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ON THE PERIODICITY OF TWO RATIONAL RECURSIVE SEQUENCES*

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In this paper, we study the rational recursive sequence:

$$x_{n+1} = \frac{a_n}{x_{n-1}} \text{ for } n = 0, 1, 2, \ldots$$

and

$$x_{n+1} = \frac{b_n x_n}{x_{n-1}} \text{ for } n = 0, 1, 2, \ldots,$$

and obtain some interesting results on periodic cycles about the above two difference equations.

Key Words : Rational Recursive Sequence; Difference Equations; Periodic Cycles; Period

1. INTRODUCTION

In the monograph of V L Kocic and G Ladas[1], they give a Research Project 5.2.3 (see [1, p 141])

Assume that the sequences $\{a_n\}$ and $\{b_n\}$ are periodic. Investigate the behaviour of solutions of the rational recursive sequence

$$x_{n+1} = \frac{a_n + b_n x_n}{x_{n-1}} \text{ for } n = 0, 1, 2, \ldots$$

To this end, we consider the following two difference equations :

$$x_{n+1} = \frac{a_n}{x_{n-1}} \text{ for } n = 0, 1, 2, \ldots \quad \ldots (1)$$

and

$$x_{n+1} = \frac{b_n x_n}{x_{n-1}} \text{ for } n = 0, 1, 2, \ldots \quad \ldots (2)$$

where $\{a_n\}$ and $\{b_n\}$ are both periodic with period p, p a positive integer.

In this paper, we investigate the periodic cycles of eqs. (1) and (2) in Sections 2 and 3, respectively, and obtain some fascinating properties.

2 MAIN RESULTS ABOUT EQ. (1)

Lemma 2.1 — Assume that $a_i > 0$ for $i = 1, \ldots, p$. Then the following statements are all true :

(*a*) If $p = 1$, then every positive solution of eq. (1) is periodic with period 4;

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(*b*) If $p = 2$, then every positive solution of eq. (1) is periodic with period 4;

(*c*) If $p = 3$, then every positive solution of eq. (1) is periodic with period 12;

(*d*) If $p = 4$, then every positive solution of eq. (1) is not periodic.

PROOF : (*a*) From 1, we have

$$x_{n+1} = \frac{a}{x_{n-1}} = \frac{a}{\dfrac{a}{x_{n-3}}} = x_{n-3} \text{ for } n = 2, 3, \ldots$$

Hence, the proof of (*a*) is complete.

(*b*) Without loss of generality, we assume that

$$x_{n+1} = \frac{a_1}{x_{n-1}}, x_{n+2} = \frac{a_2}{x_n} \text{ and } x_{n+3} = \frac{a_1}{x_{n+1}} = x_{n-1}.$$

Then, we obtain

$$x_{n+4} = \frac{a_2}{x_{n+2}} = x_n \text{ for } n = 1, 2, \ldots .$$

Thus, the proof of (*b*) is complete.

(*c*) Without loss of generality, we assume that

$$x_{n+1} = \frac{a_1}{x_{n-1}}, x_{n+2} = \frac{a_2}{x_n}, x_{n+3} = \frac{a_3}{x_{n+1}} = \frac{a_3 x_{n-1}}{a_1} \text{ and } x_{n+4} = \frac{a_1}{x_{n+2}} = \frac{a_1 x_n}{a_2}.$$

By induction, we get

$$x_{n+5} = \frac{a_2}{x_{n+3}} = \frac{a_1 a_2}{a_3 x_{n-1}}, x_{n+6} = \frac{a_3}{x_{n+4}} = \frac{a_2 a_3}{a_1 x_n}, x_{n+7} = \frac{a_1}{x_{n+5}} = \frac{a_3 x_{n-1}}{a_2},$$

$$x_{n+8} = \frac{a_2}{x_{n+6}} = \frac{a_1 x_n}{a_3}, x_{n+9} = \frac{a_3}{x_{n+7}} = \frac{a_2}{x_{n-1}}, x_{n+10} = \frac{a_1}{x_{n+8}} = \frac{a_3}{x_n},$$

$$x_{n+11} = \frac{a_2}{x_{n+9}} = x_{n-1} \text{ and } x_{n+12} = \frac{a_3}{x_{n+10}} = x_n \text{ for } n = 1, 2, \ldots .$$

Hence, the proof of (*c*) is complete.

(*d*) Without loss of generality, we assume that

$$x_{n+1} = \frac{a_1}{x_{n-1}}, x_{n+2} = \frac{a_2}{x_n}, x_{n+3} = \frac{a_3}{x_{n+1}} = \frac{a_3 x_{n-1}}{a_1}, x_{n+4} = \frac{a_4}{x_{n+2}} = \frac{a_4 x_n}{a_2}$$

and

$$x_{n+5} = \frac{a_1}{x_{n+3}} = \frac{a_1^2}{a_3 x_{n-1}}.$$

By induction, we have

$$x_{n+6} = \frac{a_2}{x_{n+4}} = \frac{a_2^2}{a_4 x_n}, x_{n+7} = \frac{a_3}{x_{n+5}} = \left(\frac{a_3}{a_1}\right)^2 x_{n-1}, x_{n+8} = \frac{a_4}{x_{n+6}} = \left(\frac{a_4}{a_2}\right)^2 x_n, \ldots .$$

It is easy to see that if either $\frac{a_3}{a_1} \neq 1$ or $\frac{a_4}{a_2} \neq 1$, then every positive solution of eq. (1) is not periodic.

On the other hand, if

$$\frac{a_3}{a_1} = 1 \text{ and } \frac{a_4}{a_2} = 1,$$

then we have $a_1 = a_3$ and $a_2 = a_4$. Therefore, $\{a_n\}$ is periodic with period 2. This is a contradiction.

Thus, the proof is complete.

Lemma 2.2 — Assume that $a_i > 0$ for $i = 1, ..., p$ and $p > 4$. Let k be a positive integer. Then the following statements are all true :

(*a*) If $p = 4k + 1$, then every positive solution of eq. (1) is periodic with period $4p$.

(*b*) If $p = 4k + 2$, then every positive solution of eq. (1) is periodic with period $2p$.

(*c*) If $p = 4k + 3$, then every positive solution of eq. (1) is periodic with period $4p$.

(*d*) If $p = 4k + 4$, then every positive solution of eq. (1) is periodic with period p for

$$\frac{a_1 a_5 \dots a_{p-3}}{a_3 a_7 \dots a_{p-1}} = \frac{a_2 a_6 \dots a_{p-2}}{a_4 a_8 \dots a_p} = 1$$

and is not periodic for

$$\text{either } \frac{a_1 a_5 \dots a_{p-3}}{a_3 a_7 \dots a_{p-1}} \neq 1 \text{ or } \frac{a_2 a_6 \dots a_{p-2}}{a_4 a_8 \dots a_p} \neq 1.$$

PROOF : (*a*) Without loss of generality, we assume that

$$x_{n+1} = \frac{a_1}{x_{n-1}},\ x_{n+2} = \frac{a_2}{x_n},\ x_{n+3} = \frac{a_3}{x_{n+1}} = \frac{a_3 x_{n-1}}{a_1},$$

$$x_{n+4} = \frac{a_4}{x_{n+2}} = \frac{a_4 x_n}{a_2} \qquad \dots (3)$$

and

$$x_{n+5} = \frac{a_5}{x_{n+3}} = \frac{a_1 a_5}{a_3 x_{n-1}}. \qquad \dots (4)$$

Now, we claim that for $q = 4l + 1$, with l a positive integer,

$$x_{n+q-1} = \frac{a_4 a_8 \dots a_{q-1} x_n}{a_2 a_6 \dots a_{q-3}} \qquad \dots (5)$$

and

$$x_{n+q} = \frac{a_1 a_5 \dots a_q}{a_3 a_7 \dots a_{q-2} x_{n-1}}. \qquad \dots (6)$$

In fact, it is obvious that (5) and (6) hold for $l = 1$. We assume that (5) and (6) hold for $l = l'$, i.e.,

$$x_{n+q'-1} = \frac{a_4 a_8 \dots a_{q'-1} x_n}{a_2 a_6 \dots a_{q'-3}} \qquad \dots (7)$$

and

$$x_{n+q'} = \frac{a_1 a_5 \dots a_{q'}}{a_3 a_7 \dots a_{q'-2} x_{n-1}}, \qquad \dots (8)$$

where $q' = 4l' + 1$.

From (1), (7) and (8), we get

$$x_{n+q'+1} = \frac{a_{q'+1}}{x_{n+q'-1}} = \frac{a_2 a_6 \ldots a_{q'-3} a_{q'+1}}{a_4 a_8 \ldots a_{q'-1} x_n},$$

$$x_{n+q'+2} = \frac{a_{q'+2}}{x_{n+q'}} = \frac{a_3 a_7 \ldots a_{q'-2} a_{q'+2} x_{n-1}}{a_1 a_5 \ldots a_{q'}},$$

$$x_{n+q'+3} = \frac{a_{q'+3}}{x_{n+q'+1}} = \frac{a_4 a_8 \ldots a_{q'-1} a_{q'+3} x_n}{a_2 a_6 \ldots a_{q'-3} a_{q'+1}}$$

and

$$x_{n+q'+4} = \frac{a_{q'+4}}{x_{n+q'+2}} = \frac{a_1 a_5 \ldots a_{q'} a_{q'+4}}{a_3 a_7 \ldots a_{q'-2} a_{q'+2} x_{n-1}}.$$

So (5) and (6) hold for $l = l' + 1$. By induction, (5) and (6) hold for all $q = 4l + 1$, where l is a positive integer.

Observe that (5) and (6) hold if we replace n by $n + Np$, where N is a nonnegative integer and p is as defined in Statement (a).

Set $q = p$ and replace n by $n + p$ in (5) and (6). Then we have

$$\begin{aligned} x_{n+2p-1} &= \frac{a_4 a_8 \ldots a_{p-1} x_{n+p}}{a_2 a_6 \ldots a_{p-3}} \\ &= \frac{a_4 a_8 \ldots a_{p-1}}{a_2 a_6 \ldots a_{p-3}} \cdot \frac{a_1 a_5 \ldots a_p}{a_3 a_7 \ldots a_{p-2} x_{n-1}} \\ &= \frac{a_1 a_4 a_5 \ldots a_{p-1} a_p}{a_2 a_3 a_6 a_7 \ldots a_{p-3} a_{p-2} x_{n-1}} \end{aligned} \quad \ldots (9)$$

and

$$\begin{aligned} x_{n+2p} &= \frac{a_1 a_5 \ldots a_p}{a_3 a_7 \ldots a_{p-2} x_{n+p-1}} \\ &= \frac{a_1 a_5 \ldots a_p}{a_3 a_7 \ldots a_{p-2}} \cdot \frac{a_2 a_6 \ldots a_{p-3}}{a_4 a_8 \ldots a_{p-1} x_n} \\ &= \frac{a_1 a_2 a_5 a_6 \ldots a_{p-3} a_p}{a_3 a_4 a_7 a_8 \ldots a_{p-2} a_{p-1} x_n}, \end{aligned} \quad \ldots (10)$$

respectively.

Observe that (9) and (10) hold if we replace n by $n + Np$, where N is a nonnegative integer. Replace n by $n + 2p$ in (9) and (10). Then we have

$$x_{n+4p-1} = \frac{a_1 a_4 a_5 \ldots a_{p-1} a_p}{a_2 a_3 a_6 a_7 \ldots a_{p-3} a_{p-2} x_{n+2p-1}} = x_{n-1}$$

and

$$x_{n+4p} = \frac{a_1 a_2 a_5 a_6 \ldots a_{p-3} a_p}{a_3 a_4 a_7 a_8 \ldots a_{p-2} a_{p-1} x_{n+2p}} = x_n.$$

Furthermore, we have

$$x_{n+4p+1} = \frac{a_1}{x_{n+4p-1}} = \frac{a_1}{x_{n-1}} = x_{n+1},$$

$$x_{n+4p+2} = \frac{a_2}{x_{n+4p}} = \frac{a_2}{x_n} = x_{n+2},$$

$$\ldots,$$

$$x_{n+8p-1} = x_{n-1},\ x_{n+8p} = x_n.$$

Thus, the proof of (*a*) is complete.

(*b*) Without loss of generality, we assume that

$$x_{n+1} = \frac{a_1}{x_{n-1}},\ x_{n+2} = \frac{a_2}{x_n},\ x_{n+3} = \frac{a_3}{x_{n+1}} = \frac{a_3 x_{n-1}}{a_1},\ x_{n+4} = \frac{a_4}{x_{n+2}} = \frac{a_4 x_n}{a_2},$$

$$x_{n+5} = \frac{a_5}{x_{n+3}} = \frac{a_1 a_5}{a_3 x_{n-1}} \qquad \text{... (11)}$$

and

$$x_{n+6} = \frac{a_6}{x_{n+4}} = \frac{a_2 a_6}{a_4 x_n} \qquad \text{... (12)}$$

Now, we claim that for $q = 4l + 2$, with l is a positive integer,

$$x_{n+q-1} = \frac{a_1 a_5 \ldots a_{q-1}}{a_3 a_7 \ldots a_{q-3} x_{n-1}} \qquad \text{... (13)}$$

and

$$x_{n+q} = \frac{a_2 a_6 \ldots a_q}{a_4 a_8 \ldots a_{q-2}} x_n. \qquad \text{... (14)}$$

In fact, it is obvious that (13) and (14) hold for $l = 1$. We assume that (13) and (14) hold for $l = l'$, i.e.,

$$x_{n+q'-1} = \frac{a_1 a_5 \ldots a_{q'-1}}{a_3 a_7 \ldots a_{q'-3} x_{n-1}} \qquad \text{... (15)}$$

and

$$x_{n+q'} = \frac{a_2 a_6 \ldots a_{q'}}{a_4 a_8 \ldots a_{q'-2} x_n}, \qquad \text{... (16)}$$

where $q' = 4l' + 2$.

By (1), (15) and (16), we have

$$x_{n+q'+1} = \frac{a_{q'+1}}{x_{n+q'-1}} = \frac{a_3 a_7 \ldots a_{q'-3} a_{q'+1} x_{n-1}}{a_1 a_5 \ldots a_{q'-1}},$$

$$x_{n+q'+2} = \frac{a_{q'+2}}{x_{n+q'}} = \frac{a_4 a_8 \ldots a_{q'-2} a_{q'+2} x_n}{a_2 a_6 \ldots a_{q'-4} a_{q'}},$$

$$x_{n+q'+3} = \frac{a_{q'+2}}{x_{n+q'+1}} = \frac{a_1 a_5 \ldots a_{q'-1} a_{q'+3}}{a_3 a_7 \ldots a_{q'-3} a_{q'+1} x_{n-1}}$$

and

$$x_{n+q'+4} = \frac{a_{q'+4}}{x_{n+q'+2}} = \frac{a_2 a_6 \ldots a_{q'} a_{q'+4}}{a_4 a_8 \ldots a_{q'-2} a_{q'+2} x_n}.$$

So, (13) and (14) hold for $l = l' + 1$. By induction, (13) and (14) hold for all $q = 4l + 2$, where l is a positive integer.

Observe that (13) and (14) hold if we replace n by $n + Np$, where N is a nonnegative integer and p is as defined in Statement (b).

Set $q = p$ and replace n by $n + p$ in (13) and (14). Then we have

$$x_{n+2p-1} = \frac{a_1 a_5 \ldots a_{p-1}}{a_3 a_7 \ldots a_{p-3} x_{n+p-1}} = x_{n-1}$$

and

$$x_{n+2p} = \frac{a_2 a_6 \ldots a_p}{a_4 a_8 \ldots a_{p-2} x_{n+p}} = x_n.$$

By induction, we have

$$x_{n+2p+1} = \frac{a_1}{x_{n+2p-1}} = \frac{a_1}{x_{n-1}} = x_{n+1},$$

$$x_{n+2p+1} = \frac{a_2}{x_{n+2p}} = \frac{a_2}{x_n} = x_{n+2},$$

$$\ldots,$$

$$x_{n+4p-1} = x_{n-1}, x_{n+4p} = x_n.$$

Thus, the proof of (b) is complete.

(c) Without loss of generality, we assume that

$$x_{n+1} = \frac{a_1}{a_{n-1}}, x_{n+2} = \frac{a_2}{x_n}, x_{n+3} = \frac{a_3}{x_{n+1}} = \frac{a_3 x_{n-1}}{a_1}, x_{n+4} = \frac{a_4}{x_{n+2}} = \frac{a_4 x_n}{a_2},$$

$$x_{n+5} = \frac{a_5}{x_{n+3}} = \frac{a_1 a_5}{a_3 x_{n-1}}, x_{n+6} = \frac{a_6}{x_{n+4}} = \frac{a_2 a_6}{a_4 x_n}$$

and

$$x_{n+7} = \frac{a_7}{x_{n+5}} = \frac{a_3 a_7 x_{n-1}}{a_1 a_5}.$$

Now, we claim that for $q = 4l + 3$, with l is a positive integer,

$$x_{n+q-1} = \frac{a_2 a_6 \ldots a_{q'-1}}{a_4 a_8 \ldots a_{q-3} x_n} \qquad \ldots (17)$$

and

$$x_{n+q} = \frac{a_3 a_7 \ldots a_q}{a_1 a_5 \ldots a_{q-2}} x_{n-1}. \qquad \ldots (18)$$

In fact, it is obvious that (17) and (18) hold for $l = 1$. We assume that (13) and (14) hold for $l = l'$, i.e.,

$$x_{n+q'-1} = \frac{a_2 a_6 \ldots a_{q'-1}}{a_4 a_8 \ldots a_{q'-3} x_n}, \qquad \ldots (19)$$

and

$$x_{n+q'} = \frac{a_3 a_7 \ldots a_{q'} x_{n-1}}{a_1 a_5 \ldots a_{q'-2}}, \qquad \ldots (20)$$

where $q' = 4l' + 3.$

By (1), (19) and (20), we obtain

$$x_{n+q'+1} = \frac{a_{q'+1}}{x_{n+q'-1}} = \frac{a_4 a_8 \ldots a_{q'-3} a_{q'+1} x_n}{a_2 a_6 \ldots a_{q'-5} a_{q'-1}}$$

$$x_{n+q'+2} = \frac{a_{q'+2}}{x_{n+q'}} = \frac{a_1 a_5 \ldots a_{q'-2} a_{q'+2}}{a_3 a_7 \ldots a_{q'-4} a_{q'} x_{n-1}},$$

$$x_{n+q'+3} = \frac{a_{q'+3}}{x_{n+q'+1}} = \frac{a_2 a_6 \ldots a_{q'-1} a_{q'+3}}{a_4 a_8 \ldots a_{q'-3} a_{q'+1} x_n}$$

and
$$x_{n+q'+4} = \frac{a_{q'+4}}{x_{n+q'+2}} = \frac{a_3 a_7 \ldots a_{q'} a_{q'+4} x_{n-1}}{a_1 a_5 \ldots a_{q'-2} a_{q'+2}}.$$

So, (17) and (18) hold for $l = l' + 1$. By induction, we obtain that (17) and (18) hold for all $q = 4l + 3$, where l is a positive integer.

Observe that (17) and (18) hold if we replace n by $n + Np$, where N is a nonnegative integer and p is as defined in Statement (c).

Set $q = p$ and replace n by $n + p$ in (17) and (18). Then we have

$$\begin{aligned} x_{n+2p-1} &= \frac{a_2 a_6 \ldots a_{p-1}}{a_4 a_8 \ldots a_{p-3} x_{n+p}} \\ &= \frac{a_2 a_6 \ldots a_{p-1}}{a_4 a_8 \ldots a_{p-3}} \cdot \frac{a_1 a_5 \ldots a_{p-2}}{a_3 a_7 \ldots a_p x_{n-1}} \\ &= \frac{a_1 a_2 a_5 a_6 \ldots a_{p-2} a_{p-1}}{a_3 a_4 a_7 a_8 \ldots a_{p-4} a_{p-3} a_p x_{n-1}} \end{aligned} \quad \ldots (21)$$

and
$$\begin{aligned} x_{n+2p} &= \frac{a_3 a_7 \ldots a_p x_{n+p-1}}{a_1 a_5 \ldots a_{p-2}} \\ &= \frac{a_3 a_7 \ldots a_p}{a_1 a_5 \ldots a_{p-2}} \cdot \frac{a_2 a_6 \ldots a_{p-1}}{a_4 a_8 \ldots a_{p-3} x_n} \\ &= \frac{a_2 a_3 a_6 a_7 \ldots a_{p-1} a_p}{a_1 a_4 a_5 \ldots a_{p-3} a_{p-2} x_n}, \end{aligned} \quad \ldots (22)$$

respectively.

Observe that (21) and (22) hold if we replace n by $n + Np$, where N is a nonnegative integer. Replace n by $n + 2p$ in (21) and (22). Then we have

$$x_{n+4p-1} = \frac{a_1 a_2 a_4 a_5 \ldots a_{p-2} a_{p-1}}{a_3 a_4 a_7 a_8 \ldots a_{p-4} a_{p-3} a_p x_{n+2p-1}} = x_{n-1}$$

and
$$x_{n+4p} = \frac{a_2 a_3 a_6 a_7 \ldots a_{p-1} a_p}{a_1 a_4 a_5 \ldots a_{p-3} a_{p-2} x_{n+2p}} = x_n.$$

From induction, we get

$$x_{n+4p+1} = \frac{a_1}{x_{n+4p-1}} = \frac{a_1}{x_{n-1}} = x_{n+1}, \; x_{n+4p+2} = \frac{a_2}{x_{n+4p}} = \frac{a_2}{x_n} = x_{n+2},$$

$$\ldots,$$

$$x_{n+8p-1} = x_{n-1} \text{ and } x_{n+8p} = x_n.$$

Hence, the proof of (c) is complete.

(d) Without loss of generality, we assume that

$$x_{n+1}=\frac{a_1}{x_{n-1}},\ x_{n+2}=\frac{a_2}{x_n},\ x_{n+3}=\frac{a_3}{x_{n+1}}=\frac{a_3x_{n-1}}{a_1},\ x_{n+4}=\frac{a_4}{x_{n+2}}=\frac{a_4x_n}{a_2}.$$

$$x_{n+5}=\frac{a_5}{x_{n+3}}=\frac{a_1a_5}{a_3x_{n-1}},\ x_{n+6}=\frac{a_6}{x_{n+4}}=\frac{a_2a_6}{a_4x_n},\ x_{n+7}=\frac{a_7}{x_{n+5}}=\frac{a_3a_7x_{n-1}}{a_1a_5}$$

and $$x_{n+8}=\frac{a_8}{x_{n+6}}=\frac{a_4a_8x_n}{a_2a_6}.$$

Now, we claim that for $q = 4l + 4$, with l a positive integer,

$$x_{n+q-1}=\frac{a_3a_7\ldots a_{q-1}x_{n-1}}{a_1a_5\ldots a_{q-3}} \qquad \text{... (23)}$$

and $$x_{n+q}=\frac{a_4a_8\ldots a_qx_n}{a_2a_6\ldots a_{q-2}}. \qquad \text{... (24)}$$

In fact, it is obvious that (23) and (24) hold for $l = 1$. We assume that (23) and (24) hold for $l=l'$, i.e.,

$$x_{n+q'-1}=\frac{a_3a_7\ldots a_{q'-1}x_{n-1}}{a_1a_5\ldots a_{q'-3}} \qquad \text{... (25)}$$

and $$x_{n+q'}=\frac{a_4a_8\ldots a_{q'}x_n}{a_2a_6\ldots a_{q'-2}}, \qquad \text{... (26)}$$

where $$q'=4l'+4.$$

By (1), (25) and (26), we get

$$x_{n+q'+1}=\frac{a_{q'+1}}{x_{n+q'-1}}=\frac{a_1a_5\ldots a_{q'-3}a_{q'+1}}{a_3a_7\ldots a_{q'-5}a_{q'-1}x_{n-1}},$$

$$x_{n+q'+2}=\frac{a_{q'+2}}{x_{n+q'}}=\frac{a_2a_6\ldots a_{q'-2}a_{q'+2}}{a_4a_8\ldots a_{q'-4}a_{q'}x_n},$$

$$x_{n+q'+3}=\frac{a_{q'+3}}{x_{n+q'+1}}=\frac{a_3a_7\ldots a_{q'-1}a_{q'+3}x_{n-1}}{a_1a_5\ldots a_{q'-3}a_{q'+1}}=\frac{a_3a_7\ldots a_{q'-1}a_{q-1}x_{n-1}}{a_1a_5\ldots a_{q'-3}a_{q-3}}$$

and $$x_{n+q'+4}=\frac{a_{q'+4}}{x_{n+q'+2}}=\frac{a_4a_8\ldots a_{q'}a_{q'+4}x_n}{a_2a_6\ldots a_{q'-2}a_{q'+2}}=\frac{a_4a_8\ldots a_{q'}a_qx_n}{a_2a_6\ldots a_{q'-2}a_{q-2}}.$$

Therefore, (23) and (24) hold for $l=l'+1$. By induction, (23) and (24) hold for all $q = 4l + 4$, where l is a positive integer.

Observe that (23) and(24) hold if we replace n by $n + Np$, where N is a nonnegative integer and p is as defined in Statement (d).

Set $q = p$ and replace n by $n + p$ in (23) and (24). Then we have

$$x_{n+2p-1}=\frac{a_3a_7\ldots a_{p-1}x_{n+p-1}}{a_1a_5\ldots a_{p-3}}=\left(\frac{a_3a_7\ldots a_{p-1}}{a_1a_5\ldots a_{p-3}}\right)^2x_{n-1} \qquad \text{... (27)}$$

and $$x_{n+2p} = \frac{a_4 a_8 \dots a_p x_{n+p}}{a_2 a_6 \dots a_{p-2}} = \left(\frac{a_4 a_8 \dots a_p}{a_2 a_6 \dots a_{p-2}} \right)^2 x_n, \quad \dots (28)$$

respectively.

Similarly, we can also get

$$x_{n+3p-1} = \left(\frac{a_3 a_7 \dots a_{p-1}}{a_1 a_5 \dots a_{p-3}} \right)^3 x_{n-1},$$

$$x_{n+3p} = \left(\frac{a_4 a_8 \dots a_p}{a_2 a_6 \dots a_{p-2}} \right)^3 x_n, \ \dots .$$

Hence, if

$$\frac{a_3 a_7 \dots a_{p-1}}{a_1 a_5 \dots a_{p-3}} = \frac{a_4 a_8 \dots a_p}{a_2 a_6 \dots a_{p-2}} = 1,$$

then we have

$$x_{n+p-1} = x_{n-1}, x_{n+p} = x_n, x_{n+p+1} = \frac{a_1}{x_{n+p-1}} = \frac{a_1}{x_{n-1}} = x_{n+1}, \ \dots,$$

$$x_{n+2p-1} = x_{n-1}, x_{n+2p} = x_n;$$

if either $\frac{a_3 a_7 \dots a_{p-1}}{a_1 a_5 \dots a_{p-3}} \neq 1$ or $\frac{a_4 a_8 \dots a_p}{a_2 a_6 \dots a_{p-2}} \neq 1$,

then every positive solution of eq. (1) is not periodic.

Thus, the whole proof of the lemma is complete.

Upon combining Lemmas 2.1 and 2.2, we can obtain the following theorem.

***Theorem* A** — *Assume that $a_i > 0$ for i = 1, ..., p*. Let k be a positive integer. Then the following statements are all true :

(*a*) If $p \neq 4k$, then every positive solution of eq. (1) is periodic with period m, where m is the least common multiple of 4 and p.

(*b*) If $p = 4$, then every solution of eq. (1) is not periodic.

(*c*) If $p = 4k + 4$, then every positive solution of eq. (1) is periodic with period p for

$$\frac{a_1 a_5 \dots a_{p-3}}{a_3 a_7 \dots a_{p-1}} = \frac{a_2 a_6 \dots a_{p-2}}{a_4 a_8 \dots a_p} = 1;$$

otherwise, then every positive solution of eq. (1) is not periodic.

3. Main Results about Eq. (2)

Lemma 3.1 — Assume that $b_i > 0$ for i = 1, ..., p. Then the following statements are all true :

(*a*) If $p = 1$, then every positive solution of eq. (1) is periodic with period 6.

(*b*) If $p = 2$, then every positive solution of eq. (1) is periodic with period 6.

(*c*) If $p = 3$, then every positive solution of eq. (1) is periodic with period 6.

(*d*) If $p = 4$, then every positive solution of eq. (1) is periodic with period 12.

(*e*) If $p = 5$, then every positive solution of eq. (1) is periodic with period 30.

(*f*) If $p = 6$, then every positive solution of eq. (1) is periodic with period 6 for

$$\frac{b_4 b_5}{b_1 b_2} = \frac{b_5 b_6}{b_2 b_3} = 1;$$

otherwise, every positive solution of eq. (1) is not periodic.

PROOF : (*a*) From (2), we have

$$x_{n+1} = \frac{bx_n}{x_{n-1}},\ x_{n+2} = \frac{b\,x_{n+1}}{x_n} = \frac{b^2}{x_{n-1}},\ x_{n+3} = \frac{bx_{n+2}}{x_{n+1}} = \frac{b^2}{x_n},$$

$$x_{n+4} = \frac{b\,x_{n+3}}{x_{n+2}} = \frac{b\,x_{n-1}}{x_n},\ x_{n+5} = \frac{b\,x_{n+4}}{x_{n+3}} = x_{n-1} \text{ and } x_{n+6} = \frac{b\,x_{n+5}}{x_{n+4}} = x_n.$$

Hence, (*a*) holds.

(*b*) Without loss of generality, we assume that

$$x_{n+1} = \frac{b_1 x_n}{x_{n-1}} \text{ and } x_{n+2} = \frac{b_2 x_{n+1}}{x_n} = \frac{b_1 b_2}{x_{n-1}}.$$

Then
$$x_{n+3} = \frac{b_1 x_{n+2}}{x_{n+1}} = \frac{b_1 b_2}{x_n},\ x_{n+4} = \frac{b_2 x_{n+3}}{x_{n+2}} = \frac{b_2 x_{n-1}}{x_n},$$

$$x_{n+5} = \frac{b_1 x_{n+4}}{x_{n+3}} = x_{n-1} \text{ and } x_{n+6} = \frac{b_2 x_{n+5}}{x_{n+4}} = x_n.$$

Thus, (*b*) holds.

(*c*) Without loss of generality, we assume that

$$x_{n+1} = \frac{b_1 x_n}{x_{n-1}},\ x_{n+2} = \frac{b_2 x_{n+1}}{x_n} = \frac{b_1 b_2}{x_{n-1}} \text{ and } x_{n+3} = \frac{b_3 x_{n+2}}{x_{n+1}} = \frac{b_2 b_3}{x_n}.$$

Then
$$x_{n+4} = \frac{b_1 x_{n+3}}{x_{n+2}} = \frac{b_3 x_{n-1}}{x_n},\ x_{n+5} = \frac{b_2 x_{n+4}}{x_{n+3}} = x_{n-1} \text{ and } x_{n+6} = \frac{b_3 x_{n+5}}{x_{n+4}} = x_n.$$

Therefore, (b) holds;

(*d*) Without of loss of generality, we assume that

$$x_{n+1} = \frac{b_1 x_n}{x_{n-1}},\ x_{n+2} = \frac{b_2 x_{n+1}}{x_n} = \frac{b_1 b_2}{x_{n-1}},\ x_{n+3} = \frac{b_3 x_{n+2}}{x_{n+1}} = \frac{b_2 b_3}{x_n}$$

and
$$x_{n+4} = \frac{b_4 x_{n+3}}{x_{n+2}} = \frac{b_3 b_4 x_{n-1}}{b_1 x_n}.$$

Then
$$x_{n+5} = \frac{b_1 x_{n+4}}{x_{n+3}} = \frac{b_4 x_{n-1}}{b_2},\ x_{n+6} = \frac{b_2 x_{n+5}}{x_{n+4}} = \frac{b_1 x_n}{b_3},\ x_{n+7} = \frac{b_3 x_{n+6}}{x_{n=5}} = \frac{b_1 b_2 x_n}{b_4 x_{n-1}},$$

$$x_{n+8} = \frac{b_4 x_{n+7}}{x_{n+6}} = \frac{b_2 b_3}{b_2},\ x_{n+9} = \frac{b_1 x_{n+8}}{x_{n+7}} = \frac{b_3 b_4}{x_n},\ x_{n+10} = \frac{b_2 x_{n+9}}{x_{n+8}} = \frac{b_4 x_{n-1}}{x_n},$$

$$x_{n+11} = \frac{b_3 x_{n+10}}{x_{n+9}} = x_{n-1} \text{ and } x_{n+12} = \frac{b_4 x_{n+11}}{x_{n+10}} = x_n.$$

Part (*d*) now follows.

(*e*) Without loss of generality, we assume that

$$x_{n+1}=\frac{b_1 x_n}{x_{n-1}},\ x_{n+2}=\frac{b_2 x_{n+1}}{x_n}+\frac{b_1 b_2}{x_{n-1}},\ x_{n+3}=\frac{b_3 x_{n+2}}{x_{n+1}}=\frac{b_2 b_3}{x_n},$$

$$x_{n+4}=\frac{b_4 x_{n+3}}{x_{n+2}}=\frac{b_3 b_4 x_{n-1}}{b_1 x_n} \text{ and } x_{n+5}=\frac{b_5 x_{n+4}}{x_{n+3}}=\frac{b_4 b_5 x_{n-1}}{b_1 b_2}.$$

Then
$$x_{n+6}=\frac{b_1 x_{n+5}}{x_{n+4}}=\frac{b_1 b_5 x_n}{b_2 b_3},\ x_{n+7}=\frac{b_2 x_{n+6}}{x_{n+5}}=\frac{b_1^2 b_2 x_n}{b_3 b_4 x_{n-1}},$$

$$x_{n+8}=\frac{b_3 x_{n+7}}{x_{n+6}}=\frac{b_1 b_2^2 b_3}{b_4 b_5 x_{n-1}},\ x_{n+9}=\frac{b_4 x_{n+8}}{x_{n+7}}=\frac{b_2 b_3^2 b_4}{b_1 b_5 x_n},$$

$$x_{n+10}=\frac{b_5 x_{n+9}}{x_{n+8}}=\frac{b_3 b_4^2 b_5 x_{n-1}}{b_1^2 b_2 x_n},\ x_{n+11}=\frac{b_1 x_{n+10}}{x_{n+9}}=\frac{b_4 b_5^2 x_{n-1}}{b_2^2 b_3},$$

$$x_{n+12}=\frac{b_2 x_{n+11}}{x_{n+10}}=\frac{b_1^2 b_5 x_n}{b_3^2 b_4},\ x_{n+13}=\frac{b_3 x_{n+12}}{x_{n+11}}=\frac{b_1^2 b_2^2 x_n}{b_4^2 b_5 x_{n-1}},$$

$$x_{n+14}=\frac{b_4 x_{n+13}}{x_{n+12}}=\frac{b_2^2 b_3^2}{b_5^2 b_4 x_{n-1}},\ x_{n+15}=\frac{b_5 x_{n+14}}{x_{n+13}}=\frac{b_3^2 b_4^2}{b_1^2 x_n},$$

$$x_{n+16}=\frac{b_1 x_{n+15}}{x_{n+14}}=\frac{b_4^2 b_5^2 x_{n-1}}{b_1 b_2^2 x_n},\ x_{n+17}=\frac{b_2 x_{n+16}}{x_{n+15}}=\frac{b_1 b_5^2 x_{n-1}}{b_2 b_3^2},$$

$$x_{n+18}=\frac{b_3 x_{n+17}}{x_{n+16}}=\frac{b_1^2 b_2 x_n}{b_3 b_4^2},\ x_{n+19}=\frac{b_4 x_{n+18}}{x_{n+17}}=\frac{b_1 b_2^2 b_3 x_n}{b_4 b_5^2 x_{n-1}},$$

$$x_{n+20}=\frac{b_5 x_{n+19}}{x_{n+18}}=\frac{b_2 b_3^2 b_4}{b_1 b_5 x_{n-1}},\ x_{n+21}=\frac{b_1 x_{n+20}}{x_{n+19}}=\frac{b_3 b_4^2 b_5}{b_1 b_2 x_n},$$

$$x_{n+22}=\frac{b_2 n_{n+21}}{x_{n+20}}=\frac{b_4 b_5^2 x_{n-1}}{b_2 b_3 x_n},\ x_{n+23}=\frac{b_3 x_{n+22}}{x_{n+21}}=\frac{b_1 b_5 x_{n-1}}{b_3 b_4},$$

$$x_{n+24}=\frac{b_4 x_{n+23}}{x_{n+22}}=\frac{b_1 b_2 x_n}{b_4 b_5},\ x_{n+25}=\frac{b_5 x_{n+24}}{x_{n+23}}=\frac{b_2 b_3 x_n}{b_5 x_{n-1}},$$

$$x_{n+26}=\frac{b_1 x_{n+25}}{x_{n+24}}=\frac{b_3 b_4}{x_{n-1}},\ x_{n+27}=\frac{b_2 x_{n+26}}{x_{n=25}}=\frac{b_4 b_5}{x_n},$$

$$x_{n+28}=\frac{b_3 x_{n+27}}{x_{n+26}}=\frac{b_5 x_{n-1}}{x_n},\ x_{n=29}=\frac{b_4 x_{n+28}}{x_{n+27}}=x_{n-1}$$

and
$$x_{n+30}=\frac{b_5 x_{n+29}}{x_{n+28}}=x_n.$$

Hence, (e) holds.

(f) Without loss of generality, we assume that

$$x_{n+1}=\frac{b_1 x_n}{x_{n-1}},\ x_{n+2}=\frac{b_2 x_{n+1}}{x_n}=\frac{b_1 b_2}{x_{n-1}},\ x_{n+3}=\frac{b_3 x_{n+2}}{x_{n+1}}=\frac{b_2 b_3}{x_n},$$

$$x_{n+4}=\frac{b_4 x_{n+3}}{x_{n+2}}=\frac{b_3 b_4 x_{n-1}}{b_1 x_n},\ x_{n+5}=\frac{b_5 x_{n+4}}{x_{n+3}}=\frac{b_4 b_5 x_{n-1}}{b_1 b_2}.$$

and $$x_{n+6}=\frac{b_6 x_{n+5}}{x_{n+4}}=\frac{b_5 b_6 x_n}{b_2 b_3}.$$

Then $$x_{n+7}=\frac{b_1 x_{n+6}}{x_{n+5}}=\frac{b_1^2 b_6 x_n}{b_3 b_4 x_{n-1}},\ x_{n+8}=\frac{b_2 x_{n+7}}{x_{n+6}}=\frac{b_1^2 b_2^2}{b_4 b_5 x_{n-1}},$$

$$x_{n+9}=\frac{b_3 x_{n+8}}{x_{n+7}}=\frac{b_2^2 b_3^2}{b_5 b_6 x_n},\ x_{n+10}=\frac{b_4 x_{n+9}}{x_{n+8}}=\frac{b_3^2 b_4^2 x_{n-1}}{b_1^2 b_6 x_n},$$

$$x_{n+11}=\frac{b_5 x_{n+10}}{x_{n+9}}=\left(\frac{b_4 b_5}{b_1 b_2}\right)^2 x_{n-1} \text{ and } x_{n+12}=\left(\frac{b_5 b_6}{b_2 b_3}\right)^2 x_n, \ \ldots.$$

If $$\frac{b_4 b_5}{b_1 b_2}=\frac{b_5 b_6}{b_2 b_3}=1,$$

then $$\frac{b_3 b_4}{b_1 b_6}=1.$$

Hence, $$x_{n+11}=x_{n-1},\ x_{n+12}=x_n,\ x_{n+13}=x_{n+1},\ \ldots$$

On the other hand, if

either $$\frac{b_4 b_5}{b_1 b_2}\neq 1 \text{ or } \frac{b_5 b_6}{b_2 b_3}\neq 1,$$

then it is clear that every solution of eq. (2) is not periodic.

The proof of the lemma is complete.

Lemma 3.2 — Assume that $b_i > 0$ for $i = 1, \ldots, p$. Let k be a positive integer. Then the following statements are all true :

(a) If $p = 6k + 1$, then every positive solution of eq. (2) is periodic with period $6p$

(b) If $p = 6k + 2$, then every positive solution of eq. (2) is periodic with period $3p$

(c) If $p = 6k + 3$, then every positive solution of eq. (2) is periodic with period $2p$

(d) If $p = 6k + 4$, then every positive solution of eq. (2) is periodic with period $3p$

(e) If $p = 6k + 5$, then every positive solution of eq. (2) is periodic with period $6p$

(f) If $p = 6k$, then every positive solution of eq. (2) is periodic with period p for

$$\frac{b_4 b_5 \ldots b_{p-2} b_{p-1}}{b_1 b_2 \ldots b_{p-5} b_{p-4}}=\frac{b_5 b_6 \ldots b_{p-1} b_p}{b_2 b_3 \ldots b_{p-4} b_{p-3}}=1;$$

otherwise, it is not periodic.

PROOF : (a) Without loss of generality, we assume that

$$x_{n+1}=\frac{b_1 x_n}{x_{n-1}},\ x_{n+2}=\frac{b_2 x_{n+1}}{x_n}=\frac{b_1 b_2}{x_{n-1}},\ x_{n+3}=\frac{b_3 x_{n+2}}{x_{n+1}}=\frac{b_2 b_3}{x_n},$$

$$x_{n+4}=\frac{b_4 x_{n+3}}{x_{n+2}}=\frac{b_3 b_4 x_{n-1}}{b_1 x_n},\ x_{n+5}=\frac{b_5 x_{n+4}}{x_{n+3}}$$

$$=\frac{b_4 b_5 x_{n-1}}{b_1 b_2},\ x_{n+6}=\frac{b_6 x_{n=5}}{x_{n+4}}=\frac{b_5 b_6 x_n}{b_2 b_3}$$

and $$x_{n+7}=\frac{b_7 x_{n+6}}{x_{n+5}}=\frac{b_1 b_6 b_7 x_n}{b_3 b_4 x_{n-1}}.$$

Now, we claim that for $q = 6l + 1$, with l is a positive intger,

$$x_{n+q-1}=\frac{b_5 b_6 \dots b_{q-2} b_{q-1} x_n}{b_2 b_3 \dots b_{q-5} b_{q-4}} \qquad \dots (29)$$

and $$x_{n+q}=\frac{b_1 b_6 b_7 \dots b_{q-1} b_q x_n}{b_3 b_4 \dots b_{q-4} b_{q-3} x_{n-1}}. \qquad \dots (30)$$

In fact, it is clear that (29) and (30) hold for $l = 1$. We assume that (29) and (30) hold for $l+l'$, i.e.,

$$x_{n+q'-1}=\frac{b_5 b_6 \dots b_{q'-2} b_{q'-1} x_n}{b_2 b_3 \dots b_{q'-5} b_{q'-4}} \qquad \dots (31)$$

and $$x_{n+q'}=\frac{b_1 b_6 b_7 \dots b_{p'-1} b_{q'} x_n}{b_3 b_4 \dots b_{q'-4} b_{q'-3} x_{n-1}}, \qquad \dots (32)$$

where $q'=6l'+1$.

By (2), (31) and (32), we have

$$x_{n+q'+1}=\frac{b_{q'+1} x_{n+q'}}{x_{n+q'-1}}=\frac{b_1 b_2 b_7 b_8 \dots b_{q'-5} b_{q'} b_{q'+1}}{b_4 b_5 \dots b_{q'-3} b_{q'-2} x_{n-1}},$$

$$x_{n+q'+2}=\frac{b_{q'+2} x_{n+q'+1}}{x_{n+q'}}=\frac{b_2 b_3 \dots b_{q'-5} b_{q'-4} b_{q'+1} b_{q'+2}}{b_5 b_6 \dots b_{q'-2} b_{q'-1} x_n},$$

$$x_{n+q'+3}=\frac{b_{q'+3} x_{n+q'+2}}{x_{n+q'+1}}=\frac{b_3 b_4 \dots b_{q'-4} b_{q'-3} b_{q'+2} b_{q'+3} x_{n-1}}{b_1 b_6 b_7 \dots b_{q'-1} b_{q'} x_n},$$

$$x_{n+q'+4}=\frac{b_{q'+4} x_{n+q'+3}}{x_{n+q'+2}}=\frac{b_4 b_5 \dots b_{q'-3} b_{q'-2} b_{q'+3} b_{q'+4} x_{n-1}}{b_1 b_2 \dots b_{q'-6} b_{q'-5} b_{q'} b_{q'+1}},$$

$$x_{n+q'+5}=\frac{b_{q'+5} x_{n+q'+4}}{x_{n+q'=3}}=\frac{b_5 b_6 \dots b_{q'-2} b_{q'-1} b_{q+4} b_{q'=5} x_n}{b_2 b_3 \dots b_{q'-5} b_{q'-4} b_{q'+1} b_{q'+2}}$$

and $$x_{n+q'+6}=\frac{b_{q'+6} x_{n+q'+5}}{x_{n+q'+4}}=\frac{b_1 b_6 b_7 \dots b_{q'-1} b_q' b_{q'+5} b_{q'+6} x_n}{b_3 b_4 \dots b_{q'-4} b_{q'-3} b_{q'+2} b_{q'+3} x_{n-1}}.$$

Thus, (29) and (30) hold for $l=l'+1$. So, (29) and (30) hold for all $q = 6l + 1$, where l is a positive integer.

Observe that (29) and (30) hold if we replace n by $n + Np$, where N is a nonnegative integer and p is as defned in Statement (a).

Now set $q = p$ and replace n by $n + p$ in (29) and (30). Then we have

$$x_{n+2p-1} = \frac{b_5 b_6 \dots b_{p-2} b_{p-1} x_{n+p}}{b_2 b_3 \dots b_{p-5} b_{p-4}}$$

$$= \frac{b_1 b_5 b_6^2 b_7 \dots b_{p-2} b_{p-1}^2 b_p x_n}{b_2 b_3^2 b_4 \dots b_{p-5} b_{p-4}^2 b_{p-3} x_{n-1}} \quad \dots (33)$$

and

$$x_{n+2p} = \frac{b_1 b_6 b_7 \dots b_{p-1} b_p x_{n+p}}{b_3 b_4 \dots b_{p-4} b_{p-3} x_{n+p-1}}$$

$$= \frac{b_1^2 b_2 b_6 b_7^2 \dots b_{p-5} b_{p-1}^2 b_p^2}{b_3 b_4^2 b_5 \dots b_{p-4} b_{p-3}^2 b_{p-2} x_{n-1}}. \quad \dots (34)$$

Furthermore, observe that (33) and (34) hold if we replace n by $n + Np$, where N is a nonnegative integer. Replace n by $n + p$ in (33) and (34). Then we have

$$x_{n+3p-1} = \frac{b_1 b_5 b_6^2 b_7 \dots b_{p-2} b_{p-1}^2 b_p x_{n+p}}{b_2 b_3^2 b_4 \dots b_{p-5} b_{p-4}^2 b_{p-3} x_{n+p-1}}$$

$$= \frac{b_1^2 b_6^2 b_7^2 \dots b_{p-1}^2 b_p^2}{b_3^2 b_4^2 \dots b_{p-4}^2 b_{p-3}^2 x_{n-1}} \quad \dots (35)$$

and

$$x_{n+3p} = \frac{b_1^2 b_2 b_6 b_7^2 \dots b_{p-5} b_{p-1} b_p^2}{b_3 b_4^2 b_5 \dots b_{p-4} b_{p-3}^2 b_{p-2} x_{n+p-1}}$$

$$= \frac{b_1^2 b_2^2 b_7^2 b_8^2 \dots b_{p-6}^2 b_{p-5}^2 b_p^2}{b_4^2 b_5^2 \dots b_{p-3}^2 b_{p-2}^2 x_n}. \quad \dots (36)$$

Finally, observe that (35) and (36) hold if we replace n by $n + Np$, where N is a nonnegative integer. replace n by $n + p$ in (35) and (36). Then we have

$$x_{n+6p-1} = x_{n-1} \text{ and } x_{n+6p} = x_n,$$

and from (2) we also have

$$x_{n+6p+1} = \frac{b_1 x_{n+6p}}{x_{n+6p-1}} = \frac{b_1 x_n}{x_{n-1}} = x_{n+1},$$

$$x_{n+6p+2} = \frac{b_2 x_{n+6p+1}}{x_{n+6p}} = \frac{b_2 x_{n+1}}{x_n} = x_{n+2},$$

$$\dots,$$

$$x_{n+12p-1} = x_{n-1} \text{ and } x_{n+12p} = x_n.$$

Hence, the proof of part (a) is complete.

(b) Without loss of generality, we assume that

$$x_{n+1} = \frac{b_1 x_n}{x_{n-1}}, x_{n+2} = \frac{b_2 x_{n+1}}{x_n} = \frac{b_1 b_2}{x_{n-1}}, x_{n+3} = \frac{b_3 x_{n+2}}{x_{n+1}} = \frac{b_2 b_3}{x_n},$$

$$x_{n+4} = \frac{b_4 x_{n+3}}{x_{n+2}} = \frac{b_3 b_4 x_{n-1}}{b_1 x_n}, x_{n+5} = \frac{b_5 x_{n+4}}{x_{n+3}}$$

$$= \frac{b_4\, b_5\, x_{n-1}}{b_1\, b_2},\ x_{n+6} = \frac{b_6\, x_{n+5}}{x_{n+4}} = \frac{b_5\, b_6\, x_n}{b_2\, b_3},$$

$$x_{n+7} = \frac{b_7\, x_{n+6}}{x_{n+5}} = \frac{b_1\, b_6\, b_7\, x_n}{b_3\, b_4\, x_{n-1}} \text{ and } x_{n+8} = \frac{b_8\, x_{n=7}}{x_{n+6}} = \frac{b_1\, b_2\, b_7\, b_8}{b_4\, b_5\, x_{n-1}}.$$

Now, we claim that for $q = 6l + 2$, with l a positive integer.

$$x_{n+q-1} = \frac{b_1 b_6 b_7 \ldots b_{q-2}\, b_{q-1}\, x_n}{b_3 b_4 \ldots b_{q-5}\, b_{q-4}\, x_{n-1}} \qquad \text{... (37)}$$

and

$$x_{n+q} = \frac{b_1 b_2 b_7\, b_8 \ldots b_{q-2}\, b_{q-1}\, b_q}{b_4 b_5 \ldots b_{q-4}\, b_{q-3}\, x_{n-1}} \qquad \text{... (38)}$$

$$x_{n+q-1} = \frac{b_1\, b_6\, b_7 \ldots b_{q-2}\, b_{q'-1}\, x_n}{b_3\, b_4 \ldots b_{q'-5}\, b_{q'-4}\, x_{n-1}} \qquad \text{... (39)}$$

and

$$x_{n+q'} = \frac{b_1\, b_2\, b_7\, b_8 \ldots b_{q'-1}\, b_q'}{b_4\, b_5 \ldots b_{q'-4}\, b_{q'-3}\, x_{n-1}}, \qquad \text{... (40)}$$

where $q' = 6\,l' + 2.$

By (2), (39) and (40), we have

$$x_{n+q'+1} = \frac{b_{q'+1}\, x_{n+q'}}{x_{n+q'-1}} = \frac{b_2\, b_3\, b_8\, b_9 \ldots b_{q'-5}\, b_q'\, b_{q'+1}}{b_5\, b_6 \ldots b_{q'-3}\, b_{q'-2}\, x_n},$$

$$x_{n+q'+2} = \frac{b_{q'+2}\, x_{n+q'+1}}{x_{n+q'}} = \frac{b_3\, b_4 \ldots b_{q'-5}\, b_{q'-4}\, b_{q'+1}\, b_{q'+2}\, x_{n-1}}{b_1\, b_6\, b_7 \ldots b_{q'-2}\, b_{q'-1}\, x_n},$$

$$x_{n+q'+3} = \frac{b_{q'+3}\, x_{n+q'+2}}{x_{n+q'+1}} = \frac{b_4\, b_5 \ldots b_{q'-4}\, b_{q'-3}\, b_{q'+2}\, b_{q'+3}\, x_{n-1}}{b_1\, b_2\, b_7 \ldots b_8 \ldots b_{q'-1}\, b_{q'}}$$

$$x_{n+q'+4} = \frac{b_{q'+4}\, x_{n+q'+3}}{x_{n+q+2}} = \frac{b_5\, b_6 \ldots b_{q'-3}\, b_{q'-2}\, b_{q'=3}\, b_{q'+4}\, x_n}{b_2\, b_3 \ldots b_{q'-6}\, b_{q'-5}\, b_q'\, b_{q'+1}}$$

$$x_{n+q'+5} = \frac{b_{q'+5}\, x_{n+q'+4}}{x_{n+q'+3}} = \frac{b_1\, b_6\, b_7 \ldots b_{q'-2}\, b_{q'-1}\, b_{q'+4}\, b_{q'+5}\, x_n}{b_3\, b_4 \ldots b_{q'-5}\, b_{q'-4}\, b_{q'+1}\, b_{q'+2}\, x_{n-1}}$$

and

$$x_{n+q'+6} = \frac{b_{q'+6}\, x_{n+q'+5}}{x_n + q' + 4} = \frac{b_1\, b_2\, b_7\, b_8 \ldots b_{q'-1}\, b_q'\, b_{q'=5}\, b_{q'+6}}{b_4\, b_5 \ldots b_{q'-4}\, b_{q'-3}\, b_{q'+2}\, b_{q'+3}\, x_{n-1}}.$$

Thus, (37) and (38) hold for $l = l' + 1$. Therefore, (37) and (38) hold for all $q = 6l + 2$, where l is a positive integer.

Observe that (37) and (38) hold if we replace n by $n + Np$, where N is a nonnegative integer and p is as defined in statement (b).

Set $q = p$ and replace n by $n + p$ in (37) and (38). Then we have

$$x_{n+2p-1} = \frac{b_1\, b_6\, b_7 \ldots b_{p-2}\, b_{p-1}\, x_{n+p}}{b_3\, b_4 \ldots b_{p-5}\, b_{p-4}\, x_{n+p-1}}$$

$$= \frac{b_1\, b_2\, b_7\, b_8 \ldots b_{p-1}\, b_p}{b_4\, b_5 \ldots b_{p-4}\, b_{p-3}\, x_n} \qquad \text{... (41)}$$

and
$$x_{n+2p} = \frac{b_1 b_2 b_7 b_8 \dots b_{p-1} b_p}{b_4 b_5 \dots b_{p-4} b_{p-3} x_{n+p-1}}$$

$$= \frac{b_2 b_3 b_8 b_9 \dots b_{p-5} b_p x_{n-1}}{b_5 b_6 \dots b_{p-3} b_{p-2} x_n}. \quad \dots (42)$$

Observe that (41) and (42) hold if we replace n by $n + Np$, where N is a nonnegative integer. Replace n by $n + p$ in (41) and (42). Then we have

$$x_{n+3p-1} = \frac{b_1 b_2 b_7 b_8 \dots b_{p-1} b_p}{b_4 b_5 \dots b_{p-4} b_{p-3} x_{n+p}} = x_{n-1}$$

and
$$x_{n+3p} = \frac{b_2 b_3 b_8 b_9 \dots b_{p-5} b_p x_{n+p-1}}{b_5 b_6 \dots b_{p-3} b_{p-2} x_{n+p}} = x_n.$$

By induction, we have

$$x_{n+3p+1} = \frac{b_1 x_{n+3p}}{x_{n\,3p-1}} = \frac{b_1 x_n}{x_{n-1}} = x_{n+1}, \dots,$$

$$x_{n+6p-1} = x_{n-1} \text{ and } x_{n+6p} = x_n.$$

The proof of part (*b*) is complete.

(*c*) Without loss of generality, we assume that

$$x_{n+1} = \frac{b_1 x_n}{x_{n-1}}, x_{n+2} = \frac{b_2 x_{n+1}}{x_n} = \frac{b_1 b_2}{x_{n-1}}, x_{n+3} = \frac{b_3 x_{n+2}}{x_{n+1}} = \frac{b_2 b_3}{x_n},$$

$$x_{n+4} = \frac{b_4 x_{n+3}}{x_{n+2}} = \frac{b_3 b_4 x_{n-1}}{b_1 x_n}, x_{n+5} = \frac{b_5 x_{n+4}}{x_{n+3}}$$

$$= \frac{b_4 b_5 x_{n-1}}{b_1 b_2}, x_{n+6} = \frac{b_6 x_{n+5}}{x_{n+4}} = \frac{b_5 b_6 x_n}{b_2 b_3},$$

$$x_{n+7} = \frac{b_7 x_{n+6}}{x_{n+5}} = \frac{b_1 b_6 b_7 x_n}{b_3 b_4 x_{n-1}}, x_{n=8} = \frac{b_8 x_{n+7}}{x_{n+6}} = \frac{b_1 b_2 b_7 b_8}{b_4 b_5 x_{n-1}},$$

and
$$x_{n+9} = \frac{b_9 x_{n+8}}{x_{n+7}} = \frac{b_2 b_3 b_8 b_9}{b_5 b_6 x_n}$$

Now, we claim that for $q = 6\,l + 3$, with l is positive integer,

$$x_{n+q-1} = \frac{b_1 b_2 b_7 b_8 \dots b_{p-2} b_{q-1}}{b_4 b_5 \dots b_{q-5} b_{q-4} x_{n-1}} \quad \dots (43)$$

and
$$x_{n+q} = \frac{b_2 b_3 b_8 b_9 \dots b_{q-1} b_q}{b_5 b_6 \dots b_{q-4} b_{q-3} x_n}. \quad \dots (44)$$

In fact, it is clear that (43) and (44) hold for $l = 1$. We assume that (43) and (44) hold for $l = l'$, i.e.,

$$x_{n+q'-1} = \frac{b_1 b_2 b_7 b_8 \dots b_{q'-2} b_{q'-1}}{b_4 b_5 \dots b_{q'-5} b_{q'-4} x_{n-1}} \quad \dots (45)$$

and $$x_{n+q'} = \frac{b_2 b_3 b_8 b_9 \ldots b_{q'-1} b_{q'}}{b_5 b_6 \ldots b_{q'-4} b_{q'-3} x_n}, \qquad \ldots (46)$$

where $q' = 6l' + 3.$

From (2), (45) and (46), we have

$$x_{n+q'+1} = \frac{b_{q'+1} x_{n+q'}}{x_{n+q'-1}} = \frac{b_3 b_4 \ldots b_{q'-6} b_{q'-5} b_{q'} b_{q'+1} x_{n-1}}{b_1 b_6 b_7 \ldots b_{q'-3} b_{q'-2} x_n}$$

$$x_{n+2q'+2} = \frac{b_{q'+2} x_{n+q'+1}}{x_{n+q'}} = \frac{b_4 b_5 \ldots b_{q'-5} b_{q'-4} b_{q'+1} b_{q'+2} x_{n-1}}{b_1 b_2 b_7 b_8 \ldots b_{q'-2} b_{q'-1}},$$

$$x_{n+q'+3} = \frac{b_{q'+3} x_{n+q'+2}}{x_{n+q'+1}} = \frac{b_5 b_6 \ldots b_{q'-4} b_{q'-3} b_{q'+2} b_{q'+3} x_n}{b_2 b_3 b_8 b_9 \ldots b_{q'-1} b_{q'}}$$

$$x_{n+q'+4} = \frac{b_{q'+4} x_{n+q'+3}}{x_{n+q'+2}} = \frac{b_1 b_6 b_7 \ldots b_{q'-3} b_{q'-2} b_{q'+3} b_{q'+4} x_n}{b_3 b_4 \ldots b_{q'-6} b_{q'-5} b_{q'} b_{q'+1} x_{n-1}},$$

$$x_{n+q'+5} = \frac{b_{q'+5} x_{n+q'+4}}{x_{n+q'+3}} = \frac{b_1 b_2 b_7 b_8 \ldots b_{q'-2} b_{q'-1} b_{q'+4} b_{q'+5}}{b_4 b_5 \ldots b_{q'-5} b_{q'-4} b_{q'+1} b_{q'+2} x_{n-1}}$$

and $$x_{n+q'+6} = \frac{b_{q'+6} x_{n+q'+5}}{x_{n+q'+4}} = \frac{b_2 b_3 b_8 b_9 \ldots b_{q'-1} b_{q'} b_{q'+5} b_{q'+6}}{b_5 b_6 \ldots b_{q'-4} b_{q'-3} b_{q'+2} b_{q'+3} x_n}.$$

So, (43) and (44) hold for $l = l' + 1$. Therefore, (43) and (44) hold for all $q = 6l + 3$, where l is a positive integer.

Observe that (43) and (44) hold if we replace n by $n + Np$, where N is a nonnegative integer and p is as defined in Statement (*c*).

Set $q = p$ and replace n by $n + p$ in (43) and (44). Then we have

$$x_{n+2p-1} = \frac{b_1 b_2 b_7 b_8 \ldots b_{p-2} b_{p-1}}{b_4 b_5 \ldots b_{p-5} b_{p-4} x_{n+p-1}}$$

and $$x_{n+2p} = \frac{b_2 b_3 b_8 b_9 \ldots b_{p-1} b_p}{b_5 b_6 \ldots b_{p-4} b_{p-3} x_{n+p}} = x_n.$$

By induction, we obtain

$$x_{n+2p+1} = \frac{b_1 x_{n+2p}}{x_{n+2p-1}} = \frac{b_1 x_n}{x_{n-1}} = x_{n+1}, \ \ldots,$$

$$x_{n+4p-1} = x_{n-1} \text{ and } x_{n+4p} = x_n.$$

This completes the proof of part (*c*)

The proofs of parts (*d*) and (*e*) are similar to that of (*b*) and (*a*), respectively. We omit them.

(*f*) We have from Lemma 3.1.(f)

$$x_{n+5} = \frac{b_4 b_5 x_{n-1}}{b_1 b_2} \text{ and } x_{n+6} = \frac{b_5 b_6 x_n}{b_2 b_3}.$$

We claim that for $q = 6l + 6$, with l a positive integer,

$$x_{n+q-1} = \frac{b_4 b_5 \dots b_{q-2} b_{q-1} x_{n-1}}{b_1 b_2 \dots b_{q-5} b_{q-4}} \quad \dots (47)$$

and
$$x_{n+q} = \frac{b_5 b_6 \cdots b_{q-1} b_q x_n}{b_2 b_3 \cdots b_{q-4} b_{q-3}}. \quad \dots (48)$$

First of all, we know that (47) and (48) hold for $l = 1$.

Secondly, we assume that (47) and (48) hold for $l = l'$, i.e.,

$$x_{n+q'-1} = \frac{b_4 b_5 \dots b_{q'-2} b_{q'-1} x_{n-1}}{b_1 b_2 \dots b_{q'-5} b_{q'-4}} \quad \dots (49)$$

and
$$x_{n+q'} = \frac{b_5 b_6 \cdots b_{q'-1} b_{q'} x_n}{b_2 b_3 \cdots b_{q'-4} b_{q'-3}}. \quad \dots (50)$$

where $q' = 6l' + 6.$

By (2), (49) and (50), we get

$$x_{n+q'+5} = \frac{b_4 b_5 \dots b_{q'-2} b_{q'-1} b_{q'+4} b_{q'+5} x_{n-1}}{b_1 b_2 \dots b_{q'-5} b_{q'-4} b_{q'+1} b_{q'+2}}$$

and
$$x_{n+q'+6} = \frac{b_5 b_6 \cdots b_{q'-1} b_{q'} b_{q'-1} b_{q'+5} b_{q'+6} x_n}{b_2 b_3 \cdots b_{q'-4} b_{q'-3} b_{q'+2} b_{q'+3}}.$$

So, (47) and (48) hold for $l = l' + 1$. Thus, (47) and (48) hold for all $q = 6l$.

Observe that (47) and (48) hold if we replace n by $n + N p$, where N is a nonnegative integer and p is as defined in Statement (*f*).

Set $q = p$ and replace n by $n + p$ in (47) and (48). Then we have

$$x_{n+2p-1} = \frac{b_4 b_5 \dots b_{p-2} b_{p-1} x_{n+p-1}}{b_1 b_2 \dots b_{p-5} b_{p-4}} = \left(\frac{b_4 b_5 \dots b_{p-2} b_{p-1}}{b_1 b_2 \dots b_{p-5}} b_{p-4} \right)^2 x_{n-1}$$

amd
$$x_{n+2p} = \frac{b_5 b_6 \cdots b_{p-1} b_p x_{n+p}}{b_2 b_3 \cdots b_{p-4} b_{p-3}} = \left(\frac{b_5 b_6 \cdots b_{p-1} b_p}{b_2 b_3 \cdots b_{p-4} b_{p-3}} \right)^2 x_n.$$

Therefore, $x_{n+p-1} = x_{n-1}$ and $x_{n+p} = x_n$

hold for

$$\frac{b_4 b_5 \dots b_{p-2} b_{p-1}}{b_1 b_2 \dots b_{p-5} b_{p-4}} = \frac{b_5 b_6 \dots b_{p-1} b_p}{b_2 b_3 \dots b_{p-4} b_{p-3}} = 1$$

and we obtain that every positive solution of eq. (2) is periodic with period $p = 6k + 6$. It is easy for one to see that every postive solution of eq. (2) is not periodic otherwise.

This completes the proof of the lemma

By Lemmas 3.1 and 3.2, we obtain the following theorem

***Theorem* B** — *Assume that $b_i > 0$ for $i = 1, \dots, p$. Let k be a positive integer. Then the following statements are all true:*

(a) If $p \neq 6k$, then every positive solution of eq. (2) is periodic with period m, where m is the least common multiple of 6 and p.

(b) If $p = 6k$, then every positive solution of eq. (2) is periodic with period p for

$$\frac{b_4\, b_5 \ldots b_{p-2}\, b_{p-1}}{b_1\, b_2 \ldots b_{p-5}\, b_{p-4}} = \frac{b_5\, b_6 \ldots b_{p-1}\, b_p}{b_2\, b_3 \ldots b_{p-4}\, b_{p-3}} = 1;$$

otherwise, every positive solution of eq. (2) is not periodic.

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