Indian J. pure appl. Math., 34(10) : 1443-1452, October 2003

ON THE SPECTRUM AND FINE SPECTRUM OF THE COMPACT RHALY OPERATORS

M. YILDIRIM

Department of Mathematics, Faculty of Science, Cumhuriyet University, Sivas 58140, Turkey

(*Received* 23 *July* 1999; *after final revision* 10 *October* 2002; *accepted* 24 *February* 2003)

In 1989, Rhaly determined the spectrum of Rhaly operator R_a regarded as an operator on the Hilbert space l_2 normed by $\|x\| = (\Sigma_n |x_n|^2)^{1/2}$. It is the purpose of this paper to determine the spectrum of Rhaly operator R_a as an operator on the spaces bv_0 and bv.

Key Words : Rhaly Operator; Cesaro Operator; Spectrum and Point Spectrum

1. INTRODUCTION

Notation — By s; c_0; c; l_p; $b\,v$; $b\,v_0$; bs; T^*, X^*; $B(X)$; $\pi_0(M, X)$; $\sigma(T, X)$; A^t; $O(1)$; δ; δ^n are meant, respectively, the set of all sequences; the space of null sequences; convergent sequences; sequences such that $\Sigma_k |x_k|^p < \infty$; sequences such that $\sum_{k=0}^{\infty} |x_{k+1} - x_k| < \infty$; $bv_0 := bv \cap c_0$; sequences x such that $\sup_{n>0} \left| \sum_{k=0}^{n} x_k \right| < \infty$; the adjoint operator of T; the space of all continuous linear functionals on X, that is the continuous dual of X; the linear space of all bounded linear operators, say, T on X onto itself; the set of all eigenvalues of a boundeed operator T on a Banach space X; the set of the spectrum of T on X; transpose matrix of A; capital order, that is, $x_n = O(1)$ if there exists $M \in R^+$ such that $|x_n| \le M$ for all n; the constant sequences of ones; the sequences whose nth term is 1, all other terms are 0. In this paper we assume that $S := \{a_n : n = 0, 1, 2, \ldots\}$.

By Goldberg[2], if X is a Banach space, $B(X)$ denotes the collection of all bounded linear operators on X and $T \in B(X)$, then there are three possibilities for $R(T)$, the range of T:

(*i*) $R(T) = X$

(*ii*) $\overline{R(T)} = X$, but $R(T) \neq X$,

(*iii*) $\overline{R(T)} \neq X$

and three possibilities for T^{-1} :

(1) T^{-1} exists and continuous,

(2) T^{-1} exists but discontinuous,

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by $I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3$. If an operator is in state III_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exists but is discontinous.

Applying Goldberg's classification to the operator, $A := \lambda I - T$, where $\lambda \in \sigma(T, X)$ the spectrum of T, considered as an operator in $B(X)$ where $X = b\,v_0$ or $X = b\,v$, we have

(*i*) $A = \lambda I - T$ is surjective

(*ii*) $\overline{R(A)} = X$, but $R(A) \neq X$,

(*iii*) $\overline{R(A)} \neq X$

and three possibilities for T^{-1}:

(1) $A = \lambda I - T$ is injective and A^{-1} is bounded.

(2) $A = \lambda I - T$ is injective and A^{-1} is unbounded, and

(3) $A = \lambda I - T$ is not injective.

If λ is a complex number such that $A = \lambda I - T \in I_1$ or $A = \lambda I - T \in II_1$, then $\lambda \in \rho(T, X)$. All scalar values of λ not in $\rho(T, X)$ comprise the spectrum of T. The further classification of $\sigma(T, X)$ gives rise to the fine spectrum of T. That is, $\sigma(T, X)$ can be divided into the subsets $I_2\,\sigma(T, X), I_3\,\sigma(T, X), II_2\,\sigma(T, X), II_3\,\sigma(T, X), \quad III_1\,\sigma(T, X), \quad III_2\,\sigma(T, X), III_3\,\sigma(T, X)$. For example, if $A = \lambda I - T$ is in a given state, III_2 (say), then we write $\lambda \in III_2\,\sigma(T, X)$.

Given a sequence $a = (a_n)$ of scalars, the Rhaly matrix R_a is the lower triangular matrix with constant row-segments,

$$R_a = \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ a_1 & a_1 & 0 & \cdots \\ a_2 & a_2 & a_2 & \cdots \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{pmatrix} \qquad \text{... (1.1)}$$

For $a = \{1/(n+1)\}$ the spectra on $b\,v_0$ and $b\,v$ of Cesaro matrix were studied respectively in[3,4]. In 1989, Rhaly[5] determined the spectrum of Rhaly operator R_a regarded as an operator on the Hilbert space l_2 normed by $\|x\| = (\Sigma_n |x_n|^2)^{1/2}$. In previous works[10,11], the spectrum of the Rhaly operator R_a on c_0 and c were calculated. The purpose of this paper is to determine the spectrum and fine spectrum of Rhaly operators as an operator on $b\,v_0$ and $b\,v$.

2. The Spectrum and the Fine Spectrum of R_a on bv_0

From [8] any matrix A is in $B(bv_0)$ iff A has null columns and

$$\| A \|_{b\,v_0} := \sup_m \sum_n \left| \sum_{k=0}^{m} a_{nk} - a_{n-1,k} \right| < \infty. \qquad \text{... (2.1)}$$

Lemma 2.1 — Suppose $\lambda \neq a_0, a_1, \ldots$; $\lim_n (n+1)\, a_n = 0$ and $\frac{a_n}{a_{n-1}} < \frac{n}{n+1}$. Then for all $\lambda \in C$, there exist positive numbers M and H such that

$$M \leq \prod_{j=1}^{n} \left| 1 - \frac{a_j}{\lambda} \right| \leq H \qquad \text{... (2.2)}$$

PROOF : Since $1 + u \leq e^u$ for all $u \in R$, we have

$$\prod_{j=1}^{n} \left| 1 - \frac{a_j}{\lambda} \right| = \prod_{j=1}^{n} |(1 + \alpha a_j) + i \beta a_j| = \prod_{j=1}^{n} (1 + 2\alpha a_j + (\alpha^2 + \beta^2)\, a_j^2)^{1/2}$$

$$\leq \prod_{j=1}^{n} \left\{ \exp\left\{ 2\alpha a_j + (\alpha^2 + \beta^2)\, a_j^2 \right\} \right\}^{1/2} = \exp\left\{ \sum_{j=1}^{n} \alpha a_j + \sum_{j=1}^{n} O(a_j^2) \right\}$$

$$= \exp\{O(1) + O(1)\} = O(1)$$

and

$$\prod_{j=1}^{n} \left| 1 - \frac{a_j}{\lambda} \right|^{-1} = \prod_{j=1}^{n} (1 + 2\alpha a_j) + (\alpha^2 + \beta^2)\, a_j^2)^{-1/2}$$

$$\leq \exp\left\{ \prod_{j=1}^{n} -\alpha a_j + \sum_{j=1}^{n} O(a_j^2) \right\}$$

$$= \exp\{O(1) + O(1)\} = O(1),$$

where

$$-\frac{1}{\lambda} = \alpha + i\beta.$$

***Theorem* 2.2** — *Suppose* $\frac{a_n}{a_{n-1}} < \frac{n}{n+1}$ *and* $\lim_n (n+1)\, a_n = 0$. *Then* $R_a \in B(bv_0)$ *and* R_a^* *which is the adjoint operator of* R_a *on* $b\,v_0$ *is the transpose of* R_a *and* $R_a^t \in B(bs)$.

PROOF : From [6, Corollary 4], $\| R_a \|_{bv_0} = a_0$ and $R_a \in B(bv_0)$.

Using the lemma 1.4 in [3], we have

$$R_a^* = R_a^t = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ 0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_2 & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad \text{... (2.3)}$$

For any T operator on a normed space X, since $\| T \|_X = \| T^* \|_{X^*}$, we obtain $\| R_a \|_{b\,v_0} = \| R_a^* \|_{bs} = a_0$. Finally it is obtained that $R_a^* \in B(b\,v_0^*)$, i.e., $R_a^t \in B(bs)$.

***Theorem* 2.3** — *If* $\lim_n (n+1)\, a_n = 0$ *and* $\dfrac{a_n}{a_{n-1}} < \dfrac{n}{n+1}$, *then* R_a *is a compact operator on* bv_0.

PROOF : If $R_a^{(m)}$ is the Rhaly matrix with diagonal sequence $(a_0, a_1, \ldots, a_m, 0, 0, ..)$, then the $R_a^{(m)}$ is a compact operator on $b\,v_0$ and $R_a^{(m)} \in b(b\,v_0)$, the dimension of $R(R_a^{(m)})$ is finite for each m, and

$$\| (R_a - R_a^{(m)})\, x \|_{b\,v_0} = \left\| \left(0, 0, \ldots, a_{m+1} \sum_{k=0}^{m+1} x_k, a_{m+2} \sum_{k=0}^{m+2} x_k, \ldots \right) \right\|_{bv_0}$$

$$= \sum_{k=m+1}^{\infty} \left| a_n \sum_{k=0}^{n} x_k - a_{n+1} \sum_{k=0}^{n+1} x_k \right|$$

$$= \sum_{k=m+1}^{\infty} \left| (a_n - a_{n+1}) \sum_{k=0}^{n} x_k - a_{n+1} x_{n+1} \right|$$

$$\le \sum_{k=m+1}^{\infty} | a_n - a_{n+1} | \sum_{k=0}^{n} | x_k | + | a_{n+1} | \, | x_{n+1} |$$

$$\le \sup_k | x_k | \cdot \sum_{k=m+1}^{\infty} [| a_n - a_{n+1} | (n+1) + a_{n+1}]$$

$$\le \sup_k | x_k | \cdot \left\{ \sum_{k=m+1}^{\infty} [(n+1)\, a_n - (n+1)\, a_{n+1}] \sum_{k=m+1}^{\infty} a_{n+1} \right\}$$

$$= \sup_k \; |x_k| \cdot \left\{ (m+2)\, a_{m+1} + 2 \sum_{k=m+2}^{\infty} a_n - \lim_{r \to} \; (r+1)\, a_{r+1} \right\}$$

$$= \sup_k \; |x_k| \cdot \left\{ (m+2)\, a_{m+1} + 2 \sum_{k=m+2}^{\infty} a_n \right\} .$$

Thus since $\lim_n (n+1)\, a_n = 0$, $\lim_{n \to \infty} \| (R_a - R_a^{(m)}) \|_{bv_0} = 0$. Hence R_a is a compact operator on bv_0 if $\lim_n (n+1)\, a_n = 0$.

***Theorem* 2.4** — *If* $\frac{a_n}{a_{n-1}} < \frac{n}{n+1}$ *and* $\lim_n (n+1)\, a_n = 0$ *then* $\pi (R_a, b\, v_0) = S$.

PROOF : If $R_a x = \lambda x$, then $(a_0 - \lambda)\, x_0 = 0$ and $(\lambda\, a_n^{-1} - 1)\, a_n = a_{n-1}^{-1}\, \lambda\, x_{n-1}$ for all $n \geq 1$. If $\lambda = 0$, then $x = \theta$ therefore $0 \notin \pi (R_a, b\, v_0) = S$. If m is the smallest integer for which $x_m \neq 0$, then $\lambda = a_m$ and

$$x_n = \prod_{j=m+1}^{n} \frac{\lambda\, a_{j-1}^{-1}}{\lambda\, a_j^{-1} - 1}\, x_m \qquad \text{... (2.4)}$$

for $n \geq m+1$. From (2.4), we conclude that the eigenvalues of R_a are simple.

Now let us investigate for $\lambda = a_m$ whether $x \in b\, v_0$ where x is given by (2.4).

From [9, Thm.3], $(x_n) \in c_0$. Since

$$|x_n - x_{n+1}| = \left| \prod_{j=m+1}^{n} \frac{\lambda\, a_{j-1}^{-1}}{\lambda a_j^{-1} - 1} - \prod_{j=m+1}^{n+1} \frac{\lambda\, a_{j-1}^{-1}}{\lambda\, a_j^{-1} - 1} | \cdot | x_m \right|$$

$$= \left| 1 - \frac{\lambda\, a_n^{-1}}{\lambda\, a_{n+1}^{-1} - 1} \right| \prod_{j=m+1}^{n} \left| \frac{\lambda\, a_{j-1}^{-1}}{\lambda\, a_j^{-1} - 1} \right| \cdot |x_m|$$

$$= O(1) \frac{a_n}{a_m} \left| \frac{\left(1 - \frac{a_{n+1}}{a_n} \lambda - a_{n+1} \right)}{\lambda - a_{n+1}} \right|$$

and $\lim_n x_n = 0$, then we have $\sum_n |x_n - x_{n+1}| = O(1) \sum_n a_n < \infty$. Hence $x \in b\, v_0$.

Lemma 2.5 — *Suppose* $z_n = \prod_{v=1}^{n} \left(1 - \frac{a\, v}{\lambda} \right)$ *where* $\lambda \in C$ *and* $\lambda \neq 0$. *Then* (s_n) *is a bounded* sequence where $s_n = \sum_{k=1}^{n} z_k$ iff $\lambda = a_m$, $m = 0, 1, 2, \ldots$.

PROOF : Suppose $\lambda = a_m$, $m = 0, 1, 2, \ldots$ For all $n \geq m$ we have $z_n = 0$. Therefore the only if part of the Lemma is obvious.

Conversely suppose that (s_n) is a bounded sequence where $s_n = \sum_{k=1}^{n} z_k$ We have that

$\ln(1-u) = -\sum_{n=1}^{\infty} \frac{u^n}{n} = -u + O(u^2)$ and $\ln\left(\frac{1}{1-u}\right) = u + O(u^2)$ uniformly in $|u| + < 1/2, u \in C$.

Now given $\lambda \neq 0$ there is v_0 such that $\frac{|\lambda|}{a_v} \geq 2$ and

$$\ln Z_n = \sum_{\vartheta=0}^{n} \ln\left(1 - \frac{1}{\lambda a_\vartheta^{-1}}\right)$$

$$= \sum_{\vartheta=0}^{\vartheta_0 - 1} \ln\left(1 - \frac{1}{\lambda a_\vartheta^{-1}}\right) + \sum_{\vartheta=\vartheta_0}^{n} \ln\left(1 - \frac{1}{\lambda a_\vartheta^{-1}}\right)$$

$$= O(1) + \sum_{\vartheta=\vartheta_0}^{n} \left(-\frac{a_\vartheta}{\lambda} + O\left(\frac{a_\vartheta^2}{|\lambda|^2}\right)\right)$$

$$= O(1) - \frac{1}{\lambda} \sum_{\vartheta=\vartheta_0}^{n} a_\vartheta + \frac{1}{|\lambda|^2} \sum_{\vartheta=\vartheta_0}^{n} O(a_\vartheta^2) = \ln(O(1)).$$

Hence $S_n = O(1)$.

Similarly $\ln\left(\frac{1}{Z_n}\right) = O(1) + \frac{1}{\lambda} \sum_{\vartheta=\vartheta_0}^{n} a_\vartheta + \frac{1}{|\lambda|^2} \sum_{\vartheta=\vartheta_0}^{n} O(a_\vartheta^2) = (O(1) = \ln(O(1))$, and we have $\frac{1}{Z_n} = O(1)$. Using these equalities and the definition of Z_n for $\lambda \neq a_m$ (for all m), $s_n = \sum_{k=1}^{n} Z_k$ is unbounded.

Theorem 2.6 — *If* $\frac{a_n}{a_{n-1}} < \frac{n}{n+1}$ *and* $\lim_n (n+1)\, a_n = 0$, *then* $\pi_0(R_a^*, bv_0^* \cong bs) = S$.

PROOF : Since R_a^* is the transpose of R_a, for $x \neq \theta$, if $R_a^* x = \lambda x$ then $\lambda a_n^{-1} x_{n+1} = (\lambda a_n^{-1} - 1)\, x_n$. If $\lambda = 0$ then $x = \theta$, therefore $0 \notin \pi_0(R_a^*, bs)$. If $\lambda \neq 0$, from Lemma 2.5.

$x = (x_n) \in B(bs)$ where

$$x_n = \prod_{j=0}^{n-1}\left(1 - \frac{a_j}{\lambda}\right)x_0 \quad \text{... (2.5)}$$

if and only if $\lambda = a_m$ (for all m).

Now using the properties of a compact operator, we will calculate $\sigma(R_a, bv_0)$.

Theorem 2.7 — *If* $\frac{a_n}{a_{n-1}} < \frac{n}{n+1}$ *and* $\lim_n (n+1)\, a_n = 0$. *Then* $\sigma(R_a, bv_0) = S \cup \{0\}$.

PROOF : Since R_a is a compact operator and dim $(bv_0) = \infty$, $0 \in \sigma(R_a, bv_0)$ [7]. Furthermore, non-zero spectral values of R_a are eigenvalues[1]. Hence by Theorem 2.3 and Theorem 2.6 the proof is completed.

Theorem 2.8 — *Suppose* $\frac{a_n}{a_{n-1}} < \frac{n}{n+1}$ *and* $\lim_n (n+1)\, a_n = 0$. *If* $\lambda = a_m$ *and* $\lambda \neq 0$ *then* $\lambda \in III_3\, \sigma(R_a, bv_0)$.

PROOF : Since $\lambda = a_m \in \pi_0(R_a, bv_0)$, $(\lambda I - R_a)^{-1}$ does not exist, i.e., $\lambda I - R_a \in (3)$. On the other hand since $\lambda = a_m \in \pi_0(R_a^*, bs)$, $\lambda I - R_a^*$ is not one to one. So R_a has dense range on bv_0 [2, p.59). As a result $\lambda I - R_a \in III$ and hence $\lambda \in III_3\, \sigma(R_a, bv_0)$.

Theorem 2.9 — *Suppose* $\frac{a_n}{a_{n-1}} < \frac{n}{n+1}$ *and* $\lim_n (n+1)\, a_n = 0$. *Then* $0 \in II_2\, \sigma(R_a, bv_0)$.

PROOF : From Theorem 2.4, since $\pi_0(R_a, bv_0) = S$, we have $0 \notin \pi_0(R_a, bv_0)$. Thus R_a^{-1} exists and $R_a \in (1) \cup (2)$.

If $R_a^* x = y$, then it is obtained $x_n = a_n^{-1}(y_n - y_{n+1})$. Choosing $(y_n) = ((-1)^n) \in bs$ we have $(x_n) = (2(-1)^n a_n^{-1}) \notin bs$, i.e., $R(R_a^*) \neq bs$. Hence $R_a \in (2)$[2, p. 60].

From Theorem 2.6, since $0 \notin \pi_0(R_a^*, bs)$, R_a^* operator is one to one. Therefore $\overline{R(R_a)} = bv_0$[2, p. 59].

Clearly the entries of the matrix $R_a^{-1} = (b_{nk})$ are given by

$$b_{nk} = \begin{cases} a_n^{-1}, & k = n \\ -a_{n-1}^{-1}, & k = n-1 \\ 0, & \text{otherwise} \end{cases} \quad \text{... (2.6)}$$

Choose $y = (y_n) = \{(-1)^n a_n\} \in bv_0$. If $R_a^* x = y$, then $x_n = 2(-1)^n$ for all $n \geq 1$. Therefore, $x = (x_n) \notin bv_0$. Hence $R(R_a) \neq bv_0$, and consequently, $R_a \in II_1$, i.e., $0 \in II_2\, \sigma(R_a, bv_0)$.

3. The Spectrum and the Fine Spectrum of R_a on $b\,v$

From [9] any matrix A is in $B(b\,v)$ iff A holds (2.1) and $A\,\delta \in b\,v$.

***Theorem* 3.1** — *Let* $\frac{a_n}{a_{n-1}} < \frac{n}{n+1}$ *and* $\lim_n (n+1) = a_n = 0$. *Then* $R_a \in B(b\,v)$ *and* R_a^* *which is the adjoint operator of* R_a *on* $b\,v$ *is given by*

$$R_a^* = \begin{pmatrix} L & a_0 - L & a_1 - L & a_2 - L & \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_1 & a_2 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad \ldots (3.1)$$

PROOF : Suppose that $(y_n) := R_a\,\delta = (a_0, 2a_1, 3a_2, \ldots, (n+1)\,a_n, \ldots)$. We have

$$(y_n) \in b\,v \Leftrightarrow \sum_n |y_n - y_{n-1}| = \sum_{n=0}^{\infty} |(n+2)\,a_{n+1} - (n+1)\,a_n|$$

$$= \sum_{n=0}^{\infty} ((n+1)\,a_n - (n+2)\,a_{n+1}))$$

$$= \lim_{k \to \infty} (a_0 - (k+2)\,a_{k+1}) = a_0.$$

It was shown that R_a satisfied held (2.1) in Theorem 2.2.

Let $T : b\,v \to b\,v$ be an operator given by the matrix $A = (a_{nk})$. Then Okutoyi [4, Lemma 2.1] showed that $T^* : bv^* \to bv^*$ is given by

$$T^* = \begin{pmatrix} \bar{\chi} & \upsilon_0 - \bar{\chi} & \upsilon_1 - \bar{\chi} & \upsilon_2 - \bar{\chi} & \cdots \\ u_0 & a_{00} - u_0 & a_{10} - u_0 & a_{20} - u_0 & \cdots \\ u_1 & a_{01} - u_1 & a_{11} - u_1 & a_{21} - u_1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad \ldots (3.2)$$

where $\bar{\chi} = \lim_{n \to \infty} \sum_{\vartheta=0}^{\infty} a_{n\vartheta}$, $u_n = \lim_{k \to \infty} a_{kn}$ and $\upsilon_k = P_k T(\delta) = \sum_{\vartheta=0}^{\infty} P_k T(\delta^{\vartheta})$.

Choosing $A = R_a$, we obtain $\overline{\chi} = \lim_{n \to \infty} \sum_{\vartheta=0}^{\infty} a_{n\vartheta}, = \lim_{k \to \infty} (a_{n0} + a_{n1} + a_{n2} + \ldots + a_{nn}) = \lim_{n \to \infty} (n+1)a_n = 0$, $u_n = \lim_{k \to \infty} a_{kn} = \lim_{k \to \infty} a_k = 0$ and $\upsilon_k = (P_k \circ T)(\delta) = (k+1)a_k$. Hence equality (3.1) is true.

***Theorem* 3.2** — *If* $\lim_n (n+1)a_n = 0$ *and* $\frac{a_n}{a_{n-1}} < \frac{n}{n+1}$, *then* R_a *is a compact operator in* $B(b\upsilon)$.

***Theorem* 3.3** — *If* $\frac{a_n}{a_{n-1}} < \frac{n}{n+1}$ *and* $\lim_n (n+1)a_n = 0$, *then* $\pi(R_a, b\upsilon) = S$.

The proof is similar to Theorem 2.4.

***Theorem* 3.4** — *If* $\frac{a_n}{a_{n-1}} < \frac{n}{n+1}$ *and* $\lim_n (n+1)a_n = 0$, *then* $\pi(R_a^*, b\upsilon^* \cong C \oplus bs) = S \cup \{0\}$.

PROOF : If $R_a^* x = \lambda x$ where $x \neq \theta$, then the following equalities hold

$$\sum_{k=0}^{\infty} (k+1)a_k x_{k+1} = \lambda x_0 \qquad \ldots (3.3)$$

and

$$\sum_{k=n-1}^{\infty} a_k x_{k+1} = \lambda x_n \text{ for all } n \geq 1. \qquad \ldots (3.4)$$

If $\lambda = 0$, then chosing $x_0 \neq 0$ and $x_n = 0$ for $n \geq 1$ we have $(x_n) = (x_0, 0, 0, \ldots) \in bs$. Hence $0 \in \pi(R_a^* C \oplus bs)$. From (3.4), $x_{n+1} = \left(1 - \frac{a_{n-1}}{\lambda}\right) x_n$. If $\lambda = a_m$ $(m = 0, 1, 2, \ldots)$, then $x_n = 0$ for $n > m+1$. Hence $(x_n) \in bs$, i.e., $\lambda = a_m \in \pi(R_a^*, C \oplus bs)$. From (3.4), we obtain $x_n = \left(1 - \frac{a_j}{\lambda}\right) x_1$ for $n > 1$. From Lemma 2.5 the eigenvector corresponding to (x_n) belongs to bs if and only if $\lambda = a_m$ $(m = 0, 1, 2, \ldots)$. Hence there are no eigenvalues except the a_n's.

***Theorem* 3.5** — *Suppose* $\frac{a_n}{a_{n-1}} < \frac{n}{n+1}$ *and* $\lim_n (n+1)a_n = 0$. *Then* $0 \in II I_2\, \sigma(R_a, b\upsilon)$.

PROOF : Since $0 \notin \pi_0(R_a, b\upsilon_0) = S$ there exist R_a^{-1}, i.e., $R_a \in (1) \cup (2)$. If $R_a^* x = y$, then the following equalities hold :

$$\sum_{k=0}^{\infty} (k+1)a_k x_{k+1} = y_0 \qquad \text{... (3.5)}$$

and

$$\sum_{k=n-1}^{\infty} a_k x_{k+1} = y_n \text{ for all } n \geq 1. \qquad \text{... (3.6)}$$

Since $x_n = a_n^{-1}(y_n - y_{n+1})$, chosing $y_0 \neq 0$ and $y_n = (-1)^n$ for all $n \geq 1$, we have $y = (y_n) \in bs$. From (3.7), since $x_n = 2(-1)^n a_n^{-1}$, we have $(x_n) \notin bs$. Therefore, $R(R_a^*) \neq bs$, i.e., $R_a \in (2)$ [2, p. 60].

Since $0 \in \pi(R_a^*, c \oplus bs), R_a^*$ is not one to one and hence $\overline{R(R_a)} \neq bv$ [2, p. 59], i.e., $R_a \in III$. Finally, $R_a \in III_2$ and hence it is obtained $0 \in III_2\ \sigma(R_a, bv)$.

***Theorem* 3.6** — *Suppose* $\frac{a_n}{a_{n-1}} < \frac{n}{n+1}$ *and* $\lim_n (n+1)\, a_n = 0$. *If* $\lambda = a_m, \lambda \neq 0$, *then* $\lambda \in III_3\ \sigma(R_a, bv)$.

PROOF : Since $\lambda = a_m \in \pi(R_a, bv)$, then $(\lambda I - R_a)^{-1}$ doesn't exist. Hence we have $a_m I - R_a \in (3)$.

Since $\lambda = a_m \in \pi(R_a^*, C \oplus bs)$ for $m = 0, 1, 2, \ldots$, the adjoint operator $a_m I - R_a^*$ is not one to one. Therefore, $\overline{R(a_m I - R_a)} \neq bv$ [2, p. 59], i.e., $a_m I - R_a \in III$ and consequently $a_m \in III_3$ $\sigma(R_a, bv)$.

REFERENCES

1. A. Brown and E. Page, *Elements of Functional Analysis*, (Von Nostrand Reinhold Comp., 1970).
2. S. Goldberg, *Unbounded Linear Operators*, (McGraw-Hill Book Company, 1966).
3. I. J. Okutoyi, *J. Austral. Math. Soc.*, (Series A) **48** (1990), 79-86.
4. I. J. Okutoyi, *Commun. Fac. Sci. Univ. Ank. Series* A_1, Vol. **II**, (1992), 197-207.
5. H. C. Rhaly Jr., *Bull. London Math. Soc.*, **21** (1989), 399-406.
6. B. E. Rhoades, *Canad. Math. Bull.*, **35** (1992), 410-15.
7. W. Rudin, *Functional Analysis*, (McGraw-Hill Book Comp., 1973).
8. M. Stieglitz and H. Tietz, *Math. Z.*, **154** (1977), 1-16.
9. A. Wilansky, *A Summability Through Functional Analysis*, (North Holland, 1984).
10. M. Yildirim, *Indian J. Pure Appl. Math.*, **27**(8) (1996), 779-784.
11. M. Yildirim, *Indian J. Pure Appl. Math.*, **29**(12) (1998), 1301-1309.