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AN INVERSE PROBLEM OF THE THREE-DIMENSIONAL WAVE EQUATION FOR A GENERAL ANNULAR VIBRATING MEMBRANE

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This paper deals with the very interesting problem about the influence of piecewise smooth boundary conditions on the distribution of the eigenvalues of the negative Laplacian in R^3. The asymptotic expansion of the trace of the wave operator $\hat{\mu}(t) = \sum_{\upsilon=1}^{\infty} \exp(-it\mu_\upsilon^{1/2})$ for small $|t|$ and $i = \sqrt{-1}$, where $\{\mu_\nu\}_{\nu=1}^{\infty}$ are the eigenvalues of the negative Laplacian $-\nabla^2 = -\sum_{k=1}^{3}\left(\frac{\partial}{\partial x^k}\right)^2$ in the (x^1, x^2, x^3)-space, is studied for an annular vibrating membrane Ω in R^3 together with its smooth inner boundary surface S_1 and its smooth outer boundary surface S_2. In the present paper, a finite number of Dirichlet, Neumann and Robin boundary conditions on the piecewise smooth components S_j^* $(j = 1, ..., m)$ of S_1 and S_j^* $(j = m+1, ..., n)$ of S_2 such that $S_1 = \bigcup_{j=1}^{m} S_j^*$ and $S_2 = \bigcup_{j=m+1}^{n} S_j^*$ are considered. The basic problem is to extract information on the geometry of the annular vibrating membrane Ω from complete knowledge of its eigenvalues by analyzing the asymptotic expansions of the spectral function $\hat{\mu}(t)$ for small $|t|$.

Key Words : Inverse Problem; Wave Equation; Annular Vibrating Membrane; Eigenvalues; Piecewise Smooth Boundary Conditions; Spectral Function; Heat Kernel

1. INTRODUCTION

The underlying inverse problem is to deduce some geometric quantities associated with a bounded domain in R^3 from complete knowledge of the eigenvalues of the negative Laplacian.

Let Ω be a simply connected bounded domain in R^3 with a smooth bounding surface S. Consider the Robin problem

$$-\nabla^2 u = \mu u \text{ in } \Omega, \qquad \text{... (1.1)}$$

$$\left(\frac{\partial}{\partial n} + \gamma\right) u = 0 \text{ on } S, \qquad \text{... (1.2)}$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the outward normal to S and γ is a positive constant impedance, with $\phi \in C^2(\Omega) \cap C(\bar{\Omega})$.

Denote its eigenvalues, counted according to multiplicity, by

$$0 < \mu_1 \le \mu_2 \le \ldots \le \mu_\nu \le \ldots \to \infty \text{ as } \nu \to \infty \qquad \ldots (1.3)$$

Zayed *et al.*[1] have discussed the problem (1.1)-(1.2) for small/large impedance γ and have determined some geometrical quantities of Ω using the wave equation approach by analyzing the asymptotic expansion of the spectral function

$$\hat{\mu}(t) = \sum_{\nu=1}^{\infty} \exp(-it\mu_\nu^{1/2}) \text{ as } |t| \to 0; \qquad \ldots (1.4)$$

which represents a tempered distribution for $-\infty < t < \infty$ and $i = \sqrt{-1}$.

Note that, when γ is small, the Robin boundary condition (1.2) looks approximately like the Neumann boundary condition, while when γ is large, the Robin boundary condition (1.2) looks approximately like the Dirichlet boundary condition provided $\frac{\partial}{\partial n}$ remains finite.

Case 1.1 — If $\gamma = 0$ (the Neumann problem)

$$\hat{\mu}(t) = \frac{V}{4\pi t}\delta(-|t|) + \frac{|S|}{8\pi^2|t|} \, sign\, t$$

$$+ \frac{sign\, t}{12\pi^2}\int_S H(z)\,dz + O(t\, sign\, t), \text{ as } |t| \to 0, \qquad \ldots (1.5)$$

Case 1.2 — If $\gamma \to \infty$ (the Dirichlet problem)

$$\hat{\mu}(t) = \frac{V}{4\pi t}\delta(-|t|) - \frac{|S|}{8\pi^2|t|} \, sign\, t +$$

$$\frac{sign\, t}{12\pi^2}\int_S H(z)\,dz + O(t \text{ sign } t), \text{ as } |t| \to 0, \qquad \ldots (1.6)$$

where $\delta(-|t|)$ is the Dirac delta function and

$$sign\, t = \begin{cases} 1 & t > 0, \\ 0 & t = 0, \\ -1 & t < 0. \end{cases} \qquad \ldots (1.7)$$

In these formulae V, $|S|$ and $H(z)$ are respectively, the volume, the surface area, the mean curvature of Ω, such that $H(z) = \frac{1}{2}\left[\frac{1}{R_1(z)} + \frac{1}{R_2(z)}\right]$, where $R_1(z)$ and $R_2(z)$ are the principal radii of curvature. Note that the sign $\pm$ of the second term of $\hat{\mu}(t)$ determines whether we have the Neumann or Dirichlet problem.

Case 1.3 — (the mixed problem)

If $|S_1|$ is the surface area of the component S_1 of the boundary surface S with the Neumann boundary condition, and if $|S_2|$ is the surface area of the remaining component $S_2 = S \setminus S_1$ of S with the Dirichlet boundary condition, then with reference to the articles[2,3], we get

$$\hat{\mu}(t) = \frac{V}{4\pi t}\,\delta(-|t|) + \frac{|S_1| - |S_2|)}{8\pi^2 t}\ \textit{sign}\ t$$

$$+ \frac{\textit{sign}\ t}{12\pi^2}\left\{\int_{S_1} H(z)\,dz + \int_{S_2} H(z)\,dz\right\} + O(t\ \textit{sign}\ t) \text{ as } |t| \to 0. \qquad \text{... (1.8)}$$

Note that the order term $O\ (t\ sign\ t)$ in these formulae is yet undetermined. So, in the present paper, we discuss what geometric quantities are contained in this order term in the case Ω is an annular vibrating membrane together with piecewise smooth boundary conditions (1.10) and (1.11) stated below.

The object of this paper is to discuss the following more general inverse problem: Let Ω be a general annular vibrating membrane in R^3 consisting of a simply connected bounded inner domain Ω_1 with a smooth bounding surface S_1 and a simply connected bounded outer domain $\Omega_2 \supset \overline{\Omega}_1$ with a smooth bounding surface S_2 where $\overline{\Omega}_1 = \Omega_1 \cup S_1$. Suppose that the eigenvalues (1.3) are given exactly for the Helmholtz equation

$$-\nabla^2 u = \mu u \text{ in } \Omega, \qquad \text{... (1.9)}$$

together with the following Dirichlet, Neumann and Robin boundary conditions on the piecewise smooth components S_j^* $(j = 1, \ldots, m)$ of S_1 :

$$\left\{\begin{array}{lll} u = 0, & \text{on } S_j^* & (j = 1, \ldots, k), \\ \dfrac{\partial u}{\partial n_j} = 0, & \text{on } S_j^* & (j = k+1, \ldots, l), \\ \left(\dfrac{\partial}{\partial n_j} + \gamma_j\right) u = 0 & \text{on } S_j^* & (j = l+1, \ldots, m), \end{array}\right\} \qquad \text{... (1.10)}$$

where $S_1 = \bigcup_{j=1}^{m} S_j^*$ as well as the following Dirichlet, Neumann and Robin boundary conditions on the piecewise smooth components S_j^* $(j = m+1, \ldots, n)$ of S_2 :

$$\left\{\begin{array}{lll} u=0, & \text{on } S_j^* & (j=m+1,\dots,N), \\ \dfrac{\partial u}{\partial n_j}=0, & \text{on } S_j^* & (j=N+1,\dots,p), \\ \left(\dfrac{\partial}{\partial n_j}+\gamma_j\right)u=0 & \text{on } S_j^* & (j=p+1,\dots,n), \end{array}\right\} \quad \dots (1.11)$$

where $S_2 = \bigcup_{j=m+1}^{n} S_j^*$ and γ_j are piecewise smooth positive constant impedances.

The basic problem is to determine some geometric quantities (e.g., the volume of Ω, the surface area, the mean curvature, and the Gaussian curvature) associated with the main problem (1.9)-(1.11), using the wave equation approach by analyzing the asymptotic expansions of the spectral function $\hat{\mu}(t)$ for small $|t|$.

Note that the special cases of the main problem (1.9)-(1.11) have been discussed by Zayed *et al.*[1, 2, 3], Abdel-Halim[4] and Zayed[5, 6, 7, 8]. Therefore, this problem can be considered as a more general one which does not seem to have been investigated elsewhere.

We close this section with the remark that an alternative to the spectral function (1.4) is to study the trace of the heat kernel $\Theta(t) = \sum_{\upsilon=1}^{\infty} \exp(-t\mu_\upsilon)$ as $t \to 0$ (see for example[9, 10, 11, 12, 13]). But, it is well known that the wave equation methods have given very strong result; the definitive one is that of Hormander[14] who has studied the disribution $\hat{\mu}(t) = tr\,[exp\,(-it\,P)]$ near $t = 0$ for an elliptic positive semi-definite pseudodifferential operator P in R^n of order m. Therefore, in the present paper, we concentrate our efforts on a study of the asymptotic expansion of $\hat{\mu}(t)$ as $|t| \to 0$ for the main problem (1.9)-(1.11).

2. Formulation of the Mathematical Problem

With reference to the articles[1, 2, 3], it can be easily seen that the spectral function $\hat{\mu}(t)$ associated with the main problem (1.9)-(1.11) is given by

$$\hat{\mu}(t) = \iiint_{\Omega} G(x, x; t)\, dx, \quad \dots (2.1)$$

where $G(x_1, x_2; t)$ is the Green's function for the wave equation

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right) G(x_1, x_2; t) = 0 \text{ in } \Omega \times \{-\infty < t < \infty\}, \quad \dots (2.2)$$

subject to the boundary conditions (1.10)-(1.11) and the initial conditions

$$\lim_{t \to 0} G(x_1, x_2; t) = 0, \quad \lim_{t \to 0} \frac{\partial G(x_1, x_2; t)}{\partial t} = \delta(x_1 - x_2), \quad \text{... (2.3)}$$

where $\delta(x_1 - x_2)$ is the Dirac delta function located at the source point $x_1 = x_2$. Let us write

$$G(x_1, x_2; t) = G_0(x_1, x_2; t) + \chi(x_1, x_2; t), \quad \text{... (2.4)}$$

where $$G_0(x_1, x_2; t) = \frac{1}{4\pi t} \delta(|x_1 - x_2| - |t|), \quad \text{... (2.5)}$$

is the "fundamental solution" of the wave eq. (2.2) while $\chi(x_1, x_2; t)$ is the "regular solution" chosen in such a way that $G(x_1, x_2; t)$ satisfies the piecewise smooth boundary conditions (1.10)-(1.11). On setting $x_1 = x_2 = x$, we find that

$$\hat{\mu}(t) = \frac{V}{4\pi t} \delta(-|t|) + K(t), \quad \text{... (2.6)}$$

where

$$K(t) = \iiint_{\Omega} \chi(x, x; t)\, dx. \quad \text{... (2.7)}$$

The problem now is to determine the asymptotic expansions of $K(t)$ for small $|t|$. In what follows, we shall use Fourier transforms with respect to $-\infty < t < \infty$ and use $-\infty < \eta < \infty$ as the Fourier transform parameter. Thus, we define for $i = \sqrt{-1}$ that

$$\hat{G}(x_1, x_2; \eta) = \int_{-\infty}^{+\infty} e^{-2\pi i \eta t} G(x_1, x_2; t)\, dt. \quad \text{... (2.8)}$$

An application of the Fourier transform to the wave eq. (2.2) shows that $\hat{G}(x_1, x_2; \eta)$ satisfies the reduced wave equation

$$(\nabla^2 + 4\pi^2 \eta^2) \hat{G}(x_1, x_2; \eta) = -\delta(x_1 - x_2) \text{ in } \Omega, \quad \text{... (2.9)}$$

together with boundary conditions (1.10)-(1.11).

The asymptotic expansions of $K(t)$, for small $|t|$, may then be deduced directly from the asymptotic expansions of $\hat{K}(\eta)$, for large $|\eta|$, where

$$\hat{K}(\eta) = \iiint_{\Omega} \hat{\chi}(x, x; \eta)\, dx. \quad \text{... (2.10)}$$

3. Derivation of the Results

It is well known (see for example[1,2,3]) that the reduced wave eq. (2.9) has the fundamental solution

$$\hat{G}_0(x_1, x_2; \eta) = \frac{\exp(-2\pi \eta r_{x_1 x_2})}{4\pi r_{x_1 x_2}}, \quad \text{... (3.1)}$$

where $r_{x_1 x_2} = | x_1 - x_2 |$ is the distance between the points $x_1 = (x_1^1, x_1^2, x_1^3)$ and $x_2 = (x_2^1, x_2^2, x_2^3)$ of the annular region $\Omega \subseteq R^3$. The existence of the solution (3.1) enables us to construct integral equations for $\hat{G}(x_1, x_2; \eta)$ satisfying the boundary conditions (1.10)-(1.11). Therefore, if we consider the problem (1.9)-(1.11) with the case $0 < \gamma_j << 1$ ($j = l+1, \ldots, b$), $\gamma_j >> 1$ ($j = b+1, \ldots, m$), $0 < \gamma_j << 1$ ($j = p+1, \ldots, c$), and $\gamma_j > 1$ ($j = c+1, \ldots, n$) then, Green's theorem gives the following integral equation:

$$\hat{G}(x_1, x_2; \eta) = \frac{\exp(-2\pi i \eta r_{x_1 x_2})}{4\pi r_{x_1 x_2}}$$

$$-\frac{1}{2\pi} \sum_{j=1}^{k} \int_{S_j^*} \left[\frac{\partial}{\partial n_{jy}} \hat{G}(x_1, y; \eta) \right] \frac{\exp(-2\pi i \eta r_{yx_2})}{r_{yx_2}} dy$$

$$+\frac{1}{2\pi} \sum_{j=k+1}^{l} \int_{S_j^*} \hat{G}(x_1, y; \eta) \left[\frac{\partial}{\partial n_{jy}} \frac{\exp(-2\pi i \eta r_{yx_2})}{r_{yx_2}} \right] dy$$

$$+\frac{1}{2\pi} \sum_{j=l+1}^{b} \int_{S_j^*} \hat{G}(x_1, y; \eta) \left[\left(\frac{\partial}{\partial n_{jy}} + \gamma_j \right) \frac{\exp(-2\pi i \eta r_{yx_2})}{r_{yx_2}} \right] dy$$

$$-\frac{1}{2} \sum_{j=b+1}^{m} \int_{S_j^*} \left[\frac{\partial}{\partial n_{jy}} \hat{G}(x_1, y; \eta) \right] \left(1 + \gamma_j^{-1} \frac{\partial}{\partial n_{jy}} \right) \frac{\exp(-2\pi i \eta r_{yx_2})}{r_{yx_2}} dy$$

$$+\frac{1}{2\pi} \sum_{j=m+1}^{N} \int_{S_j^*} \left[\frac{\partial}{\partial n_{jy}} \hat{G}(x_1, y; \eta) \right] \frac{\exp(-2\pi i \eta r_{yx_2})}{r_{yx_2}} dy$$

$$-\frac{1}{2\pi} \sum_{j=N+1}^{p} \int_{S_j^*} \hat{G}(x_1, y; \eta) \left[\frac{\partial}{\partial n_{jy}} \frac{\exp(-2\pi i \eta r_{yx_2})}{r_{yx_2}} \right] dy$$

$$-\frac{1}{2\pi} \sum_{j=p+1}^{c} \int_{S_j^*} \hat{G}(x_1, y; \eta) \left[\left(\frac{\partial}{\partial n_{jy}} + \gamma_j \right) \frac{\exp(-2\pi i \eta r_{yx_2})}{r_{yx_2}} \right] dy$$

$$+\frac{1}{2\pi} \sum_{j=c+1}^{n} \int_{S_j^*} \left[\frac{\partial}{\partial n_{jy}} \hat{G}(x_1, y; \eta) \right] \left(1 + \gamma_j^{-1} \frac{\partial}{\partial n_{jy}} \right) \frac{\exp(-2\pi i \eta r_{yx_2})}{r_{yx_2}} dy. \quad \ldots (3.2)$$

On applying the iteration method (see for example[1]) to the integral eq. (3.2), we obtain an explicit form (See (15)) of the Green's function $\hat{G}(x_1, x_2; \eta)$ for the main problem (1.9)-(1.11). On using argument similar to that obtained in [1, 2, 3], we deduce after some lengthy mathematical analysis, that the Green's function $\hat{G}(x_1, x_2; \eta)$ has a regular part in the following asymptotic form

$$\hat{\chi}(x_1, x_2; \eta) = \sum_{j=1}^{n} \hat{\chi}_j(x_1, x_2; \eta), \qquad \ldots (3.3)$$

where

(*a*) if x_1 and x_2 belong to sufficiently small domains $\mathcal{D}(I_j)$ $(j = 1, \ldots, k)$, then,

$$\hat{\chi}_j(x_1, x_2; \eta) = \frac{\exp(-2\pi i \eta \rho_{12})}{8\pi\rho_{12}} + O\left\{\rho_{12}^{-1} \exp(-A_j \eta i \rho_{12})\right\}, \text{ as } |\eta| \to \infty, \qquad \ldots (3.4)$$

(*b*) if x_1 and x_2 belong to sufficiently small domains $\mathcal{D}(I_j)$ $(j = k+1, \ldots, l)$, then,

$$\hat{\chi}_j(x_1, x_2; \eta) = -\frac{\exp(-2\pi i \eta \rho_{12})}{8\pi\rho_{12}} + O\left\{\rho_{12}^{-1} \exp(-A_j \eta i \rho_{12})\right\}, \text{ as } |\eta| \to \infty, \ldots (3.5)$$

(*c*) if x_1 and x_2 belong to sufficiently small domains $\mathcal{D}(I_j)$ $(j = l+1, \ldots, b)$, then,

$$\hat{\chi}_j(x_1, x_2; \eta) = -\frac{1}{8\pi}\left\{1 - \gamma_j\left(\frac{\partial}{\partial \xi_1^3}\right)^{-1}\right\}\frac{\exp(-2\pi i \eta \rho_{12})}{\rho_{12}}$$
$$+ O\left\{\rho_{12}^{-1} \exp(-A_j \eta i \rho_{12})\right\}, \text{ as } |\eta| \to \infty, \qquad \ldots (3.6)$$

(*d*) if x_1 and x_2 belong to sufficiently small domains $\mathcal{D}(I_j)$ $(j = b+1, \ldots, m)$, then,

$$\hat{\chi}_j(x_1, x_2; \eta) = \frac{1}{8\pi}\left\{1 - \gamma_j^{-1}\left(\frac{\partial}{\partial \xi_1^3}\right)\right\}\frac{\exp(-2\pi i \eta \rho_{12})}{\rho_{12}}$$
$$+ O\left\{\rho_{12}^{-1} \exp(-A_j \eta i \rho_{12})\right\}, \text{ as } |\eta| \to \infty, \qquad \ldots (3.7)$$

(*e*) if x_1 and x_2 belong to sufficiently small domains $\mathcal{D}(I_j)$ $(j = m+1, \ldots, N)$, then,

$$\hat{\chi}_j(x_1, x_2; \eta) = -\frac{\exp(-2\pi i \eta \rho_{12})}{8\pi\rho_{12}}$$
$$+ O\left\{\rho_{12}^{-1} \exp(-A_j \eta i \rho_{12})\right\},$$

as $|\eta| \to \infty$, ... (3.8)

(f) if x_1 and x_2 belong to sufficiently small domains $\mathcal{D}(I_j)$ $(j = N+1, \ldots, p)$, then,

$$\hat{\chi}_j(x_1, x_2; \eta) = \frac{\exp(-2\pi i \eta \rho_{12})}{8\pi\rho_{12}} + O\left\{\rho_{12}^{-1}\exp(-A_j \eta i \rho_{12})\right\}, \text{ as } |\eta| \to \infty, \quad \ldots (3.9)$$

(g) if x_1 and x_2 belong to sufficiently small domains $\mathcal{D}(I_j)$ $(j = p+1, \ldots, c)$, then,

$$\hat{\chi}_j(x_1, x_2; \eta) = \frac{1}{8\pi}\left\{1 - \gamma_j\left(\frac{\partial}{\partial \xi_1^3}\right)^{-1}\right\}\frac{\exp(-2\pi i \eta \rho_{12})}{\rho_{12}} + O\left\{\rho_{12}^{-1}\exp(-A_j \eta i \rho_{12})\right\}, \text{ as } |\eta| \to 0, \quad \ldots (3.10)$$

(h) if x_1 and x_2 belong to sufficiently small domains $\mathcal{D}(I_j)$ $(j = c+1, \ldots, n)$, then,

$$\hat{\chi}_j(x_1, x_2; \eta) = -\frac{1}{8\pi}\left\{1 - \gamma_j^{-1}\left(\frac{\partial}{\partial \xi_1^3}\right)\right\}\frac{\exp(-2\pi i \eta \rho_{12})}{\rho_{12}} + O\left\{\rho_{12}^{-1}\exp(-A_j \eta i \rho_{12})\right\}, \text{ as } |\eta| \to \infty, \quad \ldots (3.11)$$

where A_j $(j = 1, \ldots, n)$ are positive constants, and ρ_{12} is the distance between the points $\xi_1 = (\xi_1^1, \xi_1^2, \xi_1^3)$ and $\xi_2 = (\xi_2^1, \xi_2^2 - \xi_2^3)$ of the upper half plane $\xi^3 > 0$ while $\mathcal{D}(I_j)$ are defined as in Zayed *et al.*[1, 2, 3], Abdel-Halim[4] and Zayed[8].

With reference to the articles[1,2,3,4], it can be seen that for $\xi^3 \geq h_j > 0$, $(j = 1, \ldots, n)$ where h_j are sufficiently small numbers, that the functions $\hat{\chi}_j(x, x; \eta)$ are of order $O\left\{\exp(-4\eta i A_j h_j)\right\}$, and the integral of the function $\hat{\chi}(x, x; \eta)$ over the annular region $\Omega \subseteq R^3$ can be approximated in the following way (see (2.10)).

$$\hat{K}(\eta) = \sum_{j=m+1}^{n} \int_{S_j^*} \int_{\xi^3=0}^{h_j} \hat{\chi}_j(x, x; \eta)\left\{1 - 2\xi^3 H_1 + (\xi^3)^2 N_1\right\} d\xi^3\, dS_j^*$$

$$- \sum_{j=1}^{m} \int_{S_j^*} \int_{\xi^3=0}^{h_j} \hat{\chi}_j(x, x; \eta)\left\{1 + 2\xi^3 H_1^* + (\xi^3)^2 N_1^*\right\} d\xi^3\, dS_j^*$$

$$+ \sum_{j=1}^{n} O\left\{\exp(-4\eta i A_j h_j)\right\} \text{ as } |\eta| \to \infty. \quad \ldots (3.12)$$

If the e^{λ}-expansions of $\hat{\chi}_j(x, x; \eta)$ (see [1, 2, 3, 4]) are introduced into (3.12) and with the help of the formula (7.2) of Sec. 7 in [8], we deduce after inverting Fourier transforms and using (2.6) that the asymptotic expansion of $\hat{\mu}(t)$ has the form See Zayed[15] :

$$\hat{\mu}(t) = \frac{a_1}{t}\,\delta(-|t|) + \frac{a_2}{|t|} \operatorname{sign} t + a_3 \operatorname{sign} t + a_4\, t \operatorname{sign} t + O(t^2 \operatorname{sign} t), \quad \text{as } |t| \to 0, \qquad \dots (3.13)$$

where the coefficients $a_J\,(J = 1-4)$ can be written as follows :

$$a_1 = \frac{V}{4\pi},$$

$$a_2 = \frac{1}{8\pi^2}\left\{\left[\sum_{j=k+1}^{l} |S_j^*| + \sum_{j=l+1}^{b} |S_j^*|\right] - \left[\sum_{j=1}^{k} |S_j^*| + \sum_{j=b+1}^{m}\left(|S_j^*| - 2\gamma_j^{-1}\int_{S_j^*} H_1^*\, dS_j^*\right)\right] + \left[\sum_{j=N+1}^{P} |S_j^*| + \sum_{j=P+1}^{c} |S_j^*|\right] - \left[\sum_{j=m+1}^{N} |S_j^*| + \sum_{j=c+1}^{n}\left(|S_j^*| - 2\gamma_j^{-1}\int_{S_j^*} H_1\, dS_j^*\right)\right]\right\},$$

$$a_3 = \frac{1}{12\pi^2}\left\{\sum_{j=1}^{k}\int_{S_j^*} H_1^*\, dS_j^* + \sum_{j=k+1}^{l}\int_{S_j^*} H_1^*\, dS_j^* + \sum_{j=l+1}^{b}\int_{S_j^*} (H_1^* - 3\gamma_j)\, dS_j^* + \sum_{j=b+1}^{m}\int_{S_j^*} H_1^*\, dS_j^* + \sum_{j=m+1}^{N}\int_{S_j^*} H_1\, dS_j^* + \sum_{j=N+1}^{P}\int_{S_j^*} H_1\, dS_j^* + \left\{\sum_{j=P+1}^{c}\int_{S_j^*} (H_1 - 3\gamma_j)\, dS_j^* + \sum_{j=c+1}^{n}\int_{S_j^*} H_1\, dS_j^*\right\}\right.,$$

$$a_4 = \frac{1}{256\,\pi^2}\left\{ \sum_{j=1}^{k} \int_{S_j^*} (H_1^{*\,2} - N_1^*)\, dS_j^* + 7 \sum_{j=k+1}^{l} \int_{S_j^*} (H_1^{*\,2} - N_1^*)\, dS_j^* \right.$$

$$+\, 7 \sum_{j=l+1}^{b} \int_{S_j^*} \left[(H_1^* - 3\,\gamma_j)^2 - \left(N_1^* - \frac{26}{7}\,\gamma_j H_1^* + \frac{47}{7}\,\gamma_j^2 \right)\right] dS_j^*$$

$$+ \sum_{j=b+1}^{m} \int_{S_j^*} [(H_1^{*\,2} - (N_1^* - 16\,\gamma_j^{-1} H_1^*)]\, dS_j^*$$

$$+ \sum_{j=m+1}^{N} \int_{S_j^*} (H_1^2 - N_1)\, dS_j^* + 7 \sum_{j=N+1}^{P} \int_{S_j^*} (H_1^2 - N_1)\, dS_1^*$$

$$+\, 7 \sum_{j=P+1}^{c} \int_{S_j^*} \left[(H_1 - 3\,\gamma_j)^2 - \left(N_1 - \frac{26}{7}\,\gamma_j H_1 + \frac{47}{7}\,\gamma_j^2 \right)\right] dS_j^*$$

$$+ \left. \left\{ \sum_{j=c+1}^{n} \int_{S_j^*} [(H_1^2 - (N_1 - 16\,\gamma_j^{-1} H_1)]\, dS_j^* \right\} \right\},$$

Note that the proof of our result (3.13) has been proved in[15] more sufficiently.

In these coefficients, $|S_j^*|\,(j = 1, \ldots, m)$ are the surface areas of the components $S_j^*\,(j = 1, \ldots, m)$ of the inner boundary surface S_1 while $|S_j^*|\,(j = m + 1, \ldots, n)$ are the surface areas of the components $|S_j^*|\,(j = m + 1, \ldots, n)$ of the outer boundary surface S_2 respectively. Here H_1^* and N_1^* are respectively the mean curvature and Gaussian curvature of the inner boundary surface S_1 while H_1 and N_1 are respectively the mean curvature and Gaussian curvature of the outer boundary surface S_2.

With reference to the formulae (1.5)-(1.8) and to the articles[1-4], the asymptotic expansion (3.13) may be interpreted as follows :

(*i*) Ω is a general annular vibrating membrane in R^3 and we have the piecewise smooth boundary conditions (1.10) and (1.11) with small/large impedances γ_j.

(*ii*) For the first four terms, Ω is a general annular vibrating membrane in R^3 of volume V, the components $S_j^*\,(j = 1, \ldots, k)$ of S_1 are of surface areas $\sum_{j=1}^{k} |S_j^*|$, mean curvature H_1^* and Gaussian curvature N_1^* together with Dirichlet boundary conditions, the components S_j^* $(j = k+1, \ldots, l)$ of S_1 are of surfaces areas $\sum_{j=k+1}^{l} |S_j^*|$, mean curvature H_1^* and Gaussian curvature N_1^* together with Neumann boundary conditions, the components S_j^* $(j = l+1, \ldots, b)$ of S_1 are of surface areas $\sum_{j=l+1}^{b} |S_j^*|$, mean curvature $(H_1^* - 3\gamma_j)$ and Gaussian curvature $\left(N_1^* - \frac{26}{7}\gamma_j H_1^* + \frac{47}{7}\gamma_j^2\right)$ together with Neumann boundary conditions, and the remaining components S_j^* $(j = b+1, \ldots, m)$ of S_1 are of surface areas $\sum_{j=b+1}^{m} \left(|S_j^*| - 2\gamma_j^{-1} \int_{S_j^*} H_1^* \, dS_j^*\right)$, mean curvature H_1^* and Gaussian curvature $(N_1^* - 16\gamma_j^{-1} H_1^*)$ together with Dirichlet boundary conditions. Similarly, the components S_j^* $(j = m+1, \ldots, N)$ of S_2 are of surface areas $\sum_{j=m+1}^{N} |S_j^*|$, mean curvature H_1 and Gaussian curvature N_1 together with Dirichlet boundary conditions, the components S_j^* $(j = N+1, \ldots, p)$ of S_2 are of surface areas $\sum_{j=N+1}^{p} |S_j^*|$, mean curvature H_1 and Gaussian curvature N_1 together with Neumann boundary conditions, the components S_j^* $(j = p+1 \ldots, c)$ of S_2 are of surface areas $\sum_{j=p+1}^{c} |S_j^*|$, mean curvature $(H_1 - 3\gamma_j)$ and Gaussian curvature $\left(N_1 - \frac{26}{7}\gamma_j H_1 + \frac{47}{7}\gamma_j^2\right)$ together with Neumann boundary conditions, and the remaining components S_j^* $(j = c+1 \ldots, n)$ of S_2 are of surface areas $\sum_{j=c+1}^{n} \left(|S_j^*| - 2\gamma_j^{-1} \int_{S_j^*} H_1 \, dS_j^*\right)$, mean curvature H_1 and Gaussian curvature $(N_1 - 16\gamma_j^{-1} H_1)$ together with Dirichlet boundary conditions

(*iii*) The order term $O(t^2 \, sign \, t)$ may contain further information about the geometry of the annular region $\Omega \subseteq R^3$ and its determination is still an open problem, which has been left for the interested readers.

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