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ON THE HANKEL MATRIX OF DERANGEMENT POLYNOMIALS

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In this note, we establish a general representation of Hankel matrix of Derangement polynomials and extend the result of C. Radoux.

Key Words : **Derangement Polynomial; Hankel Matrix; Determinant**

1. INTRODUCTION

The Derangement polynomials are defined by

$$d_n(z) = \sum_{i=0}^{n} (-1)^i \frac{n!}{i!} z^{n-i}.$$

Note that $d_n(1)$ is the number of derangements, i.e. of permutations with no any fixed point, over a set of n elements.

Let

$$\tilde{D}_n(t) = \begin{pmatrix} d_t(z) & d_{t+1}(z) & \dots & d_{t+n}(z) \\ d_{t+1}(z) & d_{t+2}(z) & \dots & d_{t+n+1}(z) \\ \dots & \dots & \dots & \dots \\ d_{t+n}(z) & d_{t+n+1}(z) & \dots & d_{t+2n}(z) \end{pmatrix}_{(n+1)\times(n+1)}$$

Then C. Radoux[1] obtained that

$$\det \tilde{D}_n(0) = \left(\prod_{k=0}^{n} k! \right)^2 z^{n(n+1)}. \qquad \dots (1)$$

In this note, we establish a general representation of Hankel matrix of Derangement polynomials. From it we obtain that

$$\det \tilde{D}_n(1) = c_n \left(\prod_{k=0}^{n} k! \right)^2 z^{n(n+1)},$$

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$$\det \tilde{D}_n(2) = c_n \left(\prod_{k=0}^{n} k\,! \right)^2 z^{n(n+1)} \left(2z^2 - 2z + 1 + \sum_{k=0}^{n-1} ((n+1)_k)^2 z^{2k} c_{n-k}^2 \right).$$

where $c_n = ((2n+1)z - 1)c_{n-1} - n^2 z^2 c_{n-2}$ and $(n+1)_k = (n+1)n(n-1)\ldots(n-k+2)$.

2. The Results and Their Proofs

At first we introduce the polynomials $s_{n,k}(z) = \frac{1}{k\,!^2} \frac{d^k}{dz^k} d_n(z)$. Obviously, $s_{n,0}(z) = d_n(z)$ and $s_{n,n}(z) = 1$.

Using induction C. Radoux[2] obtained that

$$s_{m+n,0}(z) = \sum_{k=0}^{\min(m,n)} k\,!^2 z^{2k} s_{m,k}(z)\, s_{n,k}(z). \qquad \ldots (2)$$

Now we give the following Lemma which will be used later.

Lemma —

$$s_{n,k}(z) = s_{n-1,k-1}(z) + ((2k+1)z-1)\, s_{n-1,k}(z) + (k+1)^2 z^2 s_{n-1,k+1}(z) \quad (n \geq 1). \qquad \ldots (3)$$

Proof : Noting that

$$s_{n,k}(z) = \frac{1}{k\,!^2} \frac{d^k}{dz^k} d_n(z) = \frac{n\,!}{k\,!^2} \sum_{i=0}^{n-k} (-1)^i \frac{(n-i)_k}{i\,!} z^{n-k-i},$$

we have

$$s_{n-1,k-1}(z) + ((2k+1)z-1)\, s_{n-1,k}(z) + (k+1)^2 z^2 s_{n-1,k+1}(z)$$

$$= \frac{(n-1)\,!}{(k-1)\,!^2} \sum_{i=0}^{n-k} (-1)^i \frac{(n-1-i)_{k-1}}{i\,!} z^{n-k-i}$$

$$+ ((2k+1)z-1) \frac{(n-1)\,!}{k\,!^2} \sum_{i=0}^{n-1-k} (-1)^i \,.$$

$$\frac{(n-1-k)_k}{i\,!} z^{n-1-k-i} + (k+1)^2 z^2 \frac{(n-1)\,!}{(k+1)\,!^2} \sum_{i=0}^{n-k-2}$$

$$(-1)^i \frac{(n-1-i)_{k+1}}{i\,!} z^{n-k-2-i}$$

$$= (-1)^{n-k} \left(\frac{(n-1)\,!}{(k-1)\,!^2} \frac{(k-1)\,!}{(n-k)\,!} + \frac{(n-1)\,!}{k\,!^2} \frac{k\,!}{(n-k-1)\,!} \right)$$

$$+ (-1)^{n-k-1} \left(\frac{(n-1)\,!}{(k-1)\,!^2} \cdot \frac{(k)_{k-1}}{(n-k-1)\,!} + \frac{(n-1)\,!\,(2k+1)}{k\,!^2} \right.$$

$$\left.\frac{k\,!}{(n-1-k)\,!}+\frac{(n-1)\,!}{k\,!^2}\frac{(k+1)_k}{(n-k-2)\,!}\right)z$$

$$+\left(\frac{(n-1)\,!}{(k-1)\,!^2}(n-1)_{k-1}+\frac{(n-1)\,!\,(2k+1)}{k\,!^2}(n-1)_k+\frac{(n-1)\,!}{k\,!^2}(n-1)_{k+1}\right)z^{n-k}$$

$$+\sum_{i=1}^{n-k-2}\left\{\frac{(n-1)\,!}{(k-1)\,!^2}(-1)^i\frac{(n-i-1)_{k-1}}{i\,!}+\frac{(n-1)\,!\,(2k+1)}{k\,!^2}(-1)^i\frac{(n-1-i)_k}{i\,!}\right.$$

$$\left.+\frac{(n-1)\,!}{k\,!^2}(-1)^i\frac{(n-i)_k}{(i-1)\,!}+\frac{(n-1)\,!}{k\,!^2}(-1)^i\frac{(n-1-i)_{k+1}}{i\,!}\,z^{n-k-i}\right.$$

(Use identity $n\,(n-i)_k=k^2\,(n-1-i)_{k-1}+(2k+1)\;(n-1-i)_k\,i\,(n-i)_k+(n-1-i)_{k+1})$

$$=\frac{n\,!}{k\,!^2}\sum_{i=0}^{n-k}(-1)^i\frac{(n-i)_k}{i\,!}z^{n-k-i}$$

$$=s_{n,k}(z).$$ □

Let us introduce $(n+1)\times(n+1)$-matrices $W_n, M_n(t), J_n$ by

$$W_n=\begin{pmatrix} s_{0,0}(z) & 0 & 0 & \dots & 0\\ s_{1,0}(z) & s_{1,1}(z) & 0 & \dots & 0\\ s_{2,0}(z) & s_{2,1}(z) & s_{2,2}(z) & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ s_{n,0}(z) & s_{n,1}(z) & s_{n,2}(z) & \dots & s_{n,n}(z)\end{pmatrix},$$

$$M_n(t)=\begin{pmatrix} s_{t,0}(z) & s_{t,1}(z) & \dots & s_{t,t}(z) & 0 & \dots & 0 & 0\\ s_{t+1,0}(z) & s_{t+1,1}(z) & \dots & s_{t+1,t}(z) & s_{t+1,t+!}(z) & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ s_{n,0}(z) & s_{n,1}(z) & \dots & s_{n,t}(z) & s_{n,t+1}(z) & \dots & s_{n,n-1}(z) & s_{n,n}(z)\\ s_{n+1,0}(z) & s_{n+1,1}(z) & \dots & s_{n+1,t}(z) & s_{n+1,t+1}(z) & \dots & s_{n+1,n-1}(z) & s_{n+1,n}(z)\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ s_{t+n,0}(z) & s_{t+n,1}(z) & \dots & s_{t=n,t}(z) & s_{t+n,t+1}(z) & \dots & s_{t=n,n-1}(z) & s_{t+n,n}(z)\end{pmatrix}$$

and

$$J_n=\begin{pmatrix} a_0 & 1 & 0 & \dots & 0 & 0\\ b_0 & a_1 & 1 & \dots & 0 & 0\\ 0 & b_1 & a_2 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & a_{n-1} & 1\\ 0 & 0 & 0 & \dots & b_{n-1} & a_n\end{pmatrix}$$

respectively, where $a_k=(2k+1)\,z-1$ and $b_k=(k+1)^2\,z^2$.

It is easy to obtain that

$$M_n(0)=W_n,\; c_n=((2n+1)\,z-1)\,c_{n-1}-n^2\,z^2\,c_{n-2}, \qquad \dots (4)$$

where $\det J_n=c_n$, that will be needed later.

By $d_{m+n}(z) = s_{m+n,0}(z) = \sum_{k=0}^{\min(m,n)} k!^2 z^{2k} s_{m,k}(z) s_{n,k}(z)$, we get

$$D_n(t) = M_n(t) \ \text{diag} \ \{1, z^2, 2!^2 z^4, ..., n!^2 z^{2n}\} \ W_n^T. \quad ... (5)$$

Suppose $\upsilon_{n+1}(t) = (0, ..., 0, s_{n+1,n+1}(z), s_{n+2,n+1}(z), \ldots, s_{t+n-1,n+1}(z), s_{t=n,n+1}(z))$ $(t \geq 1)$ be an $n+1$-dimensional row vector, where $s_{n+1,n+1}(z)$ occurs in the $(n+2-t)$th place, and let

$$R_n(t) = (0_{(n+1)\times n}, ((\upsilon_{n+1}(t))^T)_{(n+1)\times(n+1)}.$$

It follows from (3) that

$$M_n(t+1) = M_n(t) J_n + (n+1)^2 z^2 R_n(t). \quad ... (6)$$

By repeated application of (6) the following identity follows.

$$M_n(t) = W_n J_n^t + (n+1)^2 z^2 \sum_{k=1}^{t-1} R_n(k) J_n^{t-1-k}. \quad ... (7)$$

Combining (5) and (7) we have

Theorem —

$$D_n(t) = \left(W_n J_n^t + (n+1)^2 z^2 \sum_{k=1}^{t-1} R_n(k) J_n^{t-1-k} \right) \text{diag} \ \{1, z^2, 2!^2 z^4, ..., n!^2 z^{2n}\} \ W_n^T. \quad ... (8)$$

Corollary 1 — $\tilde{D}_n(0) = W_n \, diag \ \{1, z^2, 2!^2 z^4, ..., n!^2 z^{2n}\} \ W_n^T$,

$$\tilde{D}_n(1) = W_n J_n \, diag \ \{1, z^2, 2!^2 z^4, ..., n!^2 z^{2n}\} \ W_n^T;$$

$$\det D_n(0) = \left(\prod_{k=0}^{n} k! \right)^2 z^{n(n+1)},$$

$$\det D_n(1) = c_n \left(\sum_{k=0}^{n} k! \right)^2 z^{n(n+1)}.$$

Corollary 2 — $D_n(2) = (W_n J_n^2 + (n+1)^2 z^2 R_n(1)) \ \text{diag} \ \{1, z^2, 2!^2 z^4, ..., n!^2 z^{2n}\} \ W_n^T$;

$$\det D_n(2) = \left(\prod_{k=0}^{n} k! \right)^2 z^{n(n+1)} \left(2z^2 - 2z + 1 + \sum_{k=0}^{n-1} ((n+1)_k)^2 z^{2k} c_{n-k}^2 \right).$$

PROOF : The first formula is true by taking $t = 2$ in Theorem. Now we prove the second formula. Noting that $W_n J_n^2 = M_n(2) - (n+1)^2 z^2 R_n(1)$, we have

$$\det J_n^2 = \det (M_n(2) - (n+1)^2 z^2 R_n(1)).$$

Let $T_n = \det M_n(2)$, since $\det J_n = c_n$, we get

$$T_n - (n+1)^2 z^2 T_{n-1} = c_n^2.$$

Therefore,

$$T_n = c_n^2 + (n+1)^2 z^2 T_{n-1}$$

$$= \sum_{k=0}^{n} ((n+1)_k)^2 z^{2k} c_{n-k}^2 + 2z^2 - 2z + 1.$$

From (5) we have

$$\det \tilde{D}_n(2) = \det M_n(2) \left(\prod_{k=0}^{n} k! \right)^2 z^{n(n+1)}$$

$$= \left(\prod_{k=0}^{n} k! \right)^2 z^{n(n+1)} \left(2z^2 - 2z + 1 + \sum_{k=0}^{n-1} ((n+1)_k)^2 z^{2k} c_{n-k}^2 \right). \qquad \square$$

REFERENCES

1. C. Radoux, *Europ J. Combinatorics*, **12** (1991), 327-29.
2. C. Radoux, *J. Computational and Applied Math.*, **115** (2000), 471-77.
3. Zhizheng Zhang, *Indian J. Pure and appl. Math.*, **32(11)** (2001), 1615-19.