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WAVE PROPAGATION IN A FLUID SATURATED INCOMPRESSIBLE POROUS MEDIUM

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Method of characteristics is applied to study the propagation of plane, cylindrical and spherical waves in a fluid saturated incompressible porous medium. The analytical results for discontinuities across the wave fronts are obtained. Impulse input is taken for numerical investigation.

Key Words : Porous Medium; Volume Fractions; Incompressibility Relation; Characteristics; Characteristic Equations; Discontinuity Relations

INTRODUCTION

The phenomenon of wave propagation in fluid-saturated porous media has created increased interest in the recent years. Solutions have been sought to wave problems particularly those arise in the field of geophysics, geo-mechanics and other such fields of engineering. Biot theory [3] is based on the assumption of compressible constituents and propagation of two dilatational and one rotational waves have been concluded. Many researchers have discussed various problems based on this theory[4, 5, 6]

An other interesting theory is given by Boer and Ehlers[1, 2]. In this theory both the constituents (solid, fluid) are assumed to be incompressible. One-dimensional transient wave propagation in fluid saturated incompressible porous media has been discussed by Boer and Ehlers[2]. The saturating fluid is assumed to be inviscid and the incompressibility constraint exhibits only one independent dilatational wave proagating in both the solid and fluid phases.

In this paper, following Boer and Ehlers theory, the propagation of plane, spherical and cylindrical waves in an incompressible porous medium has been discussed. Method of characteristics is to applied to solve the governing equations. The advantage of this method is that, it gives the simple description of wave fronts, path of waves and can be used for arbitrary input functions. it has been successfully used by many investigators[7-11] and is based on the ideas given in standard texts[12-15]. Integral transform method usually involves inversion difficulties and the numerical integration is often used to evaluate inversion integrals. There are certain problems where the Lapalace transform method fails to give solution[8], where as the method of characteristics yield it. A further important property of this method is that if there is a discontinuity at any point of a characteristic curve, then there must be one at all the points of this characteristic curve[15]. It is this property which identifies characteristic curve as a wavefront.

So following this method, the analytical results for discontinuities across the wave front are obtained. Procedure for numerical computation is presented and the impulsive input function is taken for numerical investigation.

FIELD EQUATIONS

The equations governing the deformation of a fluid saturated incompressible porous medium in the absence of body force are given by [1, 2] as

$$(\lambda^s + \mu^s) \text{ grad. div. } \boldsymbol{u}_S + \mu^S \text{ div. grad. } \boldsymbol{u}_S - \eta^s \text{ grad.p}$$

$$-\rho^s \frac{\partial^2 \boldsymbol{u}_S}{\partial t^2} + S_v \left(\frac{\partial \boldsymbol{u}_F}{\partial t} - \frac{\partial \boldsymbol{u}_S}{\partial t} \right) = 0 \qquad \text{... (1)}$$

$$\eta^f \text{ grad.p } - \rho^F \frac{\partial^2 \boldsymbol{u}_F}{\partial t^2} - S_v \left(\frac{\partial \boldsymbol{u}_F}{\partial t} - \frac{\partial \boldsymbol{u}_S}{\partial t} \right) = 0 \qquad \text{... (2)}$$

$$\text{div} \left(\eta^S \frac{\partial \boldsymbol{u}_S}{\partial t} + \eta^F \frac{\partial \boldsymbol{u}_F}{\partial t} \right) = 0 \qquad \text{... (3)}$$

where $S_v = \frac{(\eta^{F)} \gamma FR}{k^F}$, γ^{FR} is the effective specific weight of the fluid, k^F is the coefficient of permeability of porous medium and other symbols have their usual meanings.

where $S_v = \frac{(\eta^F)^2 \gamma^{FR}}{k^F}$, γ^{FR} is the effective specific weight of fluid, k^F is the Permeability Coefficient of the porous medium η^S, n^k are the volume fractions satisfying

$$\eta^S + \eta^F = 1 \qquad \text{... (4)}$$

and other symbols have their usual. meanings.

FORMULATION OF THE PROBLEM

We consider an infinite fluid-saturated incompressible porous medium having a cavity (speherical or cylindrical) of radius r_0. Initially the surface $r = r_0$ is not loaded. So each particle of the medium is at rest. As time progresses, a time dependent input $f(t)$ is applied at $r = r_0$ either suddenly or gradually. For the case of spherical (cylindrical) symmetry, the displacements of solid and fluid particles can be written as

$$\boldsymbol{u}_S = u^S \boldsymbol{e}_r \qquad \text{... (5)}$$

$$\boldsymbol{u}_F = u^F \boldsymbol{e}_r \qquad \text{... (6)}$$

The initial and boundary conditions are

$$u^S(r, 0) = u^F(r, 0) = 0 \qquad \text{... (7)}$$

$$\frac{\partial u^S}{\partial t}(r, 0) = \frac{\partial u^F}{\partial t}(r, 0) = 0 \qquad r > r_0 \qquad \text{... (8)}$$

$$\sigma_r(r_0, t) = -f(t) \qquad \text{... (9)}$$

$$p(r_0, t) = 0$$

where

$$\sigma_r = (\lambda^S + 2\mu^S) \frac{\partial u^S}{\partial r} + N \lambda^S \frac{u^S}{r} \qquad \text{... (10)}$$

As there is no effect of the applied load at very large distances. So

$$u^s, u^F \to 0 \text{ as } r \to \infty \qquad \text{... (11)}$$

Use of (5), (8) in (1) - (3) yields

$$(\lambda^2 + 2\mu^s)\left\{\frac{\partial^2 u^s}{\partial r^2} + N\left(\frac{1}{r}\frac{\partial u^s}{\partial r} - \frac{u^s}{r^2}\right)\right\} - \eta^s \frac{\partial p}{\partial r}$$

$$- \rho^s \frac{\partial^2 u^s}{\partial t^2} + S_v\left(\frac{\partial u^F}{\partial t} - \frac{\partial u^s}{\partial t}\right) = 0 \qquad \text{... (12)}$$

$$- \eta^F \frac{\partial p}{\partial r} - \rho^f \frac{\partial^2 u^F}{\partial t} - S_v\left(\frac{\partial u^F}{\partial t} - \frac{\partial u^s}{\partial t}\right) = 0 \qquad \text{... (13)}$$

where $$\eta^s \frac{\partial^2 u^s}{\partial r\, \partial t} + \eta^f \frac{\partial^2 u^f}{\partial r\, \partial t} = 0 \qquad \text{... (14)}$$

$N = 0$, if the wave is plane.

$N = 1$, if the wave is cylindrical,

and $N = 2$, if the wave is spherical.

For $N = 0$, the load is applied on the boundary of the half space.

Eq. (4) with the help of (7), (8) and (11) gives

$$\eta^s u^s + \eta^F u^F = 0 \qquad \text{... (15)}$$

Making use of (4) and (13) in (15), we obtain

$$c_0^2 = \frac{\partial^2 u^s}{\partial r^2} - \frac{\partial^2 u^s}{\partial t^2} = A \qquad \text{... (16)}$$

where $$c_0 = \sqrt{\frac{(\eta^F)^2 (\lambda^s + 2\mu^s)}{(\eta^s)^2 \rho^F + (\eta^F)^2 \rho^s}} \qquad \text{... (17)}$$

and $$A = \frac{S_V}{\left\{(\eta^s)^2 \rho^F + (\eta^F)^2 \rho^s\right\}} \frac{\partial u^s}{\partial t} - N c_0^2 \left(\frac{1}{r}\frac{\partial u^s}{\partial r} - \frac{u^s}{r^2}\right) \qquad \text{... (18)}$$

Solution by Method of Characteristics

Following the theory of characteristics, the physical characteristics of the eq. (16) are defined by

$$\frac{dr}{dt} = c_0 \qquad \text{... (19)}$$

$$\frac{dr}{dt} = - c_0 \qquad \text{... (20)}$$

and $r = r_0 + \overset{t}{c_0}$ is the projection of the leading wave front on the $r - t$ plane.

c_0 defined in above equations is the velocity of the wave propagating in a fluid saturated incompressible porous medum. If the pore liquid is absent or gas is filled in the pores, then ρ^F is very small as compare to ρ^s and can be neglected. So the relation (17) reduces to

$$c_0 = \sqrt{\frac{(\lambda^s + 2\mu^s)}{\rho^s}}$$

This gives the velocity of wave propagating in an incompressible porous solid, where the change in volume is due to the change in porosity and is a well-known result of the classical theory of elasticity.

In an incompressible non-porous solid medium $\eta^F \rightarrow 0$, then again from (17) we have $c_0 = 0$ and is physically acceptable as a longitudinal wave can not propagate in an incompressible medium.

Introducing the dimensionless quantities as

$$r' = \frac{r}{r_0}, u'^s = \frac{(\lambda^s + 2\mu^s)}{E}\frac{u^s}{r_0}, u'^F = \frac{(\lambda^s + 2\mu^s)}{E}\frac{u^F}{r_0},$$

$$t' = \frac{ct}{r_0}, \sigma_r' = \frac{\sigma_{rr}}{E}$$

where E is the Young's modulus of the solid phase and suppressing the dashes, yield characteristics as

$$\frac{dr}{dt} = 1 \quad \text{... (21)}$$

$$\frac{dr}{dt} = -1 \quad \text{... (22)}$$

and the characteristic equations as

$$dx - dv^s - Adt = 0 \text{ along } \frac{dr}{dt} = 1 \quad \text{... (23)}$$

$$dx + dv^s + Adt = 0 \text{ along } \frac{dr}{dt} = -1 \quad \text{... (24)}$$

where $$x = \frac{\partial u^s}{\partial r}, v^s = \frac{\partial u^s}{\partial t}, A = Lv^s - N\left(\frac{x}{r} - \frac{u^s}{r^2}\right)$$

and $$L = \frac{r_0 S_v}{c_0\{(\eta^s)^2 \rho^F + (\eta^F)^2 \rho^s\}} \quad \text{... (25)}$$

Propagation of Discontinuities

In the $r - t$ plane, we draw a characteristic of the family (21) and take two points P and Q on it which are sufficiently close to each other. Through these points, we draw the characteristics of the family (21) as shown in Fig. 2. Integrating (24) from P to Q we get

$$(x_Q - x_p) + (v_Q^S - v_P^s + \int_P^Q Adt = 0$$

Taking the limits as $P \rightarrow Q$ and as the integrand is bounded, so we have

$$[x] + [v^s] = 0 \quad \text{... (26)}$$

Similarly the discontinuity relation across $\frac{dr}{dt} = -1$ is given by

$$[x] - [v^s] = 0 \qquad \text{... (27)}$$

Eqs. (26) and (27) can also be derived by using the concept given by Jeffery and Taniuty [14]. Writing the eq. (23) for the points P and Q subtracting and in the limiting case as $P \rightarrow Q$, it can be easily derived that

$$d\,[x] - d\,[v^s] = \lim_{P \rightarrow Q} (A_Q - A_P)\, dr \qquad \text{... (28)}$$

Eq. (28) with the help of (25) and (26) and on integration yields

$$[x] = Ke^{-\frac{L}{2}r}\, r^{-\frac{N}{2}} \qquad \text{... (29)}$$

$$[v^s] = -Ke^{-\frac{L}{2}r}\, r^{-\frac{N}{2}} \qquad \text{... (30)}$$

Where K is a constant of integration and is evaluated by using the boundary conditions.

Similarly the discontinuity relation across $\frac{dr}{dt} = -1$ are

$$[x] = [v^s]\, Ke^{-\frac{L}{2}r}\, r^{-\frac{N}{2}} \qquad \text{... (31)}$$

The corresponding relations for the discontinuities in the displacement gradient and particle velocity of a fluid particle can be derived by using the incompressibility relation (15). If the pore liquid is absent then $S_V = 0$, so $L = 0$ and the relations (29), (30) and (31) take the form

$$[x] = K\, r^{-\frac{N}{2}}$$

$$[v^s] = -K\, r^{-\frac{N}{2}}$$

across $\frac{dr}{dt} = 1$ and

$$[x] = [v^s] = K\, r^{-\frac{N}{2}}$$

across $\frac{dr}{dt} = -1$.

These results can also be compared with the known results of classical theories, e.g. Chou and Koeing[7].

NUMERICAL INTEGRATION

In the new $r - t$ plane, $r = 1$ is a stright parallel to t-axis and $r = t + 1$ is the projection of the leading wave front in this plane. The medium particles outside this wave front are at rest. We divide the straight-line $r = 1$ by a number of points in such a way that the distance between any two consecutive points is same throughout and is very small. Let this distance be δt and from the geometry of the characteristics, it is clear that $\delta r = \delta t$. Through these points, we draw two families of characteristics defined by (21) and (22). The continuous region between $r = 1$ and $r = 1 + t$ is thus replaced by discrete points as shown in fig. 1. A numerical procedure involving stepwise integration along these characteristics can be employed to solve the problem for various inputs.

All above points can be divided into three categories. The points lying on the straight-line $r = 1 + t$ fall in the first category. For these points discontinuity relations are used to evaluate the unkwown quantities. The points lying on the straight-line $r = 1$ are included in second category. Here x, r and t are given and only u^s, v^s are to be evaluated. This can be done by writing characteristic eq. (24) and the continuity condition

$$du^s = \frac{\partial u^s}{\partial r} dr + \frac{\partial u^s}{\partial t} dt = xdr + v^s dt \qquad \text{... (32)}$$

in difference form. All other points are included in third category. Here the characteristic equations are written in difference form and the unknown quantities are evaluated from the algebraic equations so obtained.

To give the mathematical treatment to above technique, the suffixes i, j are attached to each mesh point as shown in Fig. 1 and it is clear that for any i, j varies from 1 to i. It provides a very good logic for the computer programming of the problem.

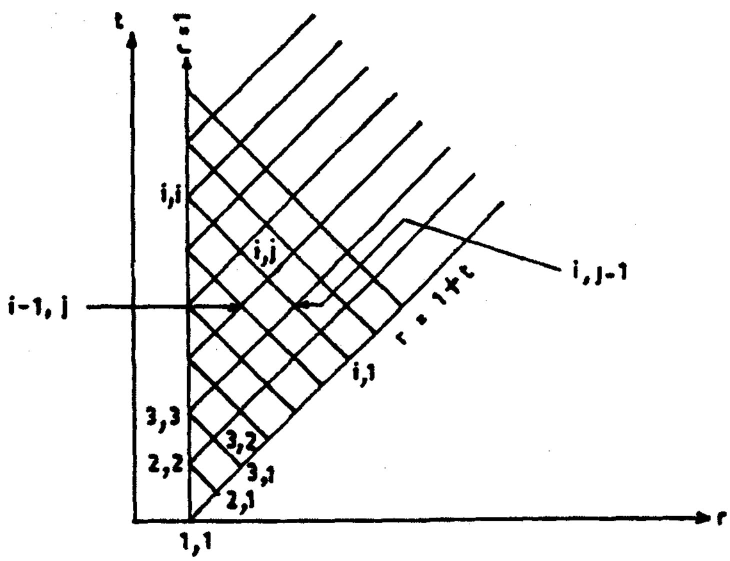

FIG. 1

Consider the point 1, 1. This point corresponds to the situation when the load is just applied. Here $t_{11} = 0, r_{11} = 1, x_{11} = -f(0), v^s_{11} = -x_{11}$ and $u^s_{11} = 0$.

For any point i, 1 of the first category,

$$t_{i1} = \left(\frac{i-1}{2}\right)\delta t,\ r_{i1} = 1 + t_{i1} \qquad \text{... (33)}$$

$$x_{i1} = Ke^{-\frac{L}{2}r_{i1}}\, r_{i1}^{-\frac{N}{2}},\ v^s_{i1} + -x_{i1} \qquad \text{... (34)}$$

Next we consider any point i, j of the third category. All the quantities are known at the neighboring point $i - 1$, j and i, $j - 1$. From Fig. 1 it is clear that

$$t_{ij} = t_{ij-1} + \frac{1}{2}\delta t \qquad \text{... (35)}$$

$$r_{ij} = r_{ij-1} - \frac{1}{2}\,\delta t \qquad \text{... (36)}$$

Writing the characteristic equations in difference form and after some simplification we get

$$x_{ij} = \frac{1}{2}\Big(x_{ij-1} + x_{i-1j} + \overset{s}{v}_{ij-1} - \overset{s}{v}_{i-1j} + A_{i-1j}$$

$$(t_{ij} - t_{i-1j}) - A_{ij-1}\,(t_{ij} - t_{ij-1}) \Big) \qquad \text{... (37)}$$

$$\overset{s}{v}_{ij} = x_{ij} - x_{ij-1} + \overset{s}{v}_{ij-1} - A_{i-1j}\,(t_{ij-i-1j}) \qquad \text{... (38)}$$

$\overset{s}{u}_{ij}$ for first and third category are evaluated by using (32).

At the last, we consider any point of second category.

For this point both the suffixes are equal. If this point is i, i; then

$$t_{ii} = (i-1)\,\delta t, \;\; r_{ii} = 1 \qquad \text{... (39)}$$

and from (8), (9) we get

$$x_{ii} = -\frac{N\lambda^s}{\lambda^s + 2\mu^s}\frac{\overset{s}{u}_{ii}}{r_{ii}} - f((i-1)\,\delta t) \qquad \text{... (40)}$$

$$\overset{s}{v}_{ii} = \overset{s}{v}_{ii-1} - (x_{ii} - x_{ii-1}) - A_{ii-1}\,(t_{ii} - t_{ii-1}). \qquad \text{... (41)}$$

These two eqs. (23) and the continuity relation (32) provide

$$\overset{s}{u}_{ii} = \frac{1}{\Delta}\,[f((i-1)\,\delta t)\,(t_{ii} - t_{ii-1} - r_{ii} + r_{ii-1}) + x_{ii-1}\,(t_{ii} - t_{ii-1} + r_{ii} - r_{ii-1})$$

$$+\; 2\,\overset{s}{v}_{ii-1}\,(t_{ii} - t_{ii-1}) - A_{ii-1}\,(t_{ii} - t_{ii-1})^2 \;+\; 2\,\overset{s}{U}_{ii-1}] \qquad \text{... (42)}$$

Where

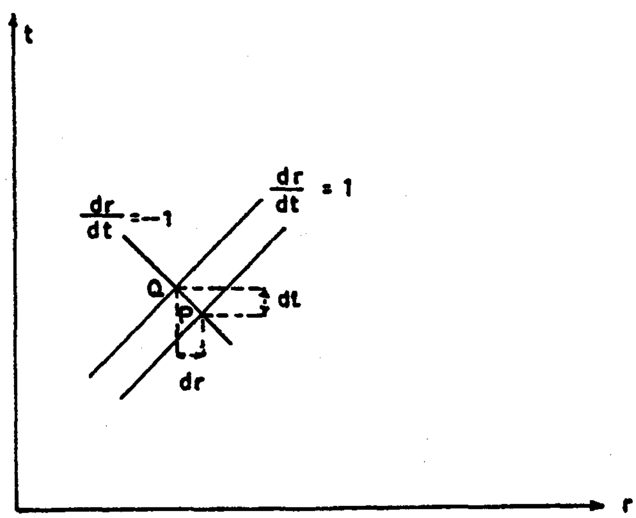

FIG. 2

$$\Delta = 2 + \frac{N \lambda^s}{(\lambda^2 + 2\mu^s)} (r_{ii} - r_{ii-1} - t_{ii} + t_{ii-1})$$

The displacement of the corresponding fluid particles can be easily obtained by using the incompressibility relation (15).

NUMERICAL RESULTS AND DISCUSSION

The numerical values of the various physical quantities are taken from Borer and Ehlers[2] as

$\lambda^s = 5.5833 \text{ MN/m}^2$

$\rho^s = 1.34 \text{ Mg/m}^3$

$\eta^s = 0.67$

$k^F = 0.01 \text{m/s}$

$\mu^s = 8.3750 \text{ MN/m}^2$

$\rho^F = 0.33 \text{ Mg/m}^3$

$\eta^F = 0.33$

$\Upsilon^{FR} = 10.0 \text{ KN/m}^3$

The variation of displacement u^s of solid particles with time have been shown in Fig. 3. It is evident that for given r, the value of u^s increases abruptly and then decreases gradually to some

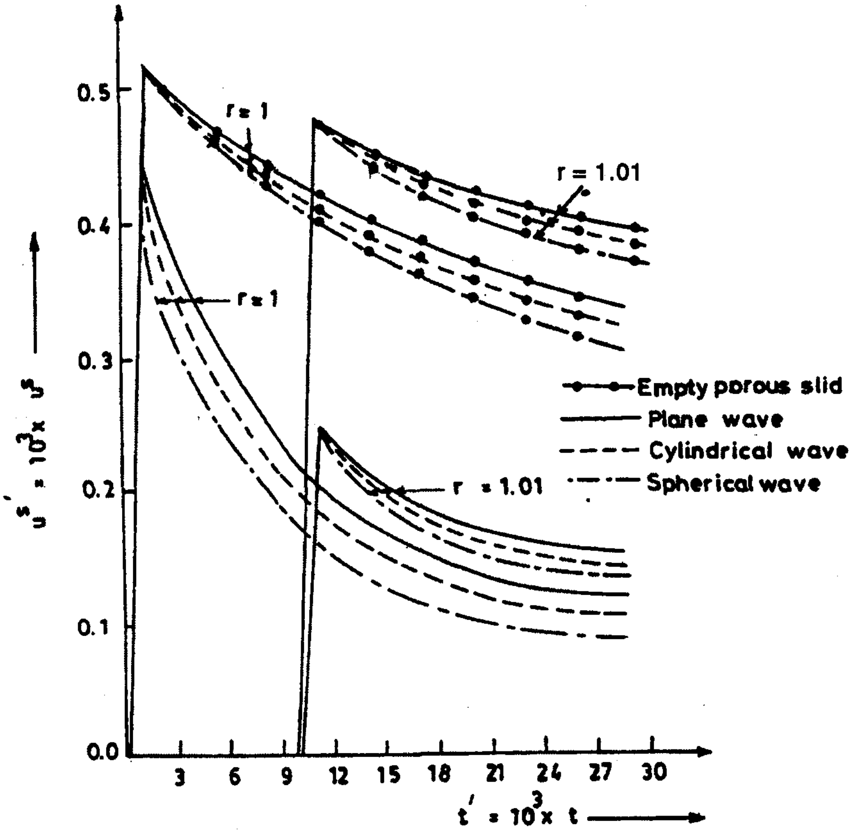

FIG. 3. Variation of u^s w.r.t. TIME

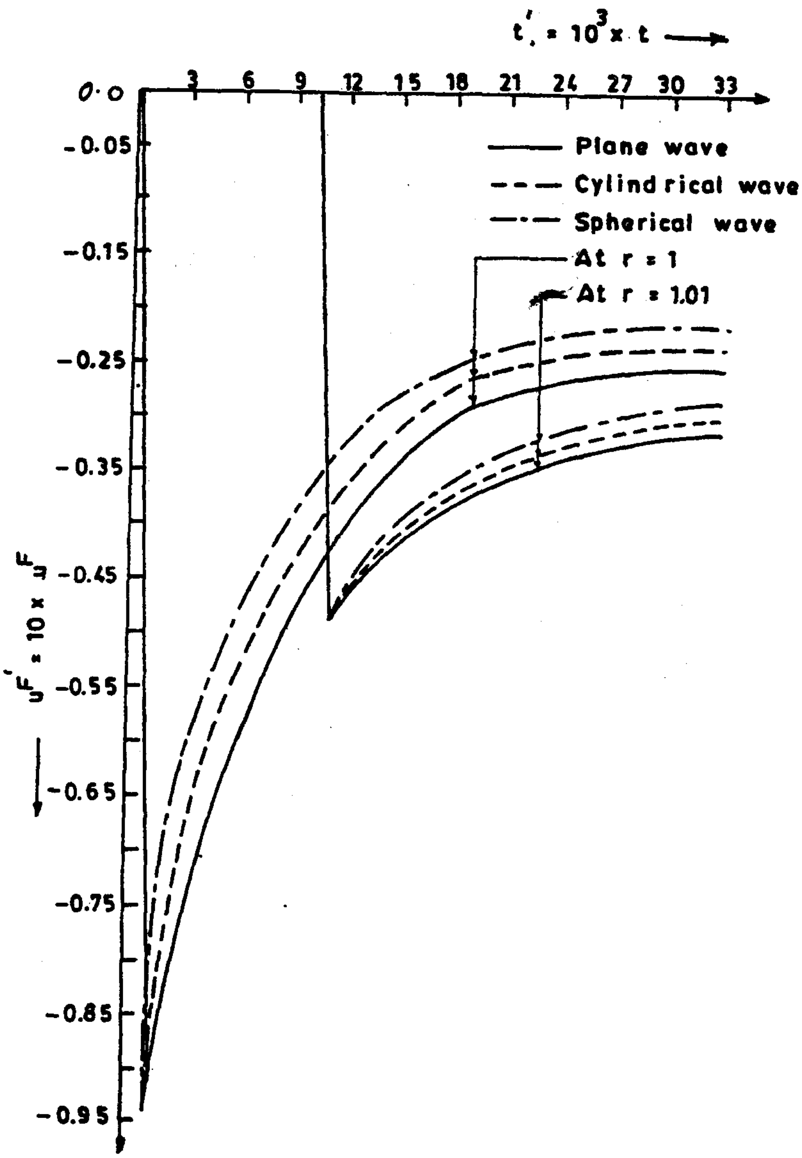

FIG. 4. Variation of u^F w.r.t. TIME

small value. It is clear that for same medium and the same input, the displacement due to cylindrical wave is more than the displacement due to spherical wave but less than that due to plane wave. If we consider the variation of solid-displacement w.r.t.r (Fig. 7), then, when time is small, the solid displacement reaches to its maximum value very quickly and then abruptly falls to zero. But latter on this displacement first increases then decreases gradually and ultimately falls to zero abruptly. The values from which the abrupt falls take place are also decreasing. This is due to the fact that the discontinuities in x and v^S attenuate according to eqs. (29), (30) and u^S is evaluated from the relation (32) where these x and v^S are used. The presence of fluid restricts the motion of solid. For the same input, if the pore liquid is absent then solid displacement is more than that if the pore liquid is present as it is evident from the Figs. 3 and 7.

For a given r, the velocity of a solid particle abruptly falls from a large positive value to some negative value then increases with time but remain negative and ultimately tends to zero (Figs. 5(a), 5(b)). Magnitude of velocity is more for spherical wave and less for plane wave. on the other

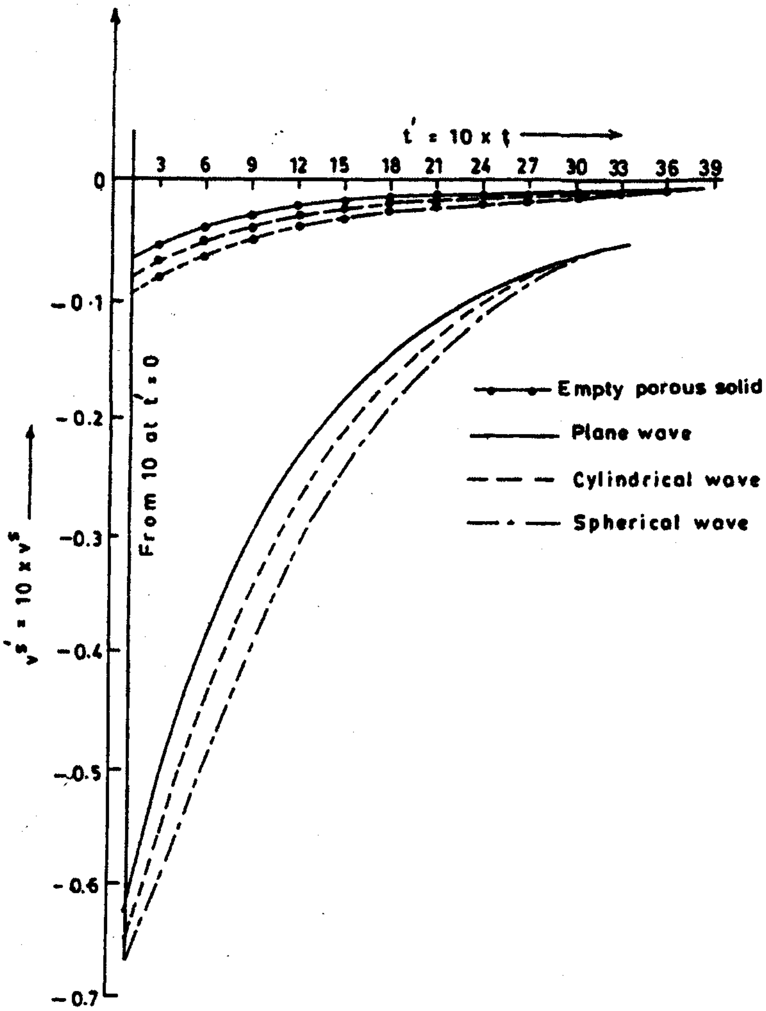

FIG. 5a. Variation of v^s w.r.t. TIME AT $r = 1$

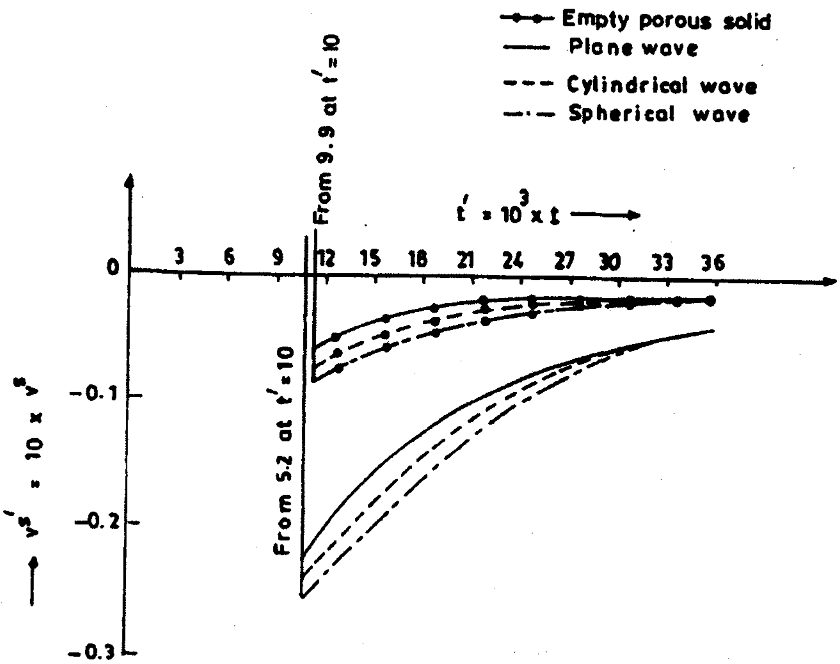

FIG. 5b. Variation of v^s w.r.t. TIME AT $r = 1.01$

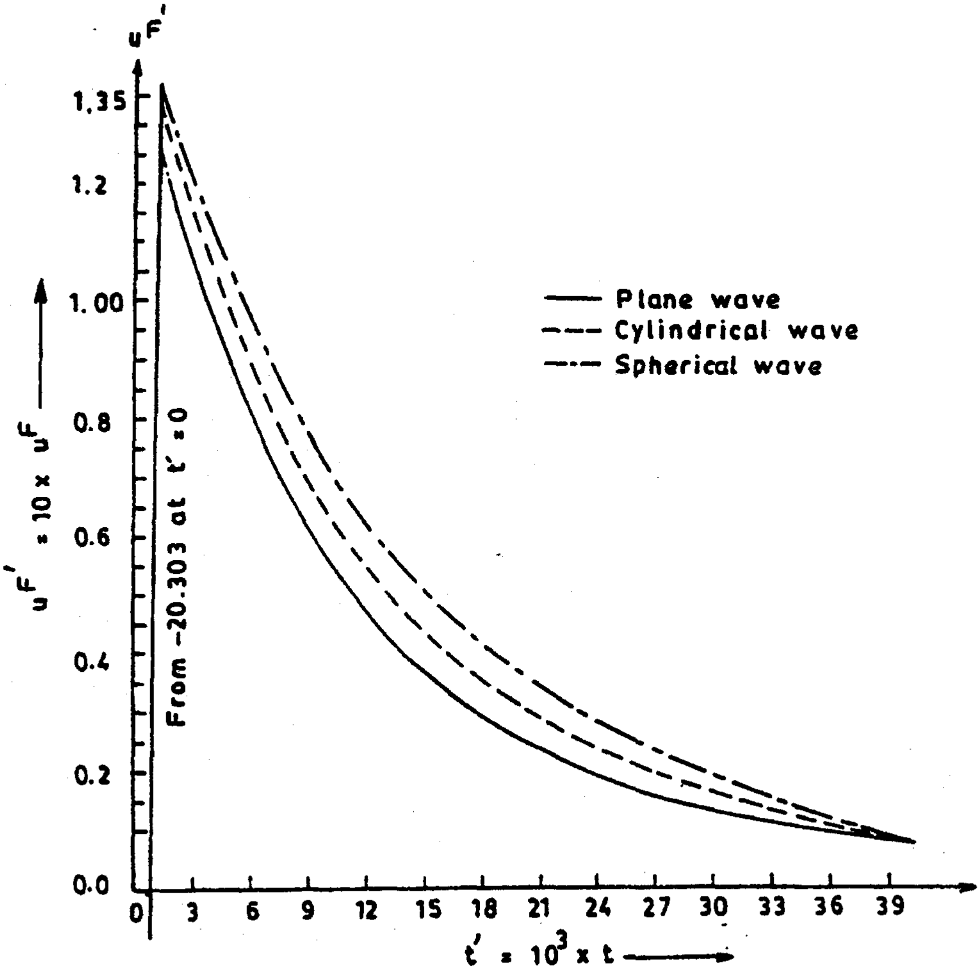

FIG. 6a. Variation of ν^F w.r.t. TIME AT $r = 1$

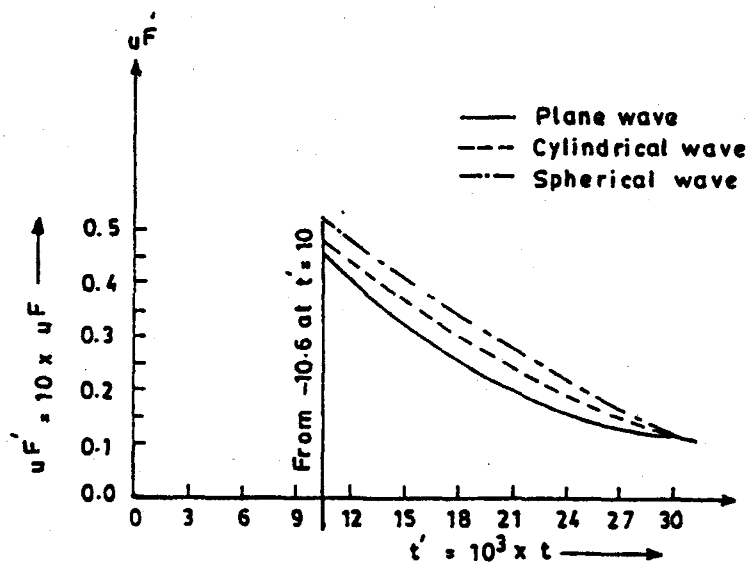

FIG. 6b. Variation of ν^F w.r.t. TIME AT $r = 1.01$

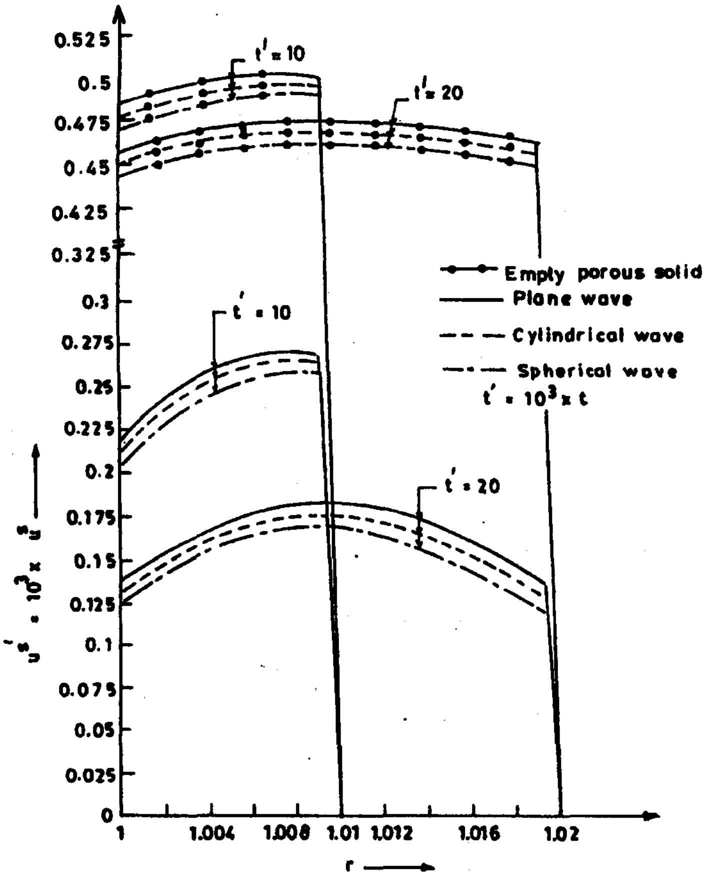

FIG. 7. Variation of u^s w.r.t. r

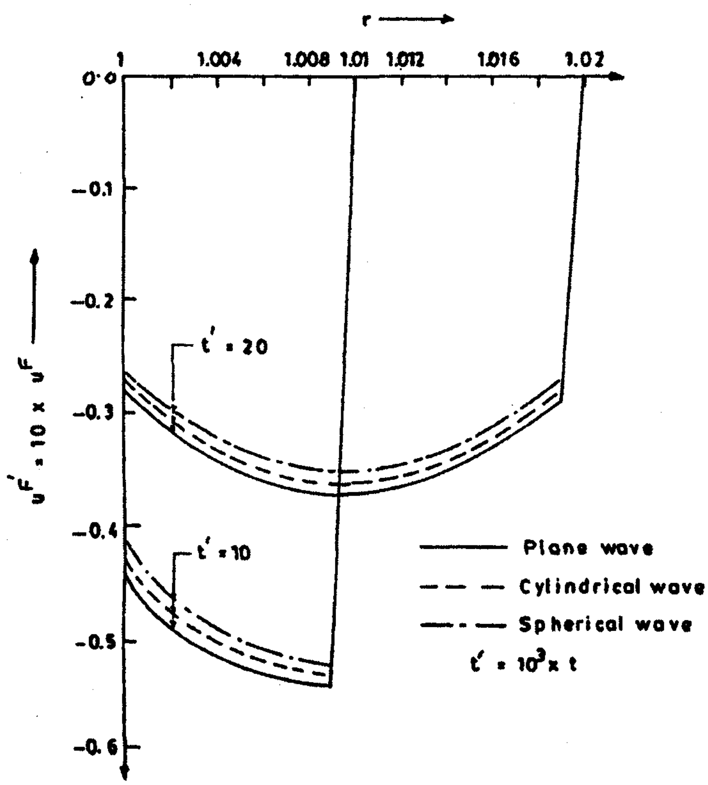

FIG. 8. Variation of u^F w.r.t. r

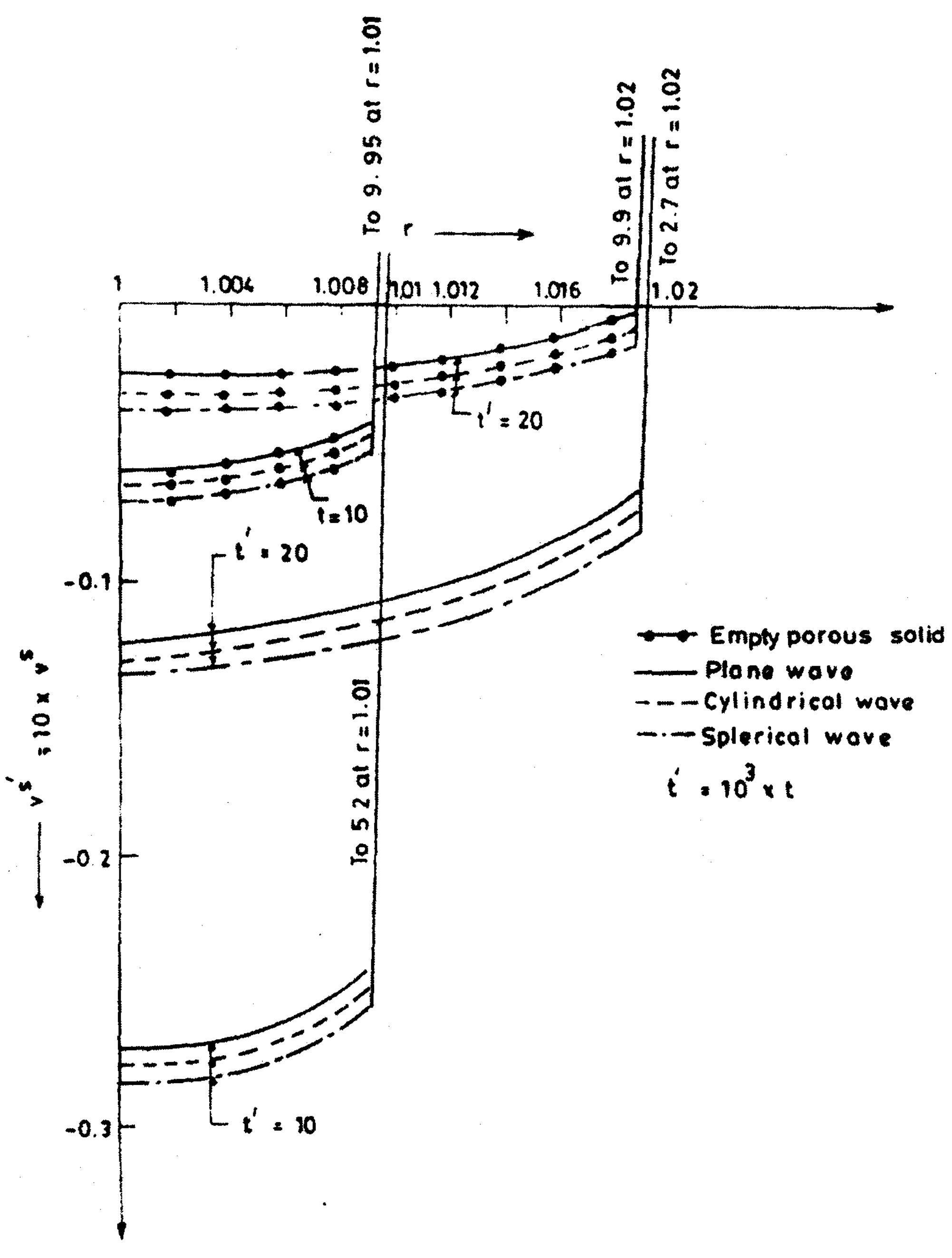

FIG. 9. Variation of v^s w.r.t. r

hand for given t, velocity increases with r but remain negative and then abruptly go to some positive value (Fig. 9) and these positive values also decrease quickly if the pore liquid is present, but slowly if the pore liquid is absent. For example when $t' = 10$, this value is 5.2 in the presence of pore liquid and is 9.95 in the case of empty porous solid; where as at $t' = 20$, these values are 27 and 9.5 respectively. The abrupt changes in all above cases correspond to points that lie on the leading wave front, which is a line of discontinuity. The variation of displacement and particle velocity is similar for plane, cylindrical and spherical wave, though their magnitudes are different.

As a result of incompressibility condition, the motion of fluid phase is opposite to that of solid phase. That is why the graphs for fluid phase (Figs 4, 6(a), 6(b), 8, 10) are mirror images of corresponding graphs for the solid phase. The magnitude of displacement and velocity of a fluid

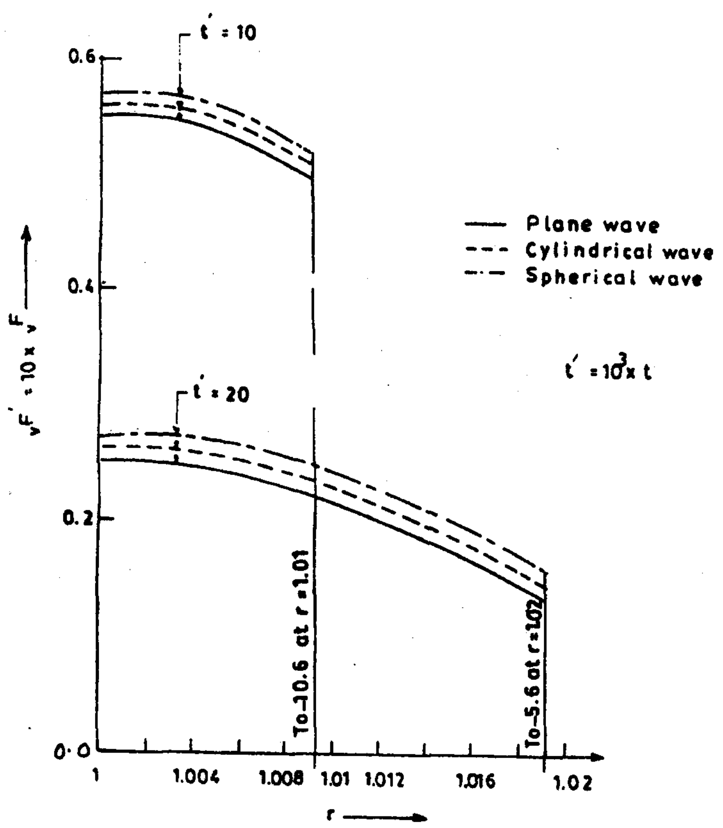

FIG. 10. Variation of v^F w.r.t. r

particle is greater than equal to or less than that of the corresponding solid particle, according as the quantity $\frac{\eta^s}{\eta^F}$ is greater than, equal to or less than 1.

CONCLUSION

Propagation of plane cylindrical and spherical waves in a fluid saturated incompressible porous medium has been discussed by using the method of characteristics. For numerical discussion a mesh size of $\delta t = 0.001$ and a total of 882 mesh points are taken. This number may be increased depending upon the memory of the computer used. In the present article only impulsive input is taken. But any type of input may be taken, for that no separate method is required, only the function defined in computer programme is to be replaced by that input function. It has been observed that the discontinuities in particle velocity and displacement gradients across the wave-front are : (*i*) inversely proportional to $e^{\frac{L}{2}r}$ for plane wave (*ii*) inversely proportional to $e^{\frac{L}{2}r}$ and square root of the radial distance for cylindrical wave and (*iii*) inversely proportional to $e^{\frac{L}{2}r}$ and the radial distance for the spherical wave. Though the nature of vibration for plane spherical and cylindrical waves is same, but the magnitude of various quantities such as the displacement and particle velocity are different as it is clear from the graphs.

REFERENCES

1. R. de Boer and W. Ehlers, Int. J. Solid Structures. **26** (1990), 43-57.

2. R. de Boer and W. Ehlers, Z.Liu, *Arch. App. Mech.* **63** (1993) 59-72.
3. M. A. Biot, *J. Acoust. Soc. Am.* **28** (1956) 168-78.
4. T. Levy, *Int. J. Engng. Sci.* **17** (1979) 1005-14.
5. J. H. Prevost, Soil Dynamics and Earthquake Engg. **4** (1985) 185-202.
6. S. K. Garg, A. H. Nafeh and A. J. Good, *J. Appl. Phys.* **45** (1974) 1968-74.
7. P. C. Chou and H. A. Koenig, *J. Appl. Mech.* **33** (1966), 159-67.
8. P. C. Chou and P. F. Gordon, *J. Acoust. Soc. Am.* **42** (1967) 36-41.
9. H. G. Hopkins, Dynamic Expansion of Spherical Cavities in Metals. Progress in Solid Mechanics. Vol. I, North Holand Publishing Company, Amsterdam, Holand, (1960) 84-164.
10. O. Lekan, Int. J. Engg. Sci. **24** (1986) 1637-54.
11. V. P. W. Shim and S. E. Quah, *J. Appl. Mech.* **65** (1998) 569-79.
12. P. R. Garabedian, Partial Differential Equations. John Wiley, New York (1964)
13. R. Courant and D. Hilbert, Method of Mathematical Physics Vol. II. Wiley Eastern Private Limited, New Delhi (1975).
14. A. Jefferey and T. Taniuty, Non-linear wave Propagation. Academic Press, New York (1964).
15. I. N. Sneddon, Elements of Partial Differential Equations, McGraw-Hill Book Company, Inc. New York (1957).