gravitational waves g_{ij} are given by the eqs. (2.24) or (2.25).

ACKNOWLEDGEMENT

The author is thankful to professor T. M. Karade of India for his constant inspiration. Thanks are also due to professor K. Kondo of Japan for his constant encouragement.

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