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PROVING FIXED POINT THEOREMS IN *D*-METRIC SPACES VIA GENERAL EXISTENCE PRINCIPLES

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In this paper two basic fixed point theorems for selfmaps of a *D*-metric space are proved using general existence principles. These theorems are then used to obtain several interesting fixed point theorems, which include some well-known results of Dhage [4], Rhoades [12], Sehie Park [9] and consequently, about two dozen other fixed point theorems as special cases.

Key Words : Commuting; *D*-Metric Space; Fixed Point; Coincidentally Commuting Maps

1. INTRODUCTION

The concept of a *D*-metric space was introduced by the first author in[1]. A nonempty set X, together with a *D*-metric function $\rho : X \times X \times X \to [0, \infty)$, is called a *D*-metric space with a *D*-metric ρ, denoted by (X, ρ), if it satisfies the following properties :

(*i*) $\rho(x, y, z) = 0 \Leftrightarrow x = y = z$ (coincidence)

(*ii*) $\rho(x, y, z) = \rho(p\ \{x, y, z\})$ (symmetry)

where p is a permutation of $\{x, y, z\}$,and

(*iii*) $\rho(x, y, z) \leq \rho(x, y, a) + \rho(x, a, z) + \rho(a, y, z)$

for each $x, y, z \in X$. (tetrahedral inequality)

A sequence $\{x_n\} \subset X$ is called *D*-Cauchyif $\lim_{m, n, p} \rho(x_m, x_n, x_p) = 0$. A complete *D*-metric space is one in which every *D*-Cauchy sequence converges to a point in it. Finally, a subset S of a *D*-metric space X is called bounded if there exists a constant $k > 0$ such that $\rho(x, y, z) \leq k$ for all $x, y, z \in S$, and the constant k is called a *D*-bound for S. The infimum of all *D*-bounds of S is called the diameter of S and is denoted by $\delta(S)$.

It is known that the *D*-metric ρ is a continuous function on X^3 in the topology of *D*-metric convergence, which is Hausdorff[3]. A few details, along with some specific examples of a *D*-metric space, appear in [2] and [3]. In a series of papers [2]-[5], the first author proved some fixed point theorems of selfmaps of a *D*-metric space satisfying certain contractive conditions. All of the results are based on the *D*-Cauchy principle developed in Dhage[5]. The general procedure for proving fixed points theorems in a *D*-metric space consists of the following three steps :

(*i*) Construction of a sequence $x_{n+1} = fx_n\ n \geq 0$, which is shown to be D-Cauchy,

(*ii*) By applying certain completeness conditions, $\{x_n\}$ is shown to be convergent, and

(*iii*) the limit point of $\{x_n\}$ is shown to be a fixed point of the map f under certain conditions prescribed on f.

In this paper we hypothesize the above procedure and prove two basic fixed point theorems using general existence principles. Our results are natural extensions of the results of Rhoades [12] and Park [9] to D-metric spaces.

2. Preliminaries

Before proceeding to the main result of this paper, we give some additional definitions and preliminaries that are needed in the sequel.

Let $f: X \to X$. The orbit of f at a point $x \in X$ is the set $O_f(x)$, defined by

$$O_f(x) = \{x, fx, f^2 x, \ldots.\} \qquad \ldots (2.1)$$

The closure of the orbit $O_f(x)$ in X is denoted by $\overline{O_f(x)}$. A D-metric space X is called f-orbitally complete if every D-Cauchy sequence $\{x_n\}$ in $O_f(x)$ converges to a point of X, for each $x \in X$. A D-metric space X is called f-orbitally bounded if $\delta(O_f(x)) < \infty$ for each $x \in X$. A mapping f on a D-metric space X is called f-orbitally continuous if $\{x_n\} \subset O_f(x)$ and $x_n \to x^*$ implies $fx_n \to fx^*$, for each $x \in X$.

Let Φ denote the class of all functions $\phi: I\!R^+ \to I\!R^+$ satisfying the following conditions :

(*i*) ϕ is continuous,

(*ii*) ϕ is nondecreasing,

(*iii*) $\phi(t) < t$ for each $t > 0$, and

(*iv*) $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t \in I\!R^+$.

The function ϕ is called a growth or control function of f and a commonly known growth function is $\phi(t) = \lambda t, 0 \leq \lambda < 1$. The following lemmas appear in [5].

Lemma 2.1 — (D-Cauchy principle) Let $\{x_n\}$ be a bounded sequence in a D-metric space X with D-bound k satisfying

$$\rho(x_n, x_{n+1}, x_m) \leq \phi^n(k)$$

for all $m > n \in N$, where $\phi: I\!R^+ \to I\!R^+$ satisfies $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t \in I\!R^+$.

Then $\{x_n\}$ is D-cauchy.

Lemma 2.2 — If $\phi \in \Phi$, then $\phi^n(0) = 0$ for each $n \in N$ and $\lim_n \phi^n(t) = 0$ for each $t > 0$.

3. First General Fixed Point Principle

***Theorem* 3.1** — *Let f be a selfmap of an f-orbitally complete D-metric space X satisfying*

$$\rho(fx, f^2x, z) \leq \phi(\rho x, fx, z)) \qquad \ldots (3.1)$$

for each $x \in X$ with $x \neq fx$ and $z \in X$, where $\phi \in \Phi$. Then (*i*) $\lim\limits_{n} f^n x = u$ exists,

(*ii*) $\rho(f^m x, f^n x, u) \le 2 \sum\limits_{j=n}^{m} \phi^j(k)$, *where* $k = \delta(O_f(x))$, *and*

(iii) u is a fixed point of f if and only if $G(x) = \rho(x, fx, x)$ *is f-orbitally lower semi-continuous at u.*

To prove the theorem we shall make use of the following lemmas.

Lemma 3.1 — Let $x \in X$ be arbitrary, and define the sequence $\{x_n\}$ by

$$x_0 = x,\ x_{n+1} = f x_n,\ n \ge 0, \quad \ldots (3.2)$$

Then, for any $m > n$

$$\rho(x_n, x_{n+1}, x_m) \le \phi^n(M)). \quad \ldots (3.3)$$

where $M := 2 \sum\limits_{i=0}^{\infty} \phi^j(q)$, and where $q = \max\{\rho(x_0, x_1, x_0),\ \rho(x_0, x_1, x_1)\}$.

PROOF : From (3.1),

$$\rho(x_n, x_{n+1}, x_m) \le \phi(\rho(x_{n-1}, x_n, x_m)) \quad \ldots (3.4)$$

for each $n = 1, 2, \ldots,$

By induction,

$$\rho(x_n, x_{n+1}, x_m) \le \phi(\rho(x_{n-1}, x_n, x_m)) \le \phi^2(\rho(x_{n-2}, x_{n-1}, x_m)) \quad \ldots (3.5)$$

$$\le \phi^n(\rho(x_0, x_1, x_m)).$$

Using property (*iii*) of the definition of a *D*-metric and (3.5).

$$\rho(x_0, x_1, x_m) \le \rho(x_0, x_1, x_{m-1}) + \rho(x_0, x_{m-1}, x_m) + \rho(x_{m-1}, x_1, x_m)$$

$$= \rho(x_0, x_1, x_{m-1}) + \rho(x_{m-1}, x_m, x_0) + \rho(x_{m-1}, x_m, x_1)$$

$$\le \rho(x_0, x_1, x_{m-1}) + \phi^{m-1}(\rho(x_0, x_1, x_0)) + \phi^{m-1}(\rho(x_0, x_1, x_1))$$

$$\le \rho(x_0, x_1, x_{m-}) + 2\phi^{m-1}(q)$$

$$\le \rho(x_0, x_1, x_{m-2}) + 2\phi^{m-2}(q) + 2\phi^{m-1}(q)$$

$$\le \rho(x_0, x_{q1}, x_1) + 2[\phi(q) + \ldots + \phi^{m-1}(q)]$$

$$< M.$$

Substituting into (3.5) yields (3.3). □

Lemma 3.2 — Each $\{x_n\}$ defined by (3.2) is bounded.

PROOF : For any integers $s \geq m \geq n$, there exist integers p and r such that

$$\rho(x_n, x_m, x_s) = \rho(x_n, x_{n+p}, x_{n+r})$$
$$\leq \rho(x_n, x_{n+1}, x_{n+r}) + \rho(x_n, x_{n+p}, x_{n-1}) + \rho(x_{n+1}, x_{n+p}, x_{n+r})$$
$$\leq \rho(x_{n+1}, x_{n+p}, x_{n+r}) + 2\,\phi^n(M)$$
$$\leq \rho(x_{n+1}, x_{n+p}, x_{n+2}) + \rho(x_{n+1}, x_{n+2}, x_{n+r})$$
$$+ \rho(x_{n+2}, x_{n+\rho}), x_{n+r}) + 2\,\phi^n(M)$$
$$\leq \rho(x_{n+2}, x_{n+p}, x_{n+r}) + 2\,\phi^{n+1}(M) + 2\,\phi^n(M)$$

$$\leq \rho(x_{n+p-1}, x_{n+p}, x_{n+r}) + \sum_{i=n}^{n+p-2} \phi^i(M)$$
$$\leq \sum_{i=n}^{n+p-1} \phi^i(M) < \sum_{i=n}^{\infty} \phi^i(M) < \infty$$

by property (iv) of Φ . □

PROOF OF THEOREM 3.1 : Let $x \in X$ and define $\{x_n\}$ by (3.2). If $x_n = x_{n+1}$ for some n, then f has a fixed point. Assume that the x_n are distinct. From Lemmas 3.2 and 3.1, $\{x_n\}$ is bounded and satisfies (3.3). By Lemma 2.1, $\{x_n\}$ is Cauchy, hence convergent since X is f-orbitally complete.

To prove (*iii*), suppose that $G(x) = \rho(x, fx, x)$ is f-orbitally lower semi-continuous at u. Then

$$\rho(u, fu, u) = G(u) \leq \lim_n G(x_n) = \lim_n \rho(x_n, fx_n, x_n) = 0,$$

and so $u = f\,u$.

Conversely, if f has a fixed point $u \in X$ then, from condition (3.1), it follows that $G(x) = \rho(x, fx, x)$ is f-orbitally lower semi-continuous at u.

Corollary 3.1 — Let f be a selfmap of an f-orbitally complete D-metric space X satisfying

$$\rho(fx, f^2x, x) \leq \lambda\, \rho(x, fx, z) \qquad \text{... (3.6)}$$

for each x in X with $x \neq fx$ and $z \in X$, where $0 \leq \lambda < 1$. Then

(*i*) $\lim_n f^n(x) = u$ exists,

(*ii*) $\rho(f^m(x), f^n(x), u) \leq 2 \sum_{j=n}^{m} \lambda^j \leq 2\frac{\lambda^n}{1-\lambda}k$, $\delta(O_f(x)) \leq k$, and

(*iii*) u is a fixed point of f iff $G(x) = \rho(x, fx, x)$ is f-orbitally lower semi-continuous at u.

The following lemma is sometimes useful in establishing hypothesis (iii) of Theorem 3.1

Lemma 3.3 — If f is a f-orbitally continuous selfmap of a D-metric space X, then the function $G : X \to I\!R^+$ defined by

$$G(x) = \rho(x, fx, x) \qquad \text{... (3.7)}$$

is f-orbitally continuous, and consequently f-orbitally lower semi-continuous on X.

PROOF : Let $x \in X$ be arbitrary and let $\{x_n\}$ be a sequence in $O_f(x)$ converging to a point $u \in X$. Suppose that f is f-orbitally continuous on X. Then $\lim\limits_n fx_n = f\left(\lim\limits_n x_n\right) = fu$, and

$$\lim_n G(x_n) = \lim_n \rho(x_n, fx_n, x_n) = \rho(u, fu, u) = G(u).$$

As every continuous function is lower semi-continuous, the function G is lower semi-continuous on $O_f(x)$ for each $x \in X$. □

Corollary 3.2 — Let f be an f-orbitally continuous selfmap of an f-orbitally complete D-metric space X satisfying

$$\min\{\rho(fx, fy, z), \rho(x, fx, z), \rho(y, fy, z)\} \leq \lambda \rho(x, y, z) \qquad \text{... (3.8)}$$

for all $x, y \in X$ with $x \neq fx, y \neq fy$ and $z \in X$, where $0 \leq \lambda < 1$. Then f has a fixed point.

PROOF : Setting $y = fx$ in (3.8), if $x = fx$, then the conclusion immediately follows. If $x \neq fx$, then, from (3.8), we obtain

$$\min\{\rho(fx, f_x^* y, z), \rho(x, fx, z), \rho(fx, f^2 x, z)\} \leq \lambda \rho(x, fx, z),$$

which implies that

$$\rho(fx, f^2 x, z) \leq \lambda \rho(x, fx, z)$$

for all $x \in X$ with $x \neq fx$ and $z \in X$, where $0 \leq \lambda < 1$.

Since f is f-orbitally continuous, lemma 3.3 implies that condition (*iii*) of Corollary 3.1 is true. Hence an application of Corollary 3.1 yields the desired result. □

Corollary 3.3 — Let f be an f-orbitally continuous selfmap of an f-orbitally complete D-metric space X satisfying

$$\rho(fx, fy, z) \leq \alpha \frac{\rho(x, fx, z)\,\rho(y, fy, z)}{\rho(x, fx, z)} + \beta \rho(x, y, z) \qquad \text{... (3.9)}$$

for all $x \in X$ with $x \neq fx, y \neq fy$ and $z \in X$, where $\alpha, \beta \geq 0$ satisfy $0 \leq \alpha + \beta < 1$. Then f has a unique fixed point.

PROOF : Setting $y = fx$ in (3.9), we obtain

$$\rho(fx, f^2 x, z) \leq \alpha \frac{\rho(x, fx, z)\,\rho(fx, f^2 x, z)}{\rho(x, fx, z)} + \beta \rho(x, fx, z), \qquad \text{... (3.10)}$$

for all $x, y \in X$ with $x \neq f^p x, y \neq f^p y$ and $z \in X$, where $\alpha, \beta \geq 0$ satsify $\alpha + \beta < 1$. Then f has a unique fixed point.

PROOF : Setting $y = fx$ in (3.9), we obtain

$$\rho(fx, f^2 x, z) \leq \alpha \frac{\rho(f, fx, z)\,\rho(y, fy, z)}{\rho(x, fx, z)} + \beta\rho(x, y, z),$$

or $$\rho(fx, f^2 x, z) \le \lambda \rho(x, fx, z)$$

for all $x, y \in X$ with $x \ne fx$ and $z \in X$, where $\lambda = \beta/(1-\alpha) < 1$, and condition (3.6) of Corollary 3.1 is satisfied.

Uniqueness readily follows from (3.9). □

Corollary 3.4 — Let f be an f-orbitally complete D-metric space X satisfying for some positive integer p,

$$\rho(f^p x, f^p y, z) \le \alpha \frac{\rho(x, f^p x, z)\, \rho(y, f^p y, z)}{\rho(x, y, z)} + \beta \rho(x, y, z)$$

for all $x, y \in X$ with $x \ne f^{px,}\, y \ne f^p x, y \ne f^p y$ and $z \in X$, where $\alpha, \beta \ge 0$ satisfy $\alpha + \beta < 1$. Then f has a unique fixed point.

PROOF : Setting $g = f^p$, inequality (3.10) reduces to (3.9). By Corollary 3.3, f^p has a unique fixed point u. Further, $f u = f(f^p u) = f^p (f u)$, and $f u$ is also a fixed point of f^p. The uniqueness of u implies that $f u = u$. □

In some specific applications of Corollary 3.1, or Theorem 3.1, it is often the case that the contractive condition is so strong that condition (*iii*) is not needed in order to establish the existence of a fixed point. The following are some examples in this direction.

Corollary 3.5 — Let f be a selfmap of an f-orbitally complete D-metric space X satisfying

$$\rho(fx, fy, z) \le \lambda \{ \rho(x, y, z), \rho(x, fx, z), \rho(y, fy, z) \} \quad \text{... (3.11)}$$

for all $x, y z \in X$, where $0 \le \lambda < 1$. Then f has a unique fixed point $u \in X$, and f is continuous at u.

PROOF : Setting $y = fx$ in (3.11), we obtain

$$\rho(fx, f^2 x, x) \le \lambda \max \{\rho(x, fx, z), \rho(x, fx, z), \rho(fx, f^2 x, z)\},$$

which implies that

$$\rho(fx, f^2, z) \le \lambda \rho(x, fx, z)$$

for all $x \in X$. Without loss of generality we may assume that $x \ne fx$, since, otherwise, x is a fixed point of f. Corollary 3.1 yields the fact that there is a point $u \in X$ such that $\lim x_n = \lim f^n x = u$. We shall show that u is the unique fixed point of f.

$$\begin{aligned} \rho(x_{n+1}, fu, u) &= \rho(fx_n, fu, u) \\ &\le \lambda \max \{ \rho(x_n, u, u), \rho(x_n, fx_n, u), \rho(u, fu, u) \} \\ &= \max \{\rho(x_n, u, u), \rho(x_n, x_{n+1}, u), \rho(u, fu, u)\} \end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality yields $u = fu$.

To prove uniqueness, suppose that u and υ are fixed points of f. Using (3.11),

$$\rho(u, \upsilon, u) \le \lambda \max \{\rho(u, \upsilon, u), \rho(u, u, \upsilon), \rho(\upsilon, \upsilon, u)\},$$

which implies that

$$\rho(u, \upsilon, u) \le \lambda \rho(u, \upsilon, \upsilon), \quad \text{... (3.12)}$$

Again using (3.11),

$$\rho(u, \upsilon, \upsilon) \le \lambda \max \{\rho(u, \upsilon, \upsilon), \rho(u, u, \upsilon), 0\},$$

which implies that

$$\rho(u, \upsilon, \upsilon) \leq \lambda \rho(u, \upsilon, u). \quad \ldots (3.13)$$

Substituting (3.13) into (3.12) yields

$$\rho(u, \upsilon, u) \leq \lambda^2 \rho(u, \upsilon, u),$$

which implies that $u = \upsilon$.

Finally, to prove the continuity of f at the fixed point, let $\{u_n\}$ be any sequence in X converging to u. It is sufficient to show that

$$\lim_{m,n} \rho(fu_m, fu_n, u) = 0. \quad \ldots (3.14)$$

$$\lim_{m,n} \rho(u_m, fu_n, u) \leq \lambda \lim_{m,n} \max$$

$$\{\rho(u_m, u_n, u), \rho(u_m, fu_m, u), \rho(u_n, fu_n, u)\}$$

$$= \lambda \max \left\{ \lim_m \rho(u_m, fu_m, u), \lim_n \rho(u_n, fu_n, u) \right\}$$

$$= \lambda \max \left\{ \lim_m \rho(u, fu_m, u), \lim_n \rho(u, fu_n, u) \right\}.$$

But, using (3.11),

$$\lim_n \rho(u_1, fu_n, u) \leq \lambda \lim_n \max \{ \rho(u, u_n, u), \rho(u, fu, u), \rho(u_n, fu_n, u) \}$$

$$= \lambda \lim_n \rho(u, fu_n, u).$$

which implies that $\lim_n \rho(u, fu_n, u) = 0$. Substituting this value in (3.14) we obtain $\lim_{m,n} \rho(fu_m, fu_n, u) = 0$ and f is continuous at u. □

Corollary 3.6 — [4] Let f be a selfmap of an f-orbitally complete D-metric space X satisfying

$$\rho(fx, fy, z) \leq \lambda \rho(x, y, z) \quad \ldots (3.15)$$

for all $x, y, z \in X$, where $0 \leq \lambda < 1$. Then f has a fixed point $u \in X$ and f is continuous at u.

Corollary 3.7 — [5] Let f be a selfmap of an f-orbitally complete D-metric space X satisfying, for some positive integer p.

$$\rho(f^p x, f^p y, z) \leq \lambda \{ \rho(x, y, z), \rho(x, f^p x, z), \rho(y, f^p y, z) \} \quad \ldots (3.16)$$

for all $x, y, z \in X$, where $0 \leq \lambda < 1$. Then f has a unique fixed point $u \in X$, f^p is continuous at u, and f is f-orbitally continuous at u.

PROOF : By Theorem 3.2, f^p has a unique fixed point $u \in X$ and f^p is continuous at u. Also, $fu = f(f^p u) = f^p(fu)$, which shows that fu is also a fixed point of f^p. From the uniqueness of u, $fu = u$. Finally, let $\{u_n\} \subset O_f(x), x \in X$, be any sequence such that $u_n \to u$. Then

$$\lim_{m,n} \rho(fu_{mp}, fu_{np}, u) = \lim_{m,n} \rho(f^p u_{(m-1)p}, f^p u_{(n-1)p}, u) = 0,$$

and so f is f-orbitally continuous at u. □

Corollary 3.8 [4] — Let f be a selfmap of an f-orbitally complete D-metric space X satisfying, for some positive integer p,

$$\rho(f^p x, f^p y, z) \leq \lambda \rho(x, y, z) \qquad \text{... (3.17)}$$

for all $x, y, z \in X$, where $0 \leq \lambda < 1$. Then f has a fixed point $u \in X, f^p$ is continuous at u, and f is f-orbitally continuous at u.

Corollary 3.9 — Let f be a selfmap of an f-orbitally complete D-metric space X satisfying

$$\rho(f^n x, f^n y, z) \leq \lambda_n \max\left\{ \rho(x, y, z), \rho(x, f^n x, z), \rho(y, f^n, z) \right\} \qquad \text{... (3.18)}$$

for all $x, y, z \in X$, where $\{\lambda_n\}$ is a sequence of positive real numbers satisfying $\Sigma\ \lambda_n < \infty$. Then f has a unique fixed point $u \in X$, and f is f-orbitally continuous at u.

PROOF : Since $\Sigma\ \lambda_n < \infty$, there exists a positive integer p and such that $\lambda_p < 1$. The desired conclusion now follows by an application of Corollary 3.7. □

Corollary 3.10 — [6] Let f be a selfmap of an f-orbitally complete D-metric space X satisfying

$$\rho(fx, fy\, z) \leq \alpha \frac{1 + \rho(x, fx, z)}{1 + \rho(x, y, z)} \rho(y, fy, z) + \beta \rho(x, y, z) \qquad \text{... (3.19)}$$

for all $x, y \in X$ with $x \neq fx$ and $z \in X$, where $\alpha, \beta \geq 0$ satisfy $\alpha + \beta < 1$. Then f has a unique fixed point $u \in X$ and f is f-orbitally continuous at u.

PROOF : Setting $y = f\,x$ in (3.19) yields

$$\begin{aligned}\rho(fx, f^2 x, z) &\leq \alpha \frac{1 + \rho(x, fx, z)}{1 + \rho(x, fx, z)} \rho(fx, f^2 x, z) + \beta \rho(x, fx, z) \\ &\leq \lambda \rho(x, fx, z),\end{aligned}$$

where $\lambda = \beta/(1 - \alpha) < 1$.

Without loss of generality we may assume that $x \neq fx$. An application of Corollary 3.1 yields that there is a point $u \in X$ such that $\lim_n\ f^n x = u$. We shall now show that u is a fixed point of f. By (3.20),

$$\rho(x_{n+1}, fu, u) \leq \alpha \frac{1 + \rho(x_n, fx_n, z)}{1 + \rho(x_n, fx_n, z)} \rho(u, fu, u) + \beta \rho(x_n, u, u).$$

Passing to the limit as $n \to \infty$, we obtain $\rho(u, fu, u) \leq \alpha \rho(u, fu, u)$, which implies that $u = f\,u$.

The uniqueness of u follows from (3.19).

To prove the continuity of f at u, let $\{u_n\}$ be a sequence in X such that $u_n \to u$. It will be sufficient to prove that $\lim_{m, n}\ \rho(fu_m, u_n, u) = 0$.

From (3.20),

$$\begin{aligned}\lim_{m, n}\ \rho(fu_m, fu_n, u) &\leq \lim_m \left(\frac{1 + \rho(u_m, fu_n, u)}{1 + \rho(u_m, u, u)} \right) \lim_n\ \rho(u, fu_n, u) \\ &= \alpha\,[1 + \lim_m \rho(u_i fu_m, u)]\ \lim_n \rho(u, fu_n, u).\end{aligned} \qquad \text{... (3.20)}$$

Also from (3.19),

$$\lim_n \rho(u, fu_n, u) = \lim_n \rho(fu, fu_n, u)$$

$$\leq \alpha \frac{1+\rho(u,fu,u)}{1+\rho(u,fu,u)} \lim_n \rho(u_n, fu_n, u) + \beta \lim_n \rho(u, u_n, u)$$

$$= \lim_n (\alpha+\beta)\, \rho(u, fu_n, u),$$

which implies that $\lim \rho(u, fu_n, u) = 0$. Substituting this fact into (3.20) yields $\lim_{m,n} \rho(fu_m, fu_n, u) = 0$. □

Corollary 3.11 — [6] Let f be a selfmap of an f-orbitally complete D-metric space X satisfying for some positive integer p.

$$\rho(f^p x, f^p y, z) \leq \alpha \frac{1+\rho(x, f^p x, z)}{1+\rho(x\, y, z)} \rho(y, f^p y, z) + \beta \rho(x, y, z) \qquad \text{... (3.21)}$$

for all $x, y, z \in X$, where $\alpha, \beta \geq 0$ satisfy $\alpha+\beta < 1$. Then f has a unique fixed point $u \in X, f^p$ is continuous at u and f is f-orbitally continuous at u.

PROOF : Without loss of generality we may assume that $x \neq fx$. The proof is similar to that of Corollary 3.7, and the conclusion follows by an application of Corollary 3.10. □

Remark 3.1 : The conclusion of Corollary 3.9 also remains true if we replace condition (3.19) by

$$\rho(f^n x, f^n, y, z) \leq \alpha_n \frac{1+\rho(x, f^n x, z)}{1+\rho(x, y, z)} \rho(y, f^n, y, z) + \beta_n \rho(x, y, z),$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive real numbers satisfying $\Sigma(\alpha_n + \beta_n) < \infty$.

Unfortunatly not every contractive condition involving a map f on a D-metric space X is a special case of Theorem 3.1. A simple example is the definition of quasi-contraction mappings given in Dhage [4],

$$\rho(fx, fy, z) \leq \lambda \max\{\rho(x, y, z), \rho(x, fx, z), \rho(y, fy, z)\, \rho(x, fy, z), \rho(y, fx, z)\}$$

However, the class of mappings (3.1) is wide enough to include a good number of definitions involving rational expressions. A modest generalization of Theorem 3.1 is the following

***Theorem* 3.2** — *Let f be a selfmap of a D-metric space X. Suppose there exists a point $x \in X$ such that*

(a) $x_n := f^n x$ has a convergent subsequence with limit a, and

(b) f satisfies $\rho(fy, f^2 y, z) \leq \phi(\rho(x, fy, z))$ for all $y \in O_f(x), y \neq fy$ and $z \in X$, where $\phi \in \Phi$.

Then, for this x,

(i) $\lim_n f^n x = u$,

(ii) $\rho(f^m x, f^n x, u) \leq 2 \sum_{j=n}^{\infty} \phi^j(k)$, where $k = \delta(O_f(x))$, and

(iii) a is a fixed point of f iff $G(x) : \rho(x, fx, x)$ is f-orbitally lower semi-continuous at a.

PROOF : From hypothesis (b) it follows that $\{f^n x\}$ is a D-Cauchy sequence in X. Since $\{f^n x\}$ has a convergent subsequence converging to a point $u \in X$, the sequence $\{f^n x\}$ coverges to u and thus (i) is satisfied. The balance of the proof is similar to that of Theorem 3.1. □

In an ordinary metric space, the second author proved the following fixed point theorem, which is a modest generalization of the fixed point theorem proved in Hicks and Rhoades[8].

Theorem R — [12] *Let f be a selfmap of a metric space (X, d). Suppose there exists a point $x \in X$ such that*

(a) $x_n := f^n x$ has a convergent subsequence with limit u, and

(b) $d(fy, f^2 y) \leq \lambda d(y, fy)$ for each $y \in O_f(x)$ and $0 \leq \lambda < 1$.

Then

(i) $\lim f^n x = u$,

(ii) $d(f^n x, u) \leq \dfrac{\lambda^n}{1-\lambda} d(x, fx)$, *and*

(iii) u is a fixed point of f iff $G(x) := d(x, fx)$ is lower semi-continuous at u.

It has been shown in Rhoades[13] that Theorem R includes several interesting fixed point theorems, proved earlier by different authors, as corollaries. We shall now show that, in certain a situation. Our Theorem 3.2 implies Theorem R.

Let X be a metric space with metric d an define two D-metrics ρ_1 and ρ_2 on X by

$$\rho_1(x, y, z) = \max\{(d(x, y), d(y, z), d) z, x)\} \quad \text{... (3.22)}$$

and

$$\rho_2(x, y, z) = d(x, y), + d(y, z) + d(z, x) \quad \text{... (3.23)}$$

for all $x, y, z \in X$.

Then the following result is implicit in Dhage [2], and explicitly mentioned in Dhage and Rhoades [6].

Proposition 3.1 — The metric d amd the D-metrics ρ_1 and ρ_2 are equivalent and generate equivalent topologies on X.

Theorem 3.3 — *Let the D-metric ρ be defined on X by either (3.22) or (3.23). Then Theorem 3.2 implies Theorem R.*

PROOF : Suppose first that the D-metric ρ is defined by (3.22). Assume that hypotheses (a) and (b) of Theorem 3.2 hold. By Proposition 3.1, the topologies generated by d and ρ are equivalent. Therefore the convergence of a subsequence of the sequence $\{x_n\} := \{f^n x\}$, $x \in X$ to the point $u \in X$ with respect to ρ implies the convergence with respect to the metric d. Further, let $\phi(t) = \lambda t, 0 \leq \lambda < 1$. Then hypothesis (b) of Theorem 3.2 becomes

$$\rho(fy, f^2 y, z) \leq \lambda \rho(y, fy, z)$$

for all $y \in \overline{O_f(x)}$ and $z \in \overline{O_f(y)}$, where $0 \leq \lambda < 1$.

Setting $z = fy$ in the above inequality yields

$$\rho(fy, f^2 y, fy) \leq \lambda \rho(y, fy, fy) :$$

i.e.,

$$d(fy, f^2 y) \leq \lambda d(y, fy)$$

for all $y \in \overline{O_f(y)}$, where $0 \leq \lambda < 1$.

Thus conditions (a) and (b) of Theorem R are satisfied. Finally, the function $G(x) = d(x, fx)$ is f-orbitally lower semi-continuous at u.

Similarly, if the D-metric is defined by (2.23), the same conclusion holds. □

4. Second Fixed Point Principle

In this section, by using a measure of noncompactness, we prove some results on the existence of fixed points via general principles for a class of mappings wider than (3.1). The measure of noncompactness of a bounded set A in X is a nonnegative real number $\alpha(A)$ defined by

$$\alpha(A) = \inf \left\{ r >= 0 : A = \bigcup_{i=1}^{n} A_i \text{ and } \delta(A_i) \leq r \text{ for all } i \right\} \qquad \dots (4.1)$$

The measure α of noncompactness enjoys the following properties :

(α_1) $\alpha(A) = \Leftrightarrow A$ is precompact,

(α_2) $\alpha(A) = \alpha(\overline{A})$, where $\overline{A}$ denotes the closure of A,

(α_3) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$, and

(α_4) $\alpha(\{A \cup B\}) = \max\{\alpha(A), \alpha(B)\}$.

Definition 1 — A mapping $f: X \to X$ is called α-condensing if, for any bounded set in A in X, $f(A)$ is bounded and $\alpha(f(A)) < \alpha(A)$ whenever $\alpha(A) > 0$.

We need the following lemma

Lemma 4.1 — Let f be an α-condensing map from an f-orbitally bounded and complete D-metric space X into itself. Then $O_f(x)$ is compact for each $x \in X$.

PROOF : Let $x \in X$ be arbitrary and define $A = O_f(x)$. Then A is bounded and complete. Suppose that A is not compact. Then $\alpha(A) > 0$.

$$A = O_f(x) = \{x, fx, f^2 x \dots\} = \{x\} \cup f(A).$$

Therefore

$$\alpha(A) = \alpha(\{x\} \cup f(A)) \leq \max\{\alpha(\{x\}), \alpha(f(A))\}$$

$$= \alpha(f(A)) < \alpha(A),$$

which is a contradiction. Hence A is precompact and A is compact since X is f-orbitally complete. □

***Theorem* 4.1** — *Let f be an α-condensing and f-orbitally continuous selfmap of an f-orbitally bounded and f-orbitally complete D-metric space X satisfying*

$$\rho(fx, f^2 x, z) \leq \rho(x, fx, z) \qquad \dots (4.2)$$

for all $x, z \in X$ for which $x \neq fx$ and $z \in X$. Then f has a fixed point.

PROOF : Define $A = \overline{O_f(x)}$ for some $x \in X$. From Lemma 4.1, A is compact, Clearly $f: A \to A$. Since f is continuous on A, and A is compact, both sides of inequality (4.2) are bounded in A.

Case I — Suppose that the right hand side of (4.2) is zero for some $x \in X$ and $z \in \overline{O_f(x)}$. Then $u = z = x$ is a fixed point of f.

Case II — Suppose that the right hand side of (4.2) is not zero. Define $T: A^2 \to I\!R^+$ by

$$T(x, z) = \frac{\rho(fx, f^2 x, z)}{\rho(x, fx, z)} \qquad \dots (4.3)$$

for $\quad x \in A$ and $z \in \overline{O_f(x)}$.

Clearly T is well defined since $x \neq fx$ for each $x \in A$. Since f is continuous, T is continuous and, from the compactness of A it follows that T attains its maximum at some point, say $(u, \upsilon) \in A^2$. Call the value c. From (4.1), $0 < c < 1$. From the definition of T,

$$\frac{\rho(fx, f^2 x, z)}{\rho(x, fx, z)} = T(x, z) \leq T(u, \upsilon) = c;$$

i.e. $\quad \rho(fx, f^2 x, z) \leq C\rho(x, fx, x)$

for $x, z \in A$ with $x \neq fx$, where $0 < c < 1$. As A is compact, it is bounded and a complete D-metric space in its own right. The desired conclusion now follows by an application of Corollary 3.1. □

Corollary 4.1 — Let f be an α-condensing and f-orbitally continuous selfmap of an f-orbitally bounded and f-robitally complete D-metric space X satisfying

$$\min\ \{\rho(fx, f^2 x, z), \rho(x, fx, z), \rho(fx^2 y, z)\} < \rho(x, y, z) \qquad \dots (4.4)$$

or $\quad \rho(fx, f^2 x, z) < \rho(x, fx, z)$

for each $x, y, z \in X$ for which $\rho(x, y, z) \neq 0$. Then f has a fixed point.

PROOF : Setting $y = fx$ in (4.4), we obtain

$$\min\ \rho(fx, f^2 x, z), \rho(x, fx, z), \rho(y, fy, z)\} < \rho(x, y z)$$

for each $x \in X, x \neq fx$ and $z \in X$. The conclusion follows by an application of Theorem 4.1. □

Corollary 4.2 — Let f be an α-condensing and f-orbitally continuous selfmap of an f-orbitally bounded and f-orbitally complete D-metric space X satisfying

$$\rho(fx, fy, z) < \max\ \{\rho(x, fx, z), \rho(y, fy, z), \rho(x, y, z)\} \qquad \dots (4.5)$$

for all $x, y, z \in X$ for which the right hand side of (4.5) is not zero. Then f has a unique fixed point.

PROOF : Setting $y = fx$ in (4.5), we obtain

$$\rho(fx, f^2 x, z) < \max\left\{\rho(x, fx, z), \rho(fx, f^2 x, z), \rho(x, fx, z)\right\}$$

for each $x, z \in X$.

A direct application of Theorem 4.1 yields that f has a fixed point u. The uniqueness of u and the continuity of f at u follow from condition (4.5). □

Corollary 4.3 — Let f be an α-condensing and f-orbitally continuous selfmap of an f-orbitally bounded and f-orbitally complete D-metric space X satisfying, for some positive integer p.

$$\rho(f^p x, f^p y, z) < \max\ \{\rho(x, f^p x, z), \rho(y, f^p y, z), \rho(x, y, z)\} \qquad \dots (4.6)$$

for all $x. y \in X$ and $z \in X$ for which the right hand side of (4.6) is not zero. Then f has a unique fixed point.

PROOF : Set $g = f^p$. Then g is an α-condensing and f-orbitally continuous selfmap of X. A direct application of Corollary 4.2 yields that g has a unique fixed point $u \in X$, which is also a unique fixed point of f. □

Corollary 4.4 — [4] Let f be an α-condensing and f-orbitally continuous selfmap of an f-orbitally bounded and f-orbitally complete D-metric space X satisfying

$$\rho(fx, fy, z) < \rho(x, y, z) \qquad \text{... (4.7)}$$

for all $x, y, z \in X$ for each x, y, z such that the right hand side of (4.7) is not zero. Then f has a unique fixed point $u \in X$ and f is continuous at u.

Corollary 4.5 — Let f be an α-condensing and f-orbitally continuous selfmap of an f-orbitally bounded and f-orbitally complete D-metric space X satisfying

$$\rho(fx, fy, z) < \alpha \frac{1 + \rho(x, fx, z)}{1 + \rho(x, y, z)} \rho(y, fy, z) + \beta \rho(x, y, z) \qquad \text{... (4.8)}$$

for all $x, y, z \in X$ for which the right hand side of (4.8) is not zero, where $\alpha, \beta \geq 0$ and satisfy $0 < \alpha + \beta \leq 1$. Then f has a unique fixed point.

PROOF : Setting $y = fx$ in (4.7) yields (4.1). Hence the desired result follows from Theorem 4.1. □

Corollary 4.6 — Let f be an α-condensing and f-orbitally continuous selfmap of an f-orbitally bounded and f-orbitally complete D-metric space X satisfying, for some positive integer p,

$$\rho(f^p, x, f^p y, z) < \alpha \frac{1 + \rho(x, f^p x, z)}{1 + \rho(x, y, z)} \rho(y, f^p y, z) + \beta \rho(x, y, z) \qquad \text{... (4.9)}$$

for all $x, y, z \in X$ for which the right hand side of (4.9) is not zero, where $\alpha, \beta \geq 0$ and satisfy $0 < \alpha + \beta \leq 1$. Then f has a unique fixed point $u \in X$.

PROOF : Set $g = f^p$. The desired conclusion follows by an application of Corollary 4.5.

Corollary 4.7 — Let f be an α-condensing and f-orbitally continuous selfmap of an f-orbitally bounded and f-orbitally complete D-metric space x satisfying

$$\rho(fx, fy, z) < \alpha \frac{\rho(x, fx, z)\, \rho(y, fy, z)}{\rho(x, y, z)} + \beta \rho(x, y, z) \qquad \text{... (4.10)}$$

for all $x, y, z \in X$ for which $\rho(x, y, z) \neq 0$, where $\alpha, \beta \geq 0$ satisfy $\alpha + \beta < 1$. Then f has a unique fixed point a and f is continuous at u.

PROOF : Setting $y = f\,x$ in inequality (4.10) reduces it to (4.1). Hence the existence of a fixed point u of f follows from Theorem 4.1. The uniqueness of u and the continuity of f follow from condition (4.10). □

Corollary 4.8 — Let f be an α-condensing and f-orbitally complete D-metric space x satisfying, for some positive integer p.

$$\rho(f^p x, f^p y, z) < \alpha \frac{\rho(x, f^p x, z)\, \rho(y, f^p y, z)}{\rho(x, y, z)} + \beta \rho(x, y, z) \qquad \text{... (4.11)}$$

for all $x, y, z \in X$ for which $\rho(x, y, z) \neq 0$, where $\alpha, \beta \geq 0$ satisfy $\alpha + \beta < 1$. Then f has a unique fixed point u.

Again we mention that not every contractive mapping on a metric space is a special case of Theorme 4.1. For example, a map $f : X \to X$ satisfying the condition

$$\rho(fx, fy, z) < \max\ \{\rho(x, y, z), \rho(x, fx, z), \rho(y, fy, z),\ \rho(x, fy, z), \rho(y, fx, z)\} \quad \text{... (4.12)}$$

is not contained in the class of mappings given by inequality (4,.1) of Theorem 4.1. However, the class of mappings satisfying (4.1) includes a good number of fixed point mappings in *D*-metric spaces.

A modest generalization of Theorem 4.1 is

***Theorem* 4.2** — *Let f be a selfmap of a metric space X. Suppose that there exists a point $x \in X$ such that*

(a) $x_n := f^n x$ has a subsequence converging to a point u.

(b) f and f^2 are f-orbitally continuous at u.

(c) $\rho(fy, f^2 y, z < \rho(y, fy, z)$ for all $y \in O_f(x)$ with $y \neq fy$ and $z \in X$.

Then u is a fixed point of f.

PROOF : Set $c_n = \rho(x_n, x_{n+1}, z)$ for each u. Then, for all $n \in I\!N$.

$$c_n > c_{n+1} > c_{n+2} > \ldots . \quad \text{... (4.13)}$$

This shows that $\{c_n\}$ is a decreasing sequence of positive real numbers which is bounded below by 0. Therefore

$$\lim_n\ c_n = c \quad \text{... (4.14)}$$

exists.

Suppose that a subsequence $\{x_{n(i)}\} = \{f^{n(i)} x\}$ of the sequence $\{x_n\} = \{f^n x\}$ converges to u.

Then we have

$$\lim_n\ f^{n(i)+1} x = \lim_n\ f f^{n(i)} x = fu.$$

$$\lim_n\ f^{n(i)+2} x = \lim_n\ f^2 f^{n(i)} x = f^2 u.$$

and

$$\lim_n\ \rho(f^{n(i)} x, f^{n(i)+1} x, u) = \rho(u, fu, u)$$

$$\lim_n\ \rho(f^{n(i)+1} x, f^{n(i)+2} x, u) = \rho(fu, f^2 u, u).$$

In view of (4.13) it follows that

$$\rho(fu, f^2 u, u) = \rho(u, fu, u).$$

which is a contradiction of (c), and hence $f\ u = u$. □

For an ordinary metric space (X, d), Sehie Park[9] proved the following fixed point theorem using a general principle.

***Theorem P*[9]** — *Let f be a selfmap of a metric space (X, d). Suppose that there exists a point $x \in X$ satisfying*

(a) $O_f(x)$ has a subsequence converging to the point u,

(b) f is f-orbitally continuous at u and $f\ u$, and

(c) f satisfies $d(dy, fz) < d(y, z)$ for all $y, z \in X$,

$$z = fy \in O_f(x), x \neq y.$$

Then u is a fixed point of f.

Some applications of Theorem P appear in [9] and [11]. We shall now show that the conclusion of Theorem P follows from out Theorem 4.2.

***Theorem* 4.3** — *Let the D-metric ρ be defined on X by either (3.22) or (3.23). Then Theorem 4.2 implies Theorem P.*

PROOF : Assume that conditions (*a*) - (*c*) of Theorem 4.2 hold. Suppose first that the *D*-metric ρ is defined by (3.22). Since the metric d and the *D*-metric ρ, by Proposition 3.1. induce equivalent topologies on X, the convergence of the subsequence of the sequence $x_n := f^n x, x \in X$, with respect to ρ, at the point $u \in X$ implies the convergence, with respect to the metric d on X. Again, f-orbital continuity of f and f^2 with respect to ρ implies the f-orbital continuity of f with respect to d on X. Thus the conditions (*a*) and (*b*) of Theorem *P* are satisfied. We now prove (*c*). By hypothesis (*c*) of Theorem 4.2, one has

$$\rho(fy, f^2 y, z) < \rho(y, fy, z)$$

for all $y \in O_f(x), y \neq fy$ and $z \in X$.

Setting $z = f\,y$ in (3.17), and using the above inequality yields

$$d(d\,y, f^2 y) = \rho(fy, f^2 y, fy) < \rho(y, fy, y) = d(fy, fy)$$

for all $y \in \overline{O_f(y)}\; y \neq fy$. Thus condition (*c*) of Theorem P is satisfied and hence the conclusion follows.

Similarly, if the *D*-metric ρ is defined by (3.23), then Theorem 4.2 also implies Theorem *P*.

□

5. CONCLUSION

From the foregoing discussion it is clear that our fixed point principle is quite general and includes more than two dozen fixed point theorems in metric and *D*-metric spaces as special cases. Further, the results of this paper may be extended to a pair of maps, three maps, and four maps in *D*-metric spaces via a general existence principle, with appropriate modifications. Some of the results along this line will be reported elsewhere.

Finally, we end this paper with an open question.

Open Question — Do the converses of Theorem 3.3 and 4.3 hold?

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