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A GENERALIZED URYSOHN IMBEDDING AND TYCHONOFF FIXED POINT THEOREM IN TOPOLOGICAL SPACE

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In this paper we prove a generalized Urysohn Imbedding Theorem and then apply it to prove fixed point theorems in a general topological space which are analogues of the Schauder and Tychonoff fixed point theorems.

Key Words : Urysohn Imbedding Theorem; Uniformity; Uniformizable; Midpoint Convexity; Fixed Point Theorem

INTRODUCTION

In this paper we prove a generalized Urysohn Imbedding Theorem. Then with the help of this theorem and the Tychonoff fixed point theorem, we prove some fixed point theorems.

We will need the following standard results and concepts of topology in our subsequent work.

1. If $h : X \to Y$ is a homeomorphism of a topological space X onto a topological space Y, then X is metrizable (resp. uniformizable) if and only if Y is metrizable (resp. uniformizable).

Indeed, if ρ is a metric on Y, then $\rho^* : X \times X \to R$ defined by $\rho^*(x, y) = \rho(hx, hy), x, y \in X$ is a metric on X. The topology generated by ρ^* coincides with the given topology on X (for a proof of this result see Kasriel [1], p. 199). Similarly, if Y is a uniform topological space generated by a family $\{\rho_\alpha : \alpha \in I\}$ of pseudometrics, where I is an index set, then the given topology on X is equivalent to a uniform topology generated by the family $\{\rho^*_\alpha : \alpha \in I\}$ of pseudometrics defined by $\rho^*_\alpha(x, y) = \rho^*_\alpha(hx, hy), x, y \in X$ for each $\alpha \in I$ (for an excellent treatment of uniform spaces we refer to Thron [5]).

Definition 1 — Let $(X, \{\rho_\alpha : \alpha \in I\})$ be a uniform topological space, where the uniform topology of X is generated by the family $\{\rho_\alpha : \alpha \in I\}$ of pseudometrics. Given two points x, y of X, with $x \neq y$, a subset $A = [x, y]$ of X is said to be a segment joining x and y if there is a homeomorphism $h : [0, 1] \to A$ of the closed interval $[0, 1]$ onto A such that $h(0) = x$ and $h(1) = y$ and for each $\alpha \in I$,

$$(*) \quad \rho_\alpha (x, h(t)) = t\, \rho_\alpha (x, y) \text{ and } \rho_\alpha (y, h(t)) = (1 - t)\, \rho_\alpha (x, y).$$

(see Tarafdar [4]).

Definition 2 — A subset K of X is said to be strongly convex (or midpoint convex or simply M-convex) with respect to $\{\rho_\alpha : \alpha \in I\}$ if, given two points, x, y of K with $x \neq y$, there is a segment $[x, y] \subset K$..

Definition 3 — A Banach space X is called strictly convex if for all $x, y \in X$, with $x \neq y$ and $\| x \| = \| y \| = 1$, it follows that $\| (1 - t)\, x + ty \| < 1$ for all $t \in [0, 1]$, or equivalently, if for all $x, y \in X$ with $\| x - y \| = \| x - z \| + \| z - y \|$ there exists $t \in [0, 1]$ such that $z = (1 - t)\, x + ty$.

Lemma 1 — Each nonempty convex subset K of a normed linear space $(X, \| \cdot \|)$ is clearly M-convex.

Indeed, if $x, y \in K$ with $x \neq y$, then $\hat{h}(t) = (1 - t)\, x + ty, t \in [0, 1]$ defines a homeomorphism $\hat{h} : [0, 1] \to [x, y] = A$ satisfying (*) of Definition 1. K is thus M-convex.

Conversely, a M-convex subset K in a strictly convex Banach space $(X, \| \cdot \|)$ with respect to the metric induced by $\| \cdot \|$ is convex.

To see this, let $x, y \in K$ with $x \neq y$. Then there is a segment $[x, y]$ in K; that is, there exists a homeomorphism $h : [0, 1] \to [x, y]$ satisfying (*) of Definition 1.

Then clearly $\rho (x, y) = \rho (x, h(t)) + \rho (h(t), y)$ for each $t \in [0, 1]$. Now let $t \in [0, 1]$ be arbitrary but fixed. Then by the strict convexity of X, there exists a $\lambda \in [0, 1]$ such that $h(t) = (1 - \lambda)\, x + \lambda y$. Now since $\rho (x, h(t)) = t\, \rho (x, y) = \lambda\, \rho (x, y)$, we have $t = \lambda$. Hence it follows that $h(t) = (1 - t)\, x + ty \in K$ for all $t \in [0, 1]$, that is, K is convex.

Lemma 2 — Each non-empty convex subset K of a locally convex Hausdorff topological vector space E generated by the family $\{p_\alpha : \alpha \in I\}$ of seminorms on E is M-convex with respect to the family $\{\rho_\alpha : \alpha \in I\}$ of pseudometrics defined by $\rho_\alpha (x, y) = p_\alpha (x - y)$ for each $\alpha \in I$.

Letting $\alpha \in I$ be arbitrary but fixed, the proof of this is exactly the same as the corresponding part of Lemma 1.

Now let K be a M-convex subset of the locally convex Hausdorff topological vector space E of the form $E = \prod_{\alpha \in I} X_\alpha$, where I is an index set and $(X_\alpha, \| \cdot \|_\alpha)$ is a strictly convex Banach space for each $\alpha \in I$ with respect to the family $\{\rho_\alpha : \alpha \in I\}$ of pseudometrics defined by $\rho_x (x, y) = \| x_\alpha - y_\alpha \|_\alpha, x = \{x_\alpha\}, y = \{y_\alpha\}$. We will prove that K is convex.

Let $x = \{x_\alpha\}$ and $y = \{y_\alpha\} \in K$ with $x \neq y$. Then there is a homeomorphism

$$h : [0, 1] \to [x,. y] \subset K$$

satisfying (*) of Definition 1.

Let $h(t) = \{h_\alpha(t)\}$ where $h_\alpha(t) \in X_\alpha$ and $t \in [0, 1]$.

Now let $\alpha \in I$.be arbitrary but fixed. Then by (*) of Definition 1.

$$\rho_\alpha(x, h(t)) + \rho_\alpha(h(t), y) = \rho_\alpha(x, y),$$

that is,

$$\| x_\alpha - h_\alpha(t) \|_\alpha + \| h_\alpha(t) - y_\alpha \|_\alpha = \| x_\alpha - y_\alpha \|_\alpha.$$

Hence, by strict convexity of X_α, there exista a $\lambda \in [0, 1]$ such that

$$h_\alpha(t) = (1 - \lambda) x_\alpha + \lambda y_\alpha.$$

Thus we have

$$\| x_\alpha - h_\alpha(t) \|_\alpha = \lambda \| x_\alpha - y_\alpha \|_\alpha = \lambda \rho_\alpha(x, y)$$

and by (*) of Definbition 1, we have

$$\rho_\alpha(x, h(t)) = \| x_\alpha - h_\alpha(t) \|_\alpha = \rho_\alpha(x, h(t)) = t \rho_\alpha(x, y) = t \| x_\alpha - y_\alpha \|_\alpha.$$

Hence, it follows $\lambda = t$ if $x_\alpha \neq y_\alpha$, and it is clear that we can assume $t = \lambda$ if $x_\alpha = y_\alpha$.

Thus for each $t \in [0, 1], h_\alpha(t) = (1 - t) x_\alpha + t y_\alpha$. As α is arbitrary, $h(t) = (1 - t) x + t y \in K$. Hence K is convex.

We will need the following generalized version of the Urysohn Imbedding Theorem. We should point out that the result of this theorem is equivalent to a result in a uniform space but the particular form we need requires proof. Also to make our paper self-contained we will include a complete proof.

***Theorem* 1** — *(Generalized Urysohn Imbedding Theorem).*

Let $\{\tau_\alpha : \alpha \in I\}$ be a family of second countable normal topologies on a non-empty set X and let $\tau = V_{\alpha \in I} \tau_\alpha$, the supremum topology of $\{\tau_\alpha : \alpha \in I\}$ be T_1 (for the definition of supremum topology see Wilansky [6] p. 90).

Then X is homeomorphic to a subspace of the locally convex Hausdorff topological vector space $E = \prod_{\alpha \in I} E_\alpha$, where $E_\alpha = l^2$ (we denote l^2 by $\| \cdot \|_2$) for each $\alpha \in I$.

We note that E is the locally convex Hausdorff topological vector space generated by the family $\{p_\alpha : \alpha \in I\}$ of seminorms of E, where for each $\alpha \in I, p_\alpha$ is defined by $p_\alpha(x) = p_\alpha(\{x_\alpha\}) = \| x_\alpha \|_2$.

Clearly, E is Hausdorff

Indeed, if $\{x_\alpha\} = x \neq y = \{y_\alpha\}$, then $x_\beta \neq y_\beta$ for some $\beta \in I$.

Hence $p_\beta(x - y) = p_\beta(x_\beta - y_\beta) = \| x_\beta - y_\beta \| = \neq 0$.

Also E is a uniform space generated by the uniformity $\{\rho_\alpha : \alpha \in I\}$ of pseudometrics on E, where each ρ_α is defined by $\rho_\alpha(x, y) = p_\alpha(x - y)$.

PROOF : Wherever possible, we follow and maintain the notations of Simmons ([3], p. 138).

Let $\alpha \in I$ be fixed but arbitrary. By normality and second countability of τ_α, there exists a countable collection of ordered pairs $(G_{\alpha(i)}, G_{\alpha(j)})$ of basic τ_α-open sets with $\overline{G}_{\alpha(i)} \subset G_{\alpha(j)}$. We enumerate these as a sequence $\{P_{\alpha(n)}\}$. Now for each ordered pair $P_{\alpha(n)} = (G_{\alpha(i)}, G_{\alpha(j)})$ there is by Urysohn Lemma a continuous function $f_{\alpha(n)}: (X, \tau_\alpha) \to [0, 1]$ such that $f_{\alpha(n)}\,\overline{(G_{\alpha(i)})} = 0$ and $f_{\alpha(n)}\,(G^c_{\alpha(j)}) = 1$

($\bar{A}$ stands for the closure of A and A^c the complement of A for any subset A of X).

Now we define the mappings

$$h_\alpha: (X, \tau_\alpha) \to l^2 \text{ by}$$

$$h_\alpha(x) = \left\{ f_{\alpha(1)}(x), \frac{1}{2} f_{\alpha(2)}(x), \ldots, \frac{1}{n} f_{\alpha(n)}(x), \ldots \right\}, x \in X.$$

Now it is easy to see that h_α is continuous from the τ_α topology of X to the norm topology of l^2 (see Simmons [3], p. 138).

Indeed, let $\varepsilon > 0$ be given and $x_0 \in X$.

Since $\sum_1^\infty \frac{1}{n^2} < \infty$, we can choose a positive integer N_0 such that

$$\sum_{n=N_0+1}^{\infty} \frac{1}{n^2} < \frac{\varepsilon^2}{2}.$$

Now

$$\| h_\alpha(y) - h_\alpha(x_0) \|^2 = \sum_{n=1}^{\infty} \frac{| f_{\alpha(n)}(y) - f_{\alpha(n)}(x_0) |^2}{n^2}$$

$$\leq \sum_{n=1}^{N_0} \frac{| f_{\alpha(n)}(y) - f_{\alpha(n)}(x_0) |^2}{n^2} + \frac{\varepsilon^2}{2}.$$

By the continuity of $f_{\alpha(1)}, f_{\alpha(2)}, \ldots, f_{\alpha(N_0)}$, there exist open neighbourhoods $U_{\alpha(1)}, U_{\alpha(2)}, \ldots, U_{\alpha(N_0)}$ of x_0 such that

$$\frac{| f_{\alpha(n)}(y) - f_{\alpha(n)}(x_0) |^2}{n^2} < \frac{\varepsilon^2}{2N_0} \text{ for all } y \in U_{\alpha(n)}, n = 1, 2, \ldots, N_0.$$

Now for each $y \in U_\alpha = \bigcap_{n=1}^{N_0} U_{\alpha(n)}$ it follows that

$$\| h_\alpha(y) - h_\alpha(x_0) \|^2 < N_0 \frac{\varepsilon^2}{2N_0} + \frac{\varepsilon^2}{2} + \varepsilon^2.$$

This proves the continuity of h_α.

We next define the mapping

$$H: (X, \tau) \to \prod_{\alpha \in I} E_\alpha = E \text{ by } H(x) = \prod_{\alpha \in I} h_\alpha(x),\ x \in X$$

It follows that H is continuous since each h_α is continuous.

We now prove that H is one-to-one. Let x and y be any two points of X with $x \neq y$. Since (X, τ) is T_1 there exists an open set G such that $x \in G$ and $y \notin G$. Now as $\tau = V_{\alpha \in I} T_\alpha$, there exist τ_{α_i}-open sets G_{α_i}, $i = 1, 2, \ldots, n$ such that $x \in \bigcap_{i=1}^{n} G_{\alpha_i} \subset G$.

As $y \notin G$ it follows that $y \notin G_{\alpha_m}$ for some $m = 1, 2, \ldots, n$.

For the sake of convenience we write $\alpha = \alpha_m$. By the normality of $\tau_{\alpha_m} = \tau_\alpha$ there exists a τ_α-basic open set $G_{\alpha(i)}$ such that $x \in G_{\alpha(i)} \subset \overline{G}_{\alpha(i)} \subset G_\alpha$.

$$\text{Let } P_{\alpha(n)} = (G_{\alpha(i)}, G_\alpha) = (G_{\alpha_m(i)}, G_{\alpha_m}).$$

Then

$$f_{\alpha(n)}(x) = 0 \text{ and } f_{\alpha(n)}(y) = 1.$$

This implies that $h_\alpha(x) \neq h_\alpha(y)$, that is,$h_{\alpha_m}(x) \neq h_{\alpha_m}(y)$.

Hence,

$$H(x) \neq H(y).$$

Finally, we prove that $H^{-1} : H(X) \to X$ is continuous.

Let $H(x_0) \in H(x)$ be arbitrary. Let G be a basic open neighbourhood of x_0. Then $G = \bigcap_{j=1}^{m} G_{\alpha_j}$, where G_{α_j} is a τ_{α_j}-open set for $j = 1, 2, \ldots, m$.

It suffices to prove the existence of $\varepsilon_j > 0$ for $j = 1, 2, \ldots, m$ such that the open set $S = \prod_{\alpha \in I} S_\alpha$ in $\prod_{\alpha \in I} E_\alpha$, where $S_\alpha = S_{\varepsilon_j}(Q_{\alpha_j}(H(x_0)))$

$$= \left\{ Q_{\alpha_j}(H(x)) \in E_{\alpha_j} : \| Q_{\alpha_j} H(x) - Q_{\alpha_j} H(x_0) \|^2 < \varepsilon_j \right\},$$

for $j = 1, 2, \ldots, m$, $S_\alpha = E_\alpha$ for $j \neq 1, 2, \ldots, m$ and Q_{α_j} is the projection of E onto E_{α_j} with $H(S) \subset G$. For $j = 1, 2, \ldots, m$ there exists by normalities of E_{α_j} a basic τ_j-open set $\hat{G}_{\alpha_j}$ with $x_0 \in \Gamma_{\alpha_\varphi} \subset \overline{G}_{\alpha_j} \subset G_{\alpha_j}$.

We now consider the pair $(\hat{G}_{\alpha_j}, \hat{G}_{\alpha_j})$ for $j = 1, 2, \ldots, m$. Then for some integers $k_1, k_2, \ldots, k_m$, we have

$$P_{\alpha_j(kj)} = (\hat{G}_{\alpha_j}, G_{\alpha_j}), j = 1, 2, \ldots, m.$$

Now for each $j = 1, 2, \ldots, m$ we choose

$$\varepsilon_j < \frac{1}{2k_j}$$

Now for $i = 1, 2, \ldots, m$

$$\| Q_{\alpha_j (k_j)}, H(x) - Q_{\alpha_j (k_j)} H(x_0) \| < \varepsilon_j \Rightarrow \sum_{n=1}^{\infty} \left| \frac{f_{\alpha_j (n)}(x) - f_{\alpha_j (n)}(x_0)}{n} \right|^2 < \left(\frac{1}{2kj} \right)^2$$

$$\Rightarrow | f_{\alpha_j (k_j)}(x) - f_{\alpha_j (kj)}(x_0) | < \frac{1}{2}.$$

But since $x_0 \in \hat{G}_{\alpha_j}, f_{\alpha_j (kj)}(x_0) = 0$. Hence, $f_{\alpha_j (k_j)}(x) < \frac{1}{2}$.

Thus $x \in G_{\alpha_j}$ as $f_{\alpha_j (kj)}(G^c_{\alpha_j}) = 0$, that is, $x \in \bigcap_{j=1}^{m} G_{\alpha_j} = G$.

Thus we have proved that $H(x) \in S \Rightarrow x \in G$. Hence H^{-1} is continuous.

Corollary 1 — (Urysohn Imbedding Theorem)

If (X, τ) is a second countable normal topological space. then $H(X)$ is homeomorphic to a subspace of l^2, that is, there is a homeomorphism $H : X \to l^2$ and X is metrizable by the metric

$$\rho^* (x, y) = \| H(x) - H(y) \|_2$$

where $\| \cdot \|_2$ is the usual norm in l^2.

PROOF : We take $I = \{1\}$ and apply Theorem 1.

***Theorem* 2** — *Let (X, τ) be a T_1-space where $\tau = V_{\alpha \in I} \tau_\alpha$ and, for each $\alpha \in I, \tau_\alpha$ is a second countable normal topology on X. Then every non-empty subset K of X is uniformizable. If K is M-convex with respect to the family $\{\rho_\alpha : \alpha \in I\}$ and $\{\rho^*_\alpha : \alpha \in I\}$, respectively given in Theorem 1, and in Lemma 1, and compact subset of X and $T : K \to K$ is a continuous mapping of K into itself, then T has a fixed point.*

PROOF : By Theorem 1, $H(X)$ is uniformizable, where $H : X \to E$ is the homeomorphism as obtained in Theorem 1 and $E = \prod_{\alpha \in I} E_\alpha, E_\alpha$ being l^2 for each $\alpha \in I$. Hence X is a uniform space generated by the family $\{\rho^*_\alpha : \alpha \in I\}$ of pseudometrics defined by

$$\rho^*_\alpha (x, y) = \rho_\alpha (H(x), H(y)) = p_\alpha (H(x) - H(y)), \ x, y \in X,$$

p_α being the seminorm in $E_\alpha = l^2$ for each $\alpha \in I$.

Since K is M-convex, $H(K)$ is M-convex. To see this, let $H(x)$ and $H(y)$ be any two points of $H(K)$ with $H(x) \neq H(y)$. Then $x, y \in K$ with $x \neq y$. Since K is M-convex there is a unique homeomorphism $h : [0, 1] \to [x, y] = A$ such that $h(0) = x$, $h(1) = y$ and for each $\alpha \in I, \rho^*_\alpha (x, h(t)) = t \rho^*_\alpha (x, y), t \in [0, 1]$ and $[x, y] \subset K$.

Then clearly $\hat{h} = H \circ h : [0, 1] \to [H(x), H(y)] = \hat{A}$ is a homeomorphism such that $H \circ h(0) = H(x)$ and $H \circ h(1) = H(y)$, and for each $\alpha \in I, \rho_\alpha (H(x), \hat{h}(t)) = \rho_\alpha (H(x), H \circ h(t)) =$ $= t \rho^*_\alpha (x, y) = \rho_\alpha (H(x), H(y))$. Hence $[H(x), H(y)] \subset H(K)$.

Thus $H(K)$ is M-convex. Thus by Lemma 1 $H(K)$ is a convex subset of E and $H(K)$ is compact by the homeomorphism of H.

Now the mapping $F = HTH^{-1} : H(K) \to H(K)$ is a continuous mapping of the compact convex subset $H(K)$ of a locally convex Hausdorff space E into itself and has a fixed point $u_0 \in H(K)$ by Tychonoff fixed point theorem. Thus $F(u_0) = HTH^{-1}(u_0) = u_0$.

Let $H^{-1}(u_0) = x_0 \in K$. Then $HT(x_0) = u_0$, that is, $Tx_0 = H^{-1}(u_0) = x_0$ i.e. x_0 is a fixed point of T.

Corollary 2 — If K is a non-empty compact and M-convex subset with respect to the metric ρ and ρ^* obtained in Corollary 1 and Lemma 1, of a second countable normal T_1-space X and $T : K \to K$ is a continuous mapping, then T has a fixed point.

PROOF : We take $I = \{1\}$ and apply Theorem 2.

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