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THE ASYMPTOTIC BEHAVIOR OF A HIGHER ORDER DELAY NONLINEAR DIFFERENCE EQUATIONS‡

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In this paper, the asymptotic behavior of solutions of the equation of the form

$$x_{n+1} = \frac{\beta x_n^2}{1 + x_{n-k}^2}, n = 0, 1, \ldots,$$

is studied under the hypothesis that β is a positive constant and the initial conditions $x_{-k}, \ldots, x_{-1}$, and x_0 are arbitrary positive numbers.

Key Words : Recursive Sequence; Monotonic Convergence; Global Asymptotic Stability; Local Asymptotic Stability

1. INTRODUCTION

In this paper we study the asymptotic behavior of delay nonlinear difference equation of the form

$$x_{n+1} = \frac{\beta x_n^2}{1 + x_{n-k}^2}, n = 0, 1, \ldots, \qquad \ldots (1)$$

where $\beta \in (0, \infty)$ and the initial conditions $x_{-k}, \ldots, x_{-1}$ and x_0 are arbitrary positive numbers.

If $a_{-k}, \ldots, a_{-1}, a_0 \in (0, \infty)$, are given, then eq. (1) has a unique soluton $\{x_n\}$ satisfying the initial conditions

$$x_{-j} = a_{-j} \text{ for } j = 0, 1, \ldots, k.$$

Clearly

$$x_n > 0 \text{ for } n \geq 0.$$

In this paper we will only investigate solutions of eq. (1) which are positive for $n \geq 0$. Such solutions will also be called positive solutions. For the general theory difference equations, one can refer to the monographs of Agarwal[1] and Kocic and Ladas[2].

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In 1993, Kocic and Ladas[2] studied the asymptotic behavior of eq. (1) with $k = 1$, and proposed an Open Problem [2, p. 159]

Open Problem — Investigate the global asymptotic ability of the rational recursive sequence

$$x_{n+1} = \frac{\beta x_n^2}{1 + x_{n-1}^2}, \quad n = 0, 1, \ldots, \qquad \ldots (2)$$

where $\beta \in (0, \infty)$ and the initial values x_{-1} and x_0 are arbitrary positive numbers.

Recently, Camouzis, Ladas, Rodrigues and Northshield[3] and Zhang, Shi and Gai[4] studied the problem and gave sufficient conditions for stability and instability of positive solutions of eq. (2). But for $k > 1$, we can not find any results on the asymptotic behaviour of positive solutions of eq. (1).

Motivated by the above Open Problem, in this paper, we consider the asymptotic behaviour of eq. (1). The results that we obtained here, include the characterization of behaviour near the origin as well as conditions implying instability and other types of asymptotic behaviour, and extend the results of Camouzis, Ladas, Rodrigues and Northshield[3] and Zhang, Shi and Gai[4] for $k = 1$.

For the sake of convenience, we list the following definitions.

Definition 1.1 (Stability) — The equilibrium point $\bar{x}$ of eq. (1) is called locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x_{-j} - \bar{x}| < \delta$ ($j = 0, 1, \ldots, k$) implies $|x_n - \bar{x}| < \varepsilon$ for all $n \geq 0$. Otherwise. $\bar{x}$ is said to be unstable.

The equilibrium point $\bar{x}$ of eq. (1) is called locally asymptotic stable if it is locally stable and there exists $\gamma > 0$ such that $|x_{-j} - \bar{x}| < \gamma$ ($j = 0, 1, \ldots, k$) implies $\lim_{n \to \infty} |x_n - \bar{x}| = 0$.

The equilibrium point $\bar{x}$ of eq. (1) is called a global attractor if $\lim_{n \to \infty} x_n = \bar{x}$ for all $x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$.

The equilibrium point $\bar{x}$ of eq. (1) is called globally asymptotically stable if $\bar{x}$ is locally asymptotically stable and a global attractor.

2. Several Lemmas

In this section we will list several lemmas which are important for proving our main results.

Lemma 2.1 — Assume that $0 < \beta < 2$, then eq. (1) has a unique non-negative equilibrium zero.

Lemma 2.2 — Assume that $\beta > 2$, then eq. (1) has three non-negative equilibria, namely,

$$\bar{x}_1 = 0, \quad \bar{x}_2 = \frac{\beta - \sqrt{\beta^2 - 4}}{2} \text{ and } \bar{x}_3 = \frac{\beta + \sqrt{\beta^2 - 4}}{2},$$

where $$\frac{1}{\beta} < \bar{x}_2 < 1 \text{ and } 1 < x_3 < \beta.$$

Lemma 2.3 — Assume that $\beta = 2$, then eq.(1) has two non-negative equilibria, namely,

$$\bar{x}_1 = 0 \text{ and } \bar{x}_2 = 1.$$

Lemma 2.4 — Assume that $\beta \in (0, \infty)$, and that

$$(k+1)^{k+1} > (2k)^k. \quad \text{... (3)}$$

Then every solution of eq. (1) is bounded from above by a positive constant. As the proof of this lemma is similar to that of Theorem 2.1 [5] or Theorem 4.1 [6], we omit it here.

3. The Case $0 < \beta < 2$

From Lemma 2.1 we know that eq. (1) has a unique non-negative equilibrium $\bar{x} = 0$. The linearized equation about zero equilibrium is

$$y_{n+1} = 0,$$

and so it is locally asymptotically stable. Now we prove that $\bar{x} = 0$ is also globally asymptotically stable.

Theorem 3.1 — *Assume that $0 < \beta < 2$ and that (3) holds, Let $\{x_n\}$ be a solution of eq. (1). Then*

$$\lim_{n \to \infty} x_n = 0. \quad \text{... (4)}$$

Proof : Let $\{x_n\}$ be a positive solution of eq. (1) with initial conditions $x_{-k}, ..., x_{-1}, x_0 \in (0, \infty)$. By using successive iteration, we see that for $n \geq 0$,

$$x_{n+1} = \frac{\beta x_n}{1 + x_{n-k}^2} x_n$$

$$= \frac{\beta^2 x_n x_{n-1}}{(1 + x_{n-k}^2)(1 + x_{n-k-1}^2)} x_{n-1}$$

$$= \cdots$$

$$= \frac{\beta^{n+1} x_n x_{n-1} \cdots x_1 x_0^2}{(1 + x_{n-k}^2)(1 + x_{n-k-1}^2) \cdots (1 + x_{-k}^2)}$$

In view of $a^2 + b^2 \geq 2ab$ for $a > 0$ and $b > 0$, we have

$$x_{n+1} \leq \frac{\beta^{n+1} x_n x_{n-1} \cdots x_1 x_0^2}{2^{n+1} x_{n-k} x_{n-k-1} \cdots x_{-k}}.$$

Set $$h_n = \frac{x_n x_{n-1} \cdots x_1 x_0^2}{x_{n-k} x_{n-k-1} \cdots x_{-k}}$$

then $$x_{n+1} \leq \left(\frac{\beta}{2}\right)^{n+1} h_n, \; n \geq 0.$$

According to Lemma 2.4, all solutions of eq. (1) are bounded. Then

$$h_n = \frac{x_n x_{n-1} \cdots x_1 x_0^2}{x_{n-k} x_{n-k-1} \cdots x_{-k}} < M,$$

where M is a positive constant. Thus, we have

$$0 < x_{n+1} \le \left(\frac{\beta}{2}\right)^{n+1} M, n \ge 0.$$

Therefore, we obtain $\lim_{n \to \infty} x_n = 0$. This completes the proof.

4. THE CASE $\beta = 2$

If $\beta = 2$, then, by Lemma 2.3, eq. (1) has two equilibria, namely,

$$\bar{x}_1 = 0 \text{ and } \bar{x}_2 = 1.$$

As in the previous cases $\bar{x}_1 = 0$ is locally asymptotic stable. In fact, we also prove that $\bar{x}_1 = 0$ is asymptotically stable and attract every trajectory of eq. (1) with initial value in the interval $\left[0, \frac{1}{2}\right)$.

***Theorem* 4.1** — *If $\beta = 2$, then the origin is asymptotically stable and attracts every trajectory of eq. (1) with initial values in the interval* $\left[0, \frac{1}{2}\right)$.

PROOF : Note that

$$x_{n+1} = \frac{2}{1 + x_{n-k}^2} x_n^2 \le 2x_n^2 \le 2^3 x_{n-1}^{2^2}.$$

This pattern continues; by induction

$$x_{n+1} \le \frac{1}{2} (2x_0)^{2^{n+1}}$$

Therefore, if the initial values (in particular, x_0) are in $\left[0, \frac{1}{2}\right)$, then the solution they generate must converge to zero. Stability is an immediate consequence of the monotonically decreasing nature of the sequence $\left\{(2x_0)^{2n+1}\right\}$. The proof is complete.

For the equilibrium point $\bar{x}_2 = 1$, we have the following result.

***Theorem* 4.2** — *If $\beta = 2$, then $\bar{x}_2 = 1$ is an unstable fixed point; in fact, in every neighborhood of $\bar{x}_2$ there are initial value generating solutions that converge monotonically to zero.*

PROOF : Let $x_0 \in (0, 1)$ and $x_{-k}, \dots, x_{-1} \ge x_0$. Then

$$x_1 = \frac{2x_0^2}{1+x_{-k}^2} \le \frac{2}{1+x_0^2} x_0^2 = 2\frac{x_0}{1+x_0^2} x_0 < x_0.$$

Therefore, $x_1 < x_0 < 1$ and $x_{-k+1}, \dots, x_0 \ge x_1$. It follows by induction that the sequence $\{x_n\}$ is monotonically decreasing in the interval (0, 1) so that it must converge to zero. Since $x_{-k}, \dots, x_{-1}, x_0$ may be chosen arbitrarily close to $\bar{x}_2 = 1$, this also proves that $\bar{x}_2 = 1$ is unstable. The proof is complete.

***Theorem* 4.3** — *Assume that $\beta = 2$ and that (3) holds, then eq. (1) has a solution which is strictly increasing to $\bar{x}_2 = 1$.*

The proof is similar to that of Theorem 5.3 in Section 5 and will be omitted.

***Theorem* 4.4** — *If $\beta = 2$, then eq. (1) has no solution which is strictly decreasing and converges to $\bar{x}_2 = 1$.*

PROOF : Assume, for the sake of contradiction, that eq. (1) has a solution which is strictly decreasing and converges to $\bar{x}_2 = 1$. Then there exists an $n_0 > 0$ such that $\bar{x}_2 = 1 < x_{n+1} < x_n$ for $n \ge n_0$ and $\lim_{n \to \infty} x_n = 1 = \bar{x}_2$.

Let $y_n = x_{n+k}/x_n$ for $n \ge n_0$. Then $0 < y_n < 1$ for $n \ge 0$. By eq. (1) we have

$$y_{n+1} = \frac{x_{n+k+1}}{x_{n+1}} = \frac{2\,x_{n+k}^2}{x_{n+1}\,(1+x_n^2)}$$

$$\le \frac{2\,x_{n+1}\,x_{n+k}}{x_{n+1}\,(1+x_n^2)} = y_n \frac{2x_n}{1+x_n^2} < y_n,$$

and so $$\lim_{n \to \infty} y_n = \lim_{n \to \infty} \frac{x_{n+k}}{x_n} = 1.$$

Hence, $y_n = 1$ for $n \ge n_0$, which implies that $x_{n+k} = x_n$ for $n \ge n_0$. This is a contradiction and completes the proof.

5. THE CASE $\beta > 2$

In this section we will consider eq. (1) for $\beta > 2$

***Theorem* 5.1** — *If $\beta > 2$, then the origin is asymptotically stable and attracts every trajectory of eq. (1) with initial values in the interval $\left[0, \frac{1}{\beta}\right)$.*

***Theorem* 5.2** — *$\beta > 2$, then $\bar{x}_2 = (\beta - \sqrt{\beta^2 - 4})/2$ is an unstable fixed point; in fact, every neighborhood of $\bar{x}_2$ there are initial value generating solutions that converge monotonically to zero.*

The proofs of Theorem 5.1 and 5.2 are similar to those of Theorem 4.1 and 4.2, we omit them here.

***Theorem* 5.3** — *Assume that $\beta > 2$ and that (3) holds. Then eq. (1) has a solution $\{x_n\}$ which is strictly increasing to $\bar{x}_2$.*

PROOF : Assume, for the sake of contradiction, that eq. (1) has no solution which is strictly increasing and converges to $\bar{x}_2$. Then we can easily see that for each solution $\{x_n\}$ of eq. (1) with

$$0 < x_{-k} < \ldots < x_0 < \bar{x}_2,$$

there exists a positive integer N (which depends on the solution), such that

$$x_{-k} < \ldots < x_0 < \ldots \leq x_N \text{ and } x_j > x_{j+1} \text{ for } j \geq N.$$

Furthermore $x_N \neq \bar{x}_2$, for otherwise

$$\bar{x}_2 > x_{N+1} = \frac{\beta x_N^2}{1 + x_{N-k}^2} \geq \frac{\beta \bar{x}_2^2}{1 + \bar{x}_2^2} = \bar{x}_2,$$

which is a contradiction.

Now for $x \in [1/\beta, \sqrt{\bar{x}_2/\beta}]$, define the sequence of continuous functions

$$f_{-k}(x) = \ldots = f_{-1}(x) = x, \quad f_0(x) = \beta x^2,$$

and
$$f_{n+1}(x) = \frac{\beta f_n^2(x)}{1 + f_{n-k}^2(x)}, \quad n = 0, 1, \ldots.$$

For each $x \in [1/\beta, \sqrt{\bar{x}_2/\beta}]$, the sequence of positive numbers $\{f_n(x)\}$ is a solution of eq. (1) and by Lemma 2.4 this sequence is bounded. Let

$$s(x) = \sup_n f_n(x)$$

In view of the hypothesis that eq. (1) has no solution which is strictly increasing and converges to $\bar{x}_2$, it follows that for every $x \in [1/\beta, \sqrt{\bar{x}/\beta}]$, there exists an integer N (which depends on x) such that

$$s(x) = f_n(x), \quad f_N(x) > f_{N+1}(x), \text{ and } f_n(x) \leq f_N(x) \text{ for } n = -k, \ldots, N-1.$$

We will now prove that the function $s(x)$ is continuous for all $x \in [1/\beta, \sqrt{\bar{x}_2/\beta}]$.

To this end let $x \in [1/\beta, \sqrt{\bar{x}_2/\beta}]$, and let

$$0 < \varepsilon < \frac{f_N(x) - f_{N+1}(x)}{2}.$$

From the continuity of $f_0, \ldots, f_{N+1}$, there exists $\delta > 0$ such that

$$|x' - x| < \delta \text{ implies } \sup_{0 \le m \le N+1} |f_m(x') - f_m(x)| < \varepsilon.$$

Note that

$$f_{N+1}(x') < f_{N+1}(x) + \varepsilon < f_N(x) - \varepsilon < f_N(x')$$

and so $\quad s(x') = f_m(x')$ for some $m \le N$.

Therefore

$$s(x) - \varepsilon = f_N(x) - \varepsilon < f_N(x') \le \sup_{0 \le m \le N} f_m(x') = s(x')$$

and
$$s(x') = \sup_{0 \le m \le N} f_m(x') < \sup_{0 \le m \le N} [f_m(x) + \varepsilon] = s(x) + \varepsilon,$$

so
$$|s(x') - s(x)| < \varepsilon$$

which establishes our claim that $s(x)$ is continuous on $[1/\beta, \sqrt{\bar{x}_2/\beta}]$.

Since
$$f_{-k}\left(\frac{1}{\beta}\right) = \cdots = f_{-1}\left(\frac{1}{\beta}\right) = \frac{1}{\beta}, \quad f_0\left(\frac{1}{\beta}\right) = \beta \cdot \frac{1}{\beta^2} = \frac{1}{\beta},$$

and
$$f_1\left(\frac{1}{\beta}\right) = \frac{\beta \cdot \frac{1}{\beta^2}}{1 + \frac{1}{\beta^2}} = \frac{\frac{1}{\beta}}{1 + \frac{1}{\beta^2}} < \frac{1}{\beta}.$$

By induction, we have

$$f_n\left(\frac{1}{\beta}\right) < \frac{1}{\beta} \text{ for } n > 0.$$

So
$$s\left(\frac{1}{\beta}\right) = \frac{1}{\beta}.$$

Also since

$$f_{-k}\left(\sqrt{\frac{\bar{x}_2}{\beta}}\right) = \cdots = f_{-1}\left(\sqrt{\frac{\bar{x}_2}{\beta}}\right) = \sqrt{\frac{\bar{x}_2}{\beta}}, \ f_0\left(\sqrt{\frac{\bar{x}_2}{\beta}}\right) = \beta \cdot \frac{\bar{x}_2}{\beta} = \bar{x}_2,$$

and
$$f_1\left(\sqrt{\frac{\bar{x}_2}{\beta}}\right) = \frac{\beta \bar{x}_2^2}{1 + \frac{\bar{x}_2^2}{\beta}} > \frac{\beta \bar{x}_2^2}{1 + \bar{x}_2^2} = \bar{x}_2.$$

So, we have

$$s\left(\sqrt{\frac{\bar{x}_2}{\beta}}\right) > \bar{x}_2.$$

By the continuity of s, there exists an $x^* \in (1/\beta, \sqrt{\bar{x}_2/\beta})$ such that

$$s(x^*) = \bar{x}_2.$$

Hence by the hypothesis that eq. (1) has no solution which is strictly increasing to $\bar{x}_2$, then there exists an N, such that

$$f_N(x^*) = \bar{x}_2, f_{N+1}(x^*) < \bar{x}_2 \text{ and } f_n(x^*) \leq \bar{x}_2 \text{ for } n = -k, ..., N-1.$$

But then

$$\bar{x}_2 > f_{N+1}(x^*) = \frac{\beta f_N^2(x^*)}{1 + f_{N-k}^2(x^*)} \geq \frac{\beta \bar{x}_2^2}{1 + \bar{x}_2^2} = \bar{x}_2$$

which is a contradiction. The proof is complete.

Remark 1 : Theorem 5.3 extends Theorem 3.1 of[3] for $k = 1$.

***Theorem* 5.4** — *Assume that $\beta > 2$ and that (3) holds. Then there exists a solution $\{x_n\}$ of eq. (1) such that*

$$x_{n-1} > \bar{x}_2 \text{ for } n = 0, 1, ...$$

PROOF : Assume for the sake of contradiction, that eq. (1) has no solution which is above $\bar{x}_2$.

For $x \in (\bar{x}_2, \infty)$, define the sequence of continuous functions

$$f_{-k}(x) = \ldots = f_{-1}(x) = x, \ f_0(x) = x^2$$

and
$$f_{n+1}(x) = \frac{\beta f_n^2(x)}{1 + f_{n-k}^2(x)}, n = 0, 1, \ldots$$

For each $x \in (\bar{x}_2, \infty)$, the sequence of positive numbers $\{f_n(x)\}$ is a solution of eq. (1) and by Lemma 2.4 this sequence is bounded.

Let $s(x) = \sup_n f_n(x)$

In view of the hypothesis that eq. (1) has no solution which is above $\bar{x}_2$, it follows that for every $x \in (\bar{x}_2, \infty)$, there exist two integers N and m (which depend on x), with $N \geq -k$ and $m \geq N$, such that

$$s(x) = \sup_{n \leq N} f_n(x),$$

$$f_n(x) > f_{n+1}(x) \text{ for } n \geq N,$$

and
$$f_m(x) < \bar{x}_2,$$

Choose
$$0 < \varepsilon < \min_{N \leq n \leq m} \left\{ \frac{f_n(x) - f_{n+1}(x)}{2}, \bar{x}_2 - f_m(x) \right\}$$

By the continuity of $f_{-k}, ..., f_{-1}, f_0, ..., f_m$, there exists $\delta > 0$ such that

$$|x - x'| < \delta \text{ implies } \sup_{n \le m} |f_n(x) - f_n(x')| < \varepsilon.$$

Note that

$$f_n(x') > f_n(x) - \varepsilon > f_{n+1}(x) + \varepsilon > f_{n+1}(x') \text{ for } n = N, ..., m.$$

and $\quad f_m(x') < f_m(x) + \varepsilon < \bar{x}_2$

and so $\quad s(x') = \sup_{n \le N} f_n(x').$

Therefore, $\quad s(x) - \varepsilon = \sup_{n \le N} f_n(x) - \varepsilon,$

$$< \sup_{n \le N} f_n(x') = s(x') < \sup_{n \le N} f_n(x) > \varepsilon = s(x) + \varepsilon$$

and so $\quad |s(x') - s(x)| < \varepsilon$

which shows that the function s is continuous on $(\bar{x}_2, \infty)$.

$$\text{As } s(\sqrt{\bar{x}_3}) > \bar{x}_3 \text{ and } s(\sqrt{\bar{x}_2}) = \sqrt{\bar{x}_2} < \bar{x}_3,$$

it follows that there exists an $x^* \in (\sqrt{\bar{x}_2}, \sqrt{\bar{x}_3})$ such that

$$s(x^*) = \bar{x}_3.$$

Hence, there exists an N such that

$$\bar{x}_3 = s(x^*) = \sup_{n \le N} f_n(x^*).$$

Also note that

$$f_0(x^*) = (x^*)^2 < \bar{x}_3, \; f_{-i}(x^*) < \bar{x}_3 \text{ for } i = 1, ..., k, \text{ and so we can assume that}$$

$$\bar{x}_3 = s(x^*) = f_m(x^*) \text{ for some } 1 \le m \le N,$$

and $\quad f_{m+1}(x^*) < \bar{x}_3, \; f_{m-k}(x^*) < \bar{x}_3.$

But then $\quad \bar{x}_3 > f_{m+1}(x^*) = \dfrac{\beta \bar{x}_3^2}{1 + f_{m-k}^2(x)} > \dfrac{\beta \bar{x}_3^2}{1 + \bar{x}_3^2} = \bar{x}_3,$

which is a contradiction. The proof is complete.

***Theorem* 5.5** — *If $\beta < 2$, then eq. (1) has no solution which is decreasing and converges to $\bar{x}_3$.*

PROOF : Assume, for the sake of contradiction, that eq. (1) has a solution which is decreasing and converges to $\bar{x}_3$. Then there exists an $n_0 > 0$ such that $\bar{x}_3 < x_{n+1} < x_n$ for $n \geq n_0$ and $\lim_{n \to \infty} x_n = \bar{x}_3$.

Let $y_n = x_{n+k}/x_n$ for $n \geq n_0$, then $0 < y_n < 1$ for $n \geq n_0$. By eq. (1), we obtain,

$$\begin{aligned} y_{n+1} &= \frac{x_{n+k+1}}{x_{n+1}} = \frac{1}{x_{n+1}} \frac{\beta x_{n+k}^2}{1 + x_n^2} \\ &\leq \frac{\beta x_{n+1} x_{n+k}}{x_{n+1}(1 + x_n^2)} = y_n \frac{\beta x_n}{1 + x_n^2} \\ &< y_n, \end{aligned}$$

and

$$\lim_{n \to \infty} y_n = \lim_{n \to \infty} \frac{x_{n+k}}{x_n} = 1.$$

Hence, $y_n = 1$ for $n \geq n_0$, which implies that $x_{n+k} = x_n$ for $n \geq n_0$. This is a contradiction and completes the proof.

***Theorem* 5.6** — *If $\beta > 2$, then eq. (1) has no solution which is increasing and converges to $\bar{x}_3$.*

PROOF : Assume, for the sake of contradiction, that eq. (1) has a solution which is increasing and converges to $\bar{x}_3$. Then there exists an $n_0 > 0$ such that $1 < x_n < x_{n+1} < \bar{x}_3$ for $n \geq n_0$ and $\lim_{n \to \infty} x_n = \bar{x}_3$.

Let $y_n = x_n/x_{n+k}$ for $n \geq n_0$. Then $0 < y_n < 1$ for $n \geq n_0$. By eq. (1), we have

$$\begin{aligned} y_{n+1} &= \frac{x_{n+1}}{x_{n+k+1}} = \frac{x_{n+1}(1 + x_n^2)}{\beta x_{n+k}^2} \\ &\leq \frac{x_{n+1}(1 + x_n^2)}{\beta x_{n+1} x_{n+k}} = y_n \frac{1 + x_n^2}{\beta x_n} \; y_n, \end{aligned}$$

and

$$\lim_{n \to \infty} y_n = \lim_{n \to \infty} \frac{x_n}{x_{n+k}} = 1.$$

Hence $y_n = 1$ for $n \geq n_0$ which implies that $x_{n+k} = x_n$ for $n \geq n_0$. This is a contradiction and the proof is complete.

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