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ON THE INVERSION OF THE KERNEL $K_{\alpha,\beta,\gamma,\nu}$ RELATED TO THE OPERATOR $\oplus^k$

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In this paper, we study the inversion of the distributional kernel $K_{\alpha,\beta,\gamma,\nu}$ related to the operator $\oplus^k$ iterated k-times, which is defined by

$$\oplus^k = \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^4 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4\right]^k$$

where $p + q = n$ is the dimension of the complex space $\mathbb{C}^n$, k is a nonnegative integer and $\alpha, \beta, \gamma, \nu$ are complex parameters. We establish that the inverse $[K_{\alpha,\beta,\gamma,\nu}]^{-1}$ of $K_{\alpha,\beta,\gamma,\nu}$ exists depending on the conditions of p and q whether they are odd or even numbers.

1. INTRODUCTION

The operator $\oplus^k$ can be factorized in the form

$$\oplus^k = \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \cdot \left[\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2} + i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k$$

$$\cdot\left[\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2} - i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k \qquad \dots (1.1)$$

$p + q = n$ is the dimension of the space $\mathbb{C}^n$, $i = \sqrt{-1}$ and k is a nonnegative integer. The operator $\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2$ is first introduced by Kananthai[1] and named the Diamond operator denoted by

$$\Diamond = \left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2 \qquad \dots (1.2)$$

Let us denote the operator L_1 and L_2 by

$$L_1 = \sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \quad \text{... (1.3)}$$

$$L_2 = \sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \quad \text{... (1.4)}$$

Thus (1.1) can be written by

$$\oplus^k = \Diamond^k L_1^k L_2^k \quad \text{... (1.5)}$$

Let us define the distributional kernel $K_{\alpha, \beta, \gamma, \nu}$ by

$$K_{\alpha, \beta, \gamma, \nu} = R_\alpha^H(u) * R_\beta^e(v) * S_\gamma(w) * T_\nu(z) \quad \text{... (1.6)}$$

where $R_\alpha^H(u), R_\beta^e(\upsilon), S_\gamma(w)$ and $T_\nu(z)$ are defined by (2.2), (2.3), (2.4) and (2.5) respectively and the symbol * denotes the convolution. Since $R_\alpha^H(u), R_\beta^e(\upsilon), S_\gamma(w)$ and $T_\nu(z)$ are all tempered distributions (see [1, p. 30-31] and p5, p. 154-155]), then the convolutions on the right hand side of (1.6) exist and give rise also to a tempered distribution. Thus $K_{\alpha, \beta, \gamma, \nu}$ is well defined and also is a tempered distribution. For $\alpha = \beta = \gamma = \nu = 2k$, we obtained $(-1)^k K_{2k, 2k, 2k, 2k}$ as an elementary solution of the operator $\oplus^k$, see [2]. That is $\oplus^k \ (-1)^k K_{2k, 2k, 2k, 2k}(x) = \delta$ where δ is the Dirac-delta distribution and $\oplus^k$ is defined by (1.5).

Let $A^{\alpha, \beta, \gamma, \nu}$ be the operator defined by the formula

$$A^{\alpha, \beta, \gamma, \nu}(f) = K_{\alpha, \beta, \gamma, \nu} * f \quad \text{... (1.7)}$$

where α, β, γ and ν are complex parameters and the symbol * denotes the convolution product and $f \in S$, where S is the Schwartz space, see [8, p. 223]. Our objective is to obtain the operator

$$B^{\alpha, \beta, \gamma, \nu} = (A^{\alpha, \beta, \gamma, \nu})^{-1}$$

such that if $\varphi = A^{\alpha, \beta, \gamma, \nu}(f)$ then $B^{\alpha, \beta, \gamma, \nu}(\varphi) = f$. From (1.7) and p and q being both odd integer, we infer

$$B^{\alpha, \beta, \gamma, \nu} = (A^{\alpha, \beta, \gamma, \nu})^{-1} = (K_{\alpha, \beta, \gamma, \nu})^{-1} = K_{-\alpha, -\beta, -\gamma, -\nu}$$

If p is odd and q is even, we obtain

$$B^{\alpha, \beta, \gamma, \nu} = (K_{\alpha, \beta, \gamma, \nu})^{-1} = \left(1 + \left(\sin \alpha \frac{\pi}{2}\right)^2\right)^{-1} K_{-\alpha, -\beta, -\gamma, -\nu}$$

and if p is even, we obtain

$$B^{\alpha,\beta,\gamma,\nu} = (K_{\alpha,\beta,\gamma,\nu})^{-1} = \left[\left(\cos\alpha\frac{\pi}{2}\right)^2\right]^{-1} K_{-\alpha,-\beta,-\gamma,-\nu}$$

for all complex α such that $\alpha \neq 2s+1$ (s = 0, 1, 2, 3, ...).

2. PRELIMINARIES

Definition 2.1 — Let $x = (x_1, x_2, \ldots, x_n)$ be a point in the n-dimensional Euclidean space $\mathbb{R}^n$ and write

$$u = x_1^2 + x_2^2 + \ldots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \ldots - x_{p+q}^2, \; p+q=n \qquad \text{... (2.1)}$$

Denote by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ the interior of the forward cone and $\bar{\Gamma}_+$ denotes its closure. For any complex number α, we define the function.

$$R_\alpha^H(u) = \begin{cases} \dfrac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{if } x \in \Gamma_+ \\[2ex] 0, & \text{if } x \neq \Gamma_+ \end{cases} \qquad (2.2)$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)}$$

The function R_α^H is first introduced by Nozaki[5] [p. 72] and is called the ultra-hyperbolic kernel of Marcel Riesz. Here $R_\alpha^H(x)$ is an ordinary function if $Re\ (\alpha) \geq n$ and is a distribution of α if $Re\ (\alpha) < n$.

Definition 2.2 — Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and write $\upsilon = x_1^2 + x_2^2 + \ldots + x_n^2$..

For any complex number β, define the function

$$R_\beta^e(\upsilon) = \frac{\upsilon^{\frac{\beta-n}{2}}}{W_n(\beta)} \qquad \text{... (2.3)}$$

where $W_n(\beta) = \dfrac{\pi^{\frac{n}{2}} 2^\alpha \Gamma(\beta)}{\Gamma\left(\frac{n-\beta}{2}\right)}$. The function $R_\beta^e(\upsilon)$ is called the elliptic kernel of Marcel Riesz and is an ordinary function if $Re\ (\beta) \geq n$ and is a dsitribution of β if $Re\ (\beta) < n$.

Definition 2.3 — Let $x = (x_1, x_2, \ldots, x_n)$ a point in the n-dimensional complex space $\mathbb{C}^n$ Write $w = x_1^2 + x_2^2 + \ldots + x_p^2 - i\,(x_{p+1}^2 + x_{p+2}^2 + \ldots + x_{p+q}^2)$ and

$$Z = x_1^2 + x_2^2 + \ldots + x_p^2 + i\,(x_{p+1}^2 + x_{p+2}^2 + \ldots + x_{p+q}^2),\ p+q=n,\ i=\sqrt{-1}$$

For any complex numbers γ and ν, define

$$S_\gamma(\omega) = \frac{\omega^{\frac{\gamma-n}{2}}}{W_n(\gamma)} \qquad \ldots (2.4)$$

and
$$T_\nu(z) = \frac{Z^{\frac{\nu-n}{2}}}{W_n(\nu)} \qquad \ldots (2.5)$$

where
$$W_n(\gamma) = \frac{\pi^{\frac{n}{2}}\, 2^\gamma\, \Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{n-\gamma}{2}\right)},\quad W_n(\nu) = \frac{\pi^{\frac{n}{2}}\, 2^\nu\, \Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{n-\nu}{2}\right)}.$$

Lemma 2.1 — (*i*) $R^e_\beta * R^e_{\beta'} = R^e_{\beta+\beta'}$, where R^e_β and $R^e_{\beta'}$ are given by (2.3).

(*ii*) $S_\gamma * S_{\gamma'} = (i)^{\frac{q}{2}} S_{\gamma+\gamma'}$ and $T_\nu * T_{\nu'} = (-i)^{\frac{a}{2}} T_{\nu+\nu'}$ where S_γ and T_ν are defined by (2.4) and (2.5) respectively.

PROOF : (*i*) See[6] [p. 20].

(*ii*) Now $\langle S_\gamma(\omega), \varphi(x) \rangle = \frac{1}{W_n(\gamma)} \int_{\mathcal{R}^n} \omega^{\frac{\gamma-n}{2}} \varphi(x)\, dx$. Making the change of variables

$x_1 = y_1, x_2 = y_2, \ldots, x_p = y_p$ and $x_{p+1} = \frac{y_{p+1}}{\sqrt{-i}},\ x_{p+2} = \frac{y_{p+2}}{\sqrt{-i}}, \ldots, x_{p+q} = \frac{y_{p+q}}{\sqrt{-i}}$ then we obtain

$$\omega = x_1^2 + x_2^2 + \ldots + x_p^2 - i\,(x_{p+1}^2 + x_{p+2}^2 + \ldots + x_{p+q}^2)$$

$$= y_1^2 + y_2^2 + \ldots + y_p^2 + y_{p+1}^2 + \ldots + y_{p+q}^2,\ p+q=n.$$

Let $r^2 = y_1^2 + y_2^2 + \ldots. + y_{p+q}^2$. Thus

$$\langle S_\gamma(\omega), \varphi(x) \rangle = \frac{1}{W_n(\gamma)} \int_{\mathcal{R}^n} r^{\gamma-n}\, \varphi \left| \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(y_1, y_2, \ldots, y_n)} \right| dy_1\, dy_2 \ldots dy_n$$

$$= \frac{(i)^{\frac{q}{2}}}{W_n(\gamma)} \int_{\mathcal{R}} r^{\gamma-n} \varphi \, dy$$

$$= \left\langle \frac{(i)^{\frac{\nu}{2}} r^{\gamma-n}}{W_n(\gamma)}, \varphi \right\rangle .$$

Now
$$S_\gamma(\omega) * S_{\gamma'}(\omega) = \frac{(i)^{\frac{q}{2}} r^{\gamma-n}}{W_n(\gamma)} * \frac{(i)^{\frac{q}{2}} r^{\gamma'-n}}{W_n(\gamma')}$$

$$= (i)^q \frac{r^{\gamma+\gamma'-n}}{W_n(\gamma+\gamma')} \text{ by } (i)$$

$$= (i)^{\frac{q}{2}} \left[\frac{(i)^{\frac{q}{2}} r^{\gamma+\gamma'-n}}{W_n(\gamma+\gamma')} \right]$$

$$= (i)^{\frac{q}{2}} S_{\gamma+\gamma'}.$$

Thus $S_\gamma * S_{\gamma'} = (i)^{\frac{q}{2}} S_{\gamma+\gamma'}$. Similarly, we obtain $T_\nu * T_{\nu'} = (-i)^{\frac{q}{2}} T_{\nu+\nu'}$.

Lemma 2.2 — (*i*) $L_1 (-1)^k (-i)^{\frac{q}{2}} S_{2k}(\omega) = \delta$, that is $(-1)^k (-i)^{\frac{q}{2}} S_{2k}(\omega)$ is the elementary solution of the operator L_1 defined by (1.3).

(*ii*) $L_2 (-1)^k (i)^{\frac{q}{2}} T_{2k}(z) = \delta$, that is $(-1)^k (i)^{\frac{q}{2}} T_{2k}(z)$ is the elementary solution of the operator L_2 defined by (1.4). Moreover from (*i*), we obtain $S_{-2k}(\omega) = (-1)^k (i)^{\frac{q}{2}} L_1^k \delta(x)$ and $T_{-2k}(z) = (-1)^k (-i)^{\frac{q}{2}} L_2^k \delta(x)$. it follows that

$$S_0(\omega) = (-i)^{q/2} \delta(x) \qquad \text{... (2.6)}$$

and
$$T_0(\omega) = (i)^{q/2} \delta(x) \qquad \text{... (2.7)}$$

PROOF : See [2, Lemma 2.2 (*ii*)]

Lemma 2.3 — (The convolutions of $R_\alpha^H(u)$)

(*i*) $R_{\alpha}^{H} * R_{\alpha'}^{H} = \dfrac{\cos\frac{\alpha\pi}{2}\cos\frac{\alpha'\pi}{2}}{\cos\left(\frac{\alpha+\alpha'}{2}\right)\pi} \cdot R_{\alpha+\alpha'}^{H}$ where R_{α}^{H} is defined by (2.2) with p is an even.

(*ii*) $R_{\alpha}^{H} * R_{\alpha'}^{H} = R_{\alpha+\alpha'}^{H} + T_{\alpha,\alpha'}$ for p is an odd, where

$$T_{\alpha,\alpha'} = T_{\alpha,\alpha'}(u \pm i0, n) = \frac{2\pi i}{4} \frac{C\left(\frac{-\alpha-\alpha'}{2}\right)}{C\left(\frac{-\alpha}{2}\right)C\left(\frac{-\alpha'}{2}\right)} [H_{\alpha+\alpha'}^{+} - H_{\alpha+\alpha'}^{-}]$$

$$C(r) = \Gamma(r)\,\Gamma(1-r)$$

$$H_r^{\pm} = H_r(u \pm i0, n) = e^{\mp\frac{r\pi i}{2}} e^{\mp\frac{q\pi i}{2}} a\left(\frac{r}{2}\right)(u \pm i0)^{\frac{r-n}{2}}$$

$$a\left(\frac{r}{2}\right) = \Gamma\left(\frac{n-r}{2}\right)\left[2^r \pi^{\frac{n}{2}} \Gamma\left(\frac{r}{2}\right)\right]^{-1}$$

$$(u \pm i0)^{\lambda} = \lim_{\varepsilon\to 0} (u + i\epsilon\, |x|^2)^{\lambda}$$

$u = u(x)$ is defined by (2.1).

PROOF : See [4, p 121-123]

Lemma 2.4 — (The inverse of convolution algebra)

(*i*) $R_{\alpha}^{H} * R_{-\alpha}^{H} = R_{\alpha-\alpha}^{H} = R_{0}^{H} = \delta$ if p and q are both odd.

(*ii*) $R_{\alpha}^{H} * R_{-\alpha}^{H} = \left(1 + \left(\sin\frac{\alpha\pi}{2}\right)^2\right)\delta(x)$ if p is odd and q is even.

(*iii*) $R_{\alpha}^{H} * R_{-\alpha}^{H} = \left(\left(\cos\frac{\alpha\pi}{2}\right)^2\right)\delta(x)$ for p is even and $\alpha \neq 2s+1, s = 0, 1, 2, 3, \ldots$

(*iv*) $R_{\beta}^{e} * R_{-\beta}^{e} = R_{\beta-\beta}^{e} = R_{0}^{e} = \delta$

(*v*) $S_{\gamma} * S_{-\gamma} = (i)^{q\delta}$

(*vi*) $T_{\nu} * T_{-\nu} = (i)^{q\delta}$

PROOF : (*i*) By Lemma 2.3 (*ii*), for p is an odd, we obtain $R^H_\alpha * R^H_{-\alpha} = R^H_0 + T_{\alpha,-\alpha}$. Since $R^H_0 = \delta$ and, by computing directly, $T_{\alpha,-\alpha} = 0$ if p and q are both odd, then $R^H_\alpha * R^H_{-\alpha} = \delta$ for p and q are both odd.

(*ii*) By Lemma 2.3 (*ii*) again, for p is an odd, we obtain $R^H_\alpha * R^H_{-\alpha} = R^H_0 + T_{\alpha,-\alpha}$. Now, for p is an odd and q is an even, we obtain $T_{\alpha,-\alpha} = \left(\sin\frac{\alpha\pi}{2}\right)^2 \delta$. Thus

$$R^H_\alpha * R^H_{-\alpha} = \delta + \left(\sin\frac{\alpha\pi}{2}\right)^2 \delta$$

$$= \left[1 + \left(\sin\frac{\alpha\pi}{2}\right)^2\right] \delta(x).$$

(*iii*) By Lemma 2.3 (*i*), for p is an even

$$R^H_\alpha * R^H_{-\alpha} = \frac{\cos\frac{\alpha\pi}{2}\cos\frac{(-\alpha)\pi}{2}}{\cos\frac{\alpha-\alpha}{2}\pi} \cdot R^H_0$$

$$= \left[\cos\frac{\alpha\pi}{2}\right]^2 R^H_0$$

$$= \left(\cos\frac{\alpha\pi}{2}\right)^2 \delta,\ \alpha \neq 2s+1,\ s = 0, 1, 2, 3, \ldots$$

(*iv*) By Lemma 2.1 (*i*), $R^e_\beta * R^e_{-\beta} = R^e_{\beta-\beta} = R^e_0 = \delta$, since $R^e_0 = \delta$ see [5, p 118].

(*v*) By Lemma 2.1 (*ii*),

$$S_\gamma * S_{-\gamma} = (i)^{\frac{q}{2}} S_{\gamma-\gamma} = (i)^{\frac{q}{2}} S_0$$

$$= (i)^{\frac{q}{2}} (i)^{\frac{q}{2}} \delta = (i)^q \delta \text{ by (2.6)}$$

(*vi*) By Lemma 2.1 (*ii*) again, we obtain

$$T_\nu * T_{-\nu} = (-i)^{\frac{q}{2}} T_{\nu-\nu} = (-i)^{\frac{q}{2}} T_0$$

$$= (-i)^{\frac{q}{2}} (-i)^{\frac{q}{2}} \delta = (-i)^q \delta \text{ by (2.7)}$$

Lemma 2.5 — Let the distributional kernel $K_{\alpha,\beta,\gamma,\nu}$ be defined by (1.6). Then the following formulas hold.

(*i*) $K_{\alpha,\beta,\gamma,v} * K_{-\alpha,-\beta,-\gamma,-v} = K_{0,0,0,0} = \delta$ for p and q are both odd numbers.

(*ii*) $K_{\alpha,\beta,\gamma,v} * K_{-\alpha,-\beta,-\gamma,-v} = \left[1 + \left(\sin\frac{\alpha\pi}{2}\right)^2\right]\delta$ for p is odd and q is even.

(*iii*) $K_{\alpha,\beta,\gamma,v} * K_{-\alpha,-\beta,-\gamma,-v} = \left(\cos\frac{\alpha\pi}{2}\right)^2 \delta$ for p is even.

PROOF : (i) By (1.6) and properties of convolutions,

$$K_{\alpha,\beta,\gamma,v} * K_{-\alpha,-\beta,-\gamma,-v} = (R^H_\alpha * R^H_{-\alpha}) * (R^e_\beta * R^e_{-\beta}) * (S_\gamma * S_{-\gamma} * (T_v * T_{-v}) \quad \dots (2.8)$$

$$= \delta * \delta * (i)^q \delta * (-i)^q \delta$$

$$= \delta * \delta * \delta * \delta = \delta \text{ by Lemma 2.4 } (i), (iv), (v), (vi)$$

(*ii*) From (2.8),

$$K_{\alpha,\beta,\gamma,v} * K_{-\alpha,-\beta,-\gamma,-v} = \left[1 + \left(\sin\frac{\alpha\pi}{2}\right)^2\right]\delta * \delta * (i)^q \delta * (-i)^q \delta$$

$$= \left[1 + \left(\sin\frac{\alpha\pi}{2}\right)^2\right]\delta * \delta * \delta * \delta$$

$$= \left[1 + \left(\sin\frac{\alpha\pi}{2}\right)^2\right]\delta \text{ by Lemma 2.4 } (ii), (iv), (v), (vi)$$

(*iii*) From (2.8) again, we obtain

$$K_{\alpha,\beta,\gamma,v} * K_{-\alpha,-\beta,-\gamma,-v} = \left(\cos\frac{\alpha\pi}{2}\right)^2 \delta * \delta * (i)^q \delta * (-i)^q \delta$$

$$= \left(\cos\frac{\alpha\pi}{2}\right)^2 \delta \text{ for } \alpha \neq 2s+1, s = 0, 1, 2, 3, \dots$$

by Lemma 2.4 (*iii*), (*iv*), (*v*), (*vi*)

3. MAIN RESULTS

Theorem — *Given $\varphi = A^{\alpha,\beta,\gamma,v}(f)$ where $A^{\alpha,\beta,\gamma,v}(f)$ is defined by (1.7) for every $f \in S$ where S is the Schwartz space of function, then there exists the operator $B^{\alpha,\beta,\gamma,v}$ such that*

(*i*) $B^{\alpha,\beta,\gamma,v}(\varphi) = f$ where

$$B^{\alpha,\beta,\gamma,v} = (A^{\alpha,\beta,\gamma,v})^{-1} = (K_{\alpha,\beta,\gamma,v})^{-1} = K_{-\alpha,-\beta,-\gamma,-v}$$

if p and q are both odd numbers.

(*ii*) $B^{\alpha,\beta,\gamma,v}(\varphi) = f$ where

$$B^{\alpha,\beta,\gamma,v} = (A^{\alpha,\beta,\gamma,v})^{-1} = (K_{\alpha,\beta,\gamma,v})^{-1} = \left(1+\left(\sin\frac{\alpha\pi}{2}\right)^2\right)^{-1} K_{-\alpha,-\beta,-\gamma,-v}$$

if p is odd and q is even.

(*iii*) $B^{\alpha,\beta,\gamma,v}(\varphi) = f$ where

$$B^{\alpha,\beta,\gamma,v} = (A^{\alpha,\beta,\gamma,v})^{-1} = (K_{\alpha,\beta,\gamma,v})^{-1} = \left[\left(\cos\frac{\alpha\pi}{2}\right)^2\right]^{-1} K_{-\alpha,-\beta,-\gamma,-v}$$

if p is even for all α such that $\alpha \neq 2s+1, s = 0, 1, 2, 3, \ldots$

PROOF : (*i*) From (1.7), we have

$$A^{\alpha,\beta,\gamma,v}(f) = K_{\alpha,\beta,\gamma,v} * f = \varphi$$

where $K_{\alpha,\beta,\gamma,v}$ is defined by (1.6). Then by Lemma 25 (*i*) we obtain

$$K_{-\alpha,-\beta,-\gamma-v} * (K_{\alpha,\beta,\gamma,v} * f) = (K_{-\alpha,-\beta,-\gamma,-v} * K_{\alpha,\beta,\gamma,v}) * f$$
$$= K_{0,0,0,0} * f = \delta * f = f$$

if p and q are both odd. Thus

$$B^{\alpha,\beta,\gamma,v} = (A^{\alpha,\beta,\gamma,v})^{-1} = (K_{\alpha,\beta,\gamma,v})^{-1} = K_{-\alpha,-\beta,-\gamma-v}$$

It follows that $B^{\alpha,\beta,\gamma,v}(\varphi) = f$.

(*ii*) By Lemma 2.5 (ii), for p is an odd and q is an even, we have

$$\left[1+\left(\sin\frac{\alpha\pi}{2}\right)^2\right]^{-1} K_{-\alpha,-\beta,-\gamma-v} * (K_{\alpha,\beta,\gamma,v} * f) = \left[1+\left(\sin\frac{\alpha\pi}{2}\right)^2\right]^{-1}$$

$$\times (K_{-\alpha,-\beta,-\gamma,-v} * K_{\alpha,\beta,\gamma,v}) * f$$

$$= \left[1+\left(\sin\frac{\alpha\pi}{2}\right)^2\right]^{-1}\left[1+\left(\sin\frac{\alpha\pi}{2}\right)^2\right] \times K_{0,0,0,0} * f$$

$$= \delta * f = f$$

Thus
$$B^{\alpha,\beta,\gamma,v} = (A^{\alpha,\beta,\gamma,v})^{-1} = (K_{\alpha,\beta,\gamma,v})^{-1}$$

$$= \left[1+\left(\sin\frac{\alpha\pi}{2}\right)^2\right]^{-1} K_{-\alpha,-\beta,-\gamma,-v}.$$

It follows that $B^{\alpha,\beta,\gamma,v}(\varphi) = f$.

(*iii*) By Lemma 2.5 (*iii*), for p is even

$$\left[\left(\cos\frac{\alpha\pi}{2}\right)^2\right]^{-1} K_{-\alpha,-\beta,-\gamma,-\nu} * (K_{\alpha,\beta,\gamma,\nu} * f) = \left[\left(\cos\frac{\alpha\pi}{2}\right)^2\right]^{-1}$$

$$\times (K_{-\alpha,-\beta,-\gamma,-\nu} * K_{\alpha,\beta,\gamma,\nu}) * f$$

$$= \left[\left(\cos\frac{\alpha\pi}{2}\right)^2\right]^{-1}\left[\left(\cos\frac{\alpha\pi}{2}\right)^2\right] \times K_{0,0,0,0} * f$$

$$= \delta * f = f$$

Thus $B^{\alpha,\beta,\gamma,\nu} = (A^{\alpha,\beta,\gamma,\nu})^{-1} = (K_{\alpha,\beta,\gamma,\nu})^{-1} = \left[\left(\cos\frac{\alpha\pi}{2}\right)^2\right]^{-1} K_{-\alpha,-\beta,-\gamma-\nu}$. It follows that $B^{\alpha,\beta,\gamma,\nu}(\varphi) = f$.

In particular, we obtain for $\alpha = \beta = \gamma = \nu = 2k$ and p and q are both odd numbers.

$$(-1)^k K_{2k,2k,2k,2k}(x) = (-1)^k R^H_{2k}(u) * (-1)^k R^e_{2k}(v)$$

$* (-1)^k (-i)^{\frac{q}{2}} S_{2k}(\omega) * (-1)^k (i)^{\frac{q}{2}} T_{2k}(z)$ is an elementary solution of the operator $\oplus^k$ defined by (1.5) and by Lemma 2.2 (*i*), (*ii*). Now

$$(-1)^k K_{2k,2k,2k,2k} * (-1)^k K_{-2k,-2k,-2k,-2k} = [(-1)^k R^H_{2k}(u) * (-1)^k R^H_{-2k}]$$

$$* \ [(-1)^k R^e_{2k}(v) * (-1)^k R^e_{-2k}(v)]$$

$$* \ [(-1)^k (-i)^{\frac{q}{2}} S_{2k}(\omega) * (-1)^k (-i)^{\frac{q}{2}} S_{-2k}]$$

$$* \ [(-1)^k (i)^{\frac{q}{2}} T_{2k}(z) * (-1)^k (i)^{\frac{q}{2}} T_{-2k}(z)]$$

$$= R^H_0 * R^e_0 * (-i)^q S_0(\omega) * (i)^q T_0(z)$$

$$= \delta * \delta * (-i)^q (i)^{q/2} \delta * (i)^q (-1)^{q/2} \delta$$

$$= \delta * \delta * \delta * \delta$$

$$= \delta \text{ by (2.6), (2.7)}$$

Thus $(-1)^k K_{-2k,-2k,-2k,-2k}$ is an inverse of the convolution algebra of the elementary solution of the operator $\oplus^k$.

REFERENCES

1. A. Kananthai, *Applied Mathematics and Computation*, **88** (1997), 27-37.
2. A. Kananthai, S. Suantai, V. Longani, *Computational Technologies*, **5** (2000), 61-67.
3. Y. Nozaki, *Kodai Mathematical Seminal Reports*, **6(2)** (1964), 69-87.
4. M. Aguirre Tellez, S. E. Trione, *Revista de la Union Mathematica Argentina*, **39** (1995), 115-24.
5. W. F. Donoghue, *Distributions and Fourer Transform*, Academic Press, (1969).
6. M. Riesz, *Acta Math.*, **81** (1945), 1-223.
7. S. E. Trione, *Trabajos de Mathematica Serie I*, (1987), preprint 116.
8. L. Schwartz, *Theorie des distributions*, Hermann, Paris (1966).