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AN ISOMORPHISM THEOREM ON DIRECTED HYPERCUBES OF DIMENSION n

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In graph theory, hypercubes of dimension n were well known. In this paper directed hypercubes were introduced. Four properties P_1, P_2, P_3, P_4 of directed graphs were identified and finally proved that a directed graph is a directed hypercube of dimension n if and only if it satisfies properties $P_1 - P_4$.

Key Words : Directed Graph; Hypercube; Dimension; Level; Graph

1. INTRODUCTION

Let $G = (V, E)$ be a directed graph, where $e = vu$ is an *arc* originating at v and terminating at u. For definitions, not included and results referred here, a reference can be made either to [1] or [3] or both.

Definition 1 — (*i*) A subgraph of G generated by a node $v_0 \in V$ is the graph $G_{v_0}(V_{v_0}, E_{v_0})$ where V_{v_0} $\{v$ / there exists a directed path from v_0 to $v\} \cup \{v_0\}$ and $E_{v_0} = \left\{ uv \in E/u, v \in V_{v_0} \right\}$. In this case, we also say that G_{v_0} is rooted at v_0 and v_0 is the root of G_{v_0}. For any $v \in V_{v_0}$, we define an integer 1 (v) called the *level* of v (in G_{v_0}) as: 1 (v) = min $\{n$ / n is the length of a path from v_0 to $v\}$. Clearly 1 $(v_0) = 0$ if v_0 is the root. We say that uv starts at level m and ends at level $m + 1$ if 1 $(u) = m$ and 1 $(v) = m + 1$.

(*ii*) Suppose G is a graph rooted at v_0. Write $n = \max \{k / l(v) = k$ for some $v \in V\}$.

For $1 \leq i \leq n$, we write $L_i = \{ v \in V / l(v) = n - i \}$ and $F_i = \left\{ uv \in E / u \in L_i \right\}$.

Definition 2 — Let X be a set with $|X| = n$. The graph Q_n with $V(Q_n) = \mathcal{P}(X)$ (the power set of X) and $E(Q_n) = \left\{ IJ / I, J \in \mathcal{P}(X), I \supseteq J \text{ and } |I| = |J| + 1 \right\}$, is called a *directed hypercube of dimension n* (or directed n-cube, in short).

Undirected hypercubes were studied by Mulder[2]. Mulder used the concept of "median graph" and proved that a graph G is an n-cube if and only if G is a regular median graph of degree n.

In this paper, we enlist four properties P_1, P_2, P_3, P_4 of directed graphs and prove that a directed graph is isomorphic to a directed n-cube if and only if it satisfies these properties.

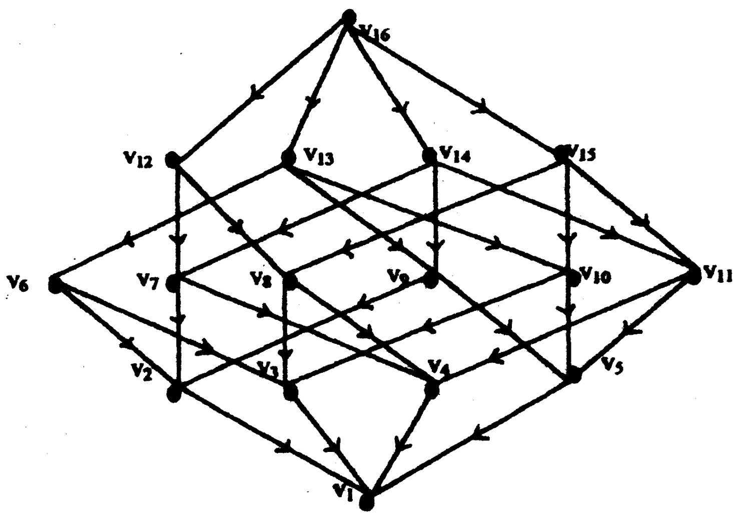
2. The Four Properties

We start this section with the following definition

Definition 1.1 — Let G be a graph rooted at v_0. We take n as it is defined in Def. 0.1 (ii). A node $v \in L_k$ is said to be *determined* by x_i, $1 \leq i \leq k$ if $L_1 \cap V_v = \{x_i/1 \leq i \leq k\}$ (here $L_1 = \{v \in V/l(v) = n-1\}$ and there is no $x \in L_k$ other than v such that $L_1 \cap V_x = \{x_i/1 \leq i \leq k\}$.

Definition 1.2 — A graph G is said to be *completely determined* if (*i*) every node $v \in L_k$ is determined by $L_1 \cap V_v$ and $|L_1 \cap V_v| = k$; (*ii*) $\{x_i/1 \leq i \leq k\} \subseteq L_1$ implies there exists $v \in L_k$ such that $\{x_i/1 \leq i \leq k\} = L_1 \cap V_v$; and (*iii*) if x, $y \in V$, $\{x_1, x_2, ..., x_{k-1}\} = L_1 \cap V_y$ and $\{x_1, x_2, ..., x_k\} = L_1 \cap V_x$ then xy.

Example 1.3 — Consider the graph given here.



(*i*) This graph is rooted at v_{16} and $n = \max\{k/1(v) = k \text{ for some } v \in V\} = 4$;

(*ii*) $L_0\{v \in V/1(v) = 4\} = \{v_1\}$;

(*iii*) $L_1 = \{v \in V/1(v) = 4 - 1 = 3\} = \{v_2, v_3, v_4, v_5\}$.

(*iv*) Here we can observe that this graph satisfies the conditions (*i*) and (*ii*) that are given in Definition 1.2.

(*v*) In above graph $V_{v_{12}} = \{v_{12}, v_7, v_8, v_2, v_3, v_4, v_1\}$, $V_{v_6} = \{v_6, v_2, v_3, v_4, v_1\}$.

Now $L_1 \cap V_{v_{12}} = \{v_2, v_3, v_4\} \supseteq \{v_2, v_3\} = L_1 \cap V_{v_6}$ and there is no arc from v_{12} to v_6. So condition (*iii*) (of the above definition) does not hold.

Now we list the properties to be satisfied for our main theorem.

$(P_1) : G = (V, E)$ is simple and having no circuits.

(P_2) : In G (V, E) the number of levels is $n + 1$ and the number of nodes at level i is ${}^nC_i$ for $0 \leq i \leq n$.

(P_3) : In $G = (V, E)$, id $(v) = m$ if $l(v) = m$; and od $(v) = m$ if 1 $(v) = n - m$. (Here id, od denote indegree, outdegree respectively)

(P_4) : $G = (V, E)$ is completely determined.

Lemma — If G satisfies P_1, P_2 and P_3 then the number of arcs starting at level $i - 1$ is same as the number of arcs ending at level i.

PROOF : The number of arcs starting at level $i - 1 = {}^nC_{i-1}(n - i + 1) = {}^nC_i\, i$ = the number of arcs ending at level i.

***Theorem* 1.5** — *If G satisfies P_1, P_2 and P_3 then every arc starting at level i – 1 ends at level i (In other words, if $x \in L_m$ and xy, then $y \in L_{m-1}$)*

PROOF : We prove this by mathematical induction.

Let P (m) be the statement: All the arcs starting at level k end at level $k + 1$ for all $k < m$. To show P (1) is true, suppose $L_n = \left\{ v_0 \right\}$ and $v \in L_{n-1}$. Since 1 $(v) = 1$, there exists an arc from v_0 to v. Since od $(v_0) = n$, id $(v) = 1$ for each $v \in L_{n-1}$, and $|L_{n-1}| = n$, we have that all the arcs starting at v_0 end at some $v \in L_{n-1}$. So $P(1)$ is true. Suppose $P(m)$ is true. If $P(m + 1)$ is not true, then since P (m) is true, there is an arc starting at level $k = m$ which does not end at level $m + 1$. Since all the arcs starting at level m are not end at level $m + 1$, by (i) there exists a node y with l $(y) = m + 1$ which cannot receive all of its id $(y) = m + 1$ arcs from level m. So there exists arc xy with $l(x) \neq m$. If $l(x) < m$ then since $P(m)$ is true, every arc starts at level $l(x)$ ends at level l $(x) + 1$ and so $l(y) = l$ $(x) + 1$. Now $m + 1 = l$ $(y) = 1$ $(x) + 1 \leq m$, a contradiction. If 1 $(x) > m$ then $m + 1 = 1$ $(y) > 1$ $(x) \geq m + 1$, a contradiction. Therefore P $(m + 1)$ is true. This completes the proof.

Corollary 1.6 — Let $x, y \in V$ and xy. Then (i) $L_1 \cap V_y \subseteq L_1 \cap V_x$, (ii) If G satisfies P_1, P_2, P_3 and F_4 then $L_1 \cap V_y \subseteq L_1 \cap V_x$ and $|L_1 \cap V_y| + 1 = |L_1 \cap V_x|$

PROOF : (i) There exists arc $xy \Rightarrow V_y \subseteq V_x \Rightarrow L_1 \cap V_y \subseteq L_1 \cap V_x$.

(ii) Let $x \in L_k$. By Theorem 1.5, we have $y \in L_{k-1}$. Now by condition (i) of the definition 1.2, it follows that $|L_1 \cap V_x| = k = 1 + (k - 1) = 1 + |L_1 \cap V_y|$.

Note 1.7 — From the Corollary 1.6, we may conclude that the converse of the statement given in condition (iii) of the definition 1.2, is true if the graph G satisfies properties $P_1 - P_4$.

3. Main Theorem

***Theorem* 2.1** — *If* $G=(V,E)$ *and* $G^1=(V^1,E^1)$ *are two finite directed graphs satisfying* P_1, P_2, P_3 *and* P_4 *then* G *is isomorphic to* G^1.

PROOF : Suppose $L_i^1=\{v\in V^1/1\,(v)=n-i\}, 0\le i\le n$ and $F_j^1=\{uv\in E^1/u\in L_i^1\}, 1\le j\le n$. Since the sets L_i, L_i^1, F_j, F_j^1 form partitions for V, V^1, E, E^1 respectively, we can define $f: \bigcup_{i=0}^{m} L_i \to \bigcup_{i=0}^{m} L_i^1$ and $g: \bigcup_{j=1}^{m} F_j \to \bigcup_{j=1}^{m} F_j^1$ by induction on m. Suppose $L_0=\{v_0\}$, $L_0^1=\{v_0^1\}$, $L_1=\{v_i/1\le i\le n\}$ and $L_1^1=\{v_i^1/1\le i\le n\}$. Now define $f(v_0)=v_0^1$ and $f(v_i)=v_i^1$ for $1\le i\le n$. Clearly f is a bijection from $L_0\bigcup L_1$ to $L_1^1\bigcup L_1^1$. By Theorem 1.5, $F_1=\{v_i v_0/1\le i\le n\}$ and $F_1^1=\{v_i^1 v_0^1\ 1\le i\le n\}$. Define $g: F_1\to F_1^1$ as $g(v_i v_0)=v_i^1 v_0^1=f(v_i)f(v_0)$. Clearly g is a bijection from F_1 to F_1^1. Suppose $f: \bigcup_{i=1}^{k-1} L_i \to \bigcup_{i=1}^{k-1} L_i^1$ and $g: \bigcup_{j=1}^{k-1} F_j \to \bigcup_{j=1}^{k-1} F_j^1$ are extended bijections with $g(xy)=f(x)f(y)$. Let $x\in L_k$. Suppose x is determined by $x_i\in L_1$, $1\le i\le k$ and consider $f(x_i)$ $1\le i\le k$. By (P_4) there exists unique $x^*\in L_k^1$ such that $L_1^1\bigcap V_x^*=\{f(x_i)\ 1\le i\le k\}$. Now define $f(x)=x^*$. Take x, y such that $x\ne y$. Suppose $L_1\bigcap V_x=\{x_i/1\le i\le k\}$ and $L_1\bigcap V_y=\{y_i/1\le i\le k\}$. Since $x\ne y$, these two sets are distinct. Since $f: L_1\to L_1^1$ is a bijection, we have $\{f(x_i)/1\le i\le k\}\ne\{f(y_i)/1\le i\le k\}$ and so $f(x)\ne f(y)$. This shows that f is one-one. Take $x^*\in L_k^1$.

Since $f: L_1\to L_1^1$ is a bijection, we can suppose that $L_1^1\bigcap V_x^*=\{f(x_1)/1\le i\le k, x_i\in L_1\}$. Now $f(x)=x^*$ where $V_x\bigcap L_1=\{x_i/1\le i\le k\}$. Therefore f is onto. Now we define $g: F_k\to F_k^1$ as follows : Let $xy\in F_k$. Suppose $V_y\bigcap L_1=\{x_i/1\le i\le k-1\}$. By Cor. 1.6, $V_y\bigcap L_1\subseteq V_x\bigcap L_1$, $|V_y\bigcap L_1|=k-1$, there exists $x_k\in L_1$ such that $V_x\bigcap L_1=\{x_i/1\le i\le k\}$. By (P_4), we have $f(x)$ $f(y)$. Define $g(xy)=f(x)f(y)$. To show g is onto, take $x^1 y^1\in F_k^1$. If $V_y^1\bigcap L_1^1=\{y_i^1/1\le i\le k-1\}$, by Corollary 1.6, there exists $y_k^1\in L_1^1$ such that $V_x^1\bigcap L_1^1=\{y_i^1/1\le i\le k\}$. Let $x, y\in V$ such that $f(x)=x^1$, $f(y)=y^1$ and $y_i\in L_1$ such that $f(y_i)=y_i^1$. Now it is clear that $V_x\bigcap L_1=\{y_i/1\le i\le k\}\supseteq$

$\{y_1/1 \le i \le k-1\} = V_y \cap L_1$ and so xy. Hence $g(xy) = x^1 y^1$. Since f is one-one, we have g is one-one. This completes the proof.

***Theorem* 2.2** — *Any finite directed graph G satisfying P_1, P_2, P_3 and P_4 is isomorphic to Q_n, the directed hypercube of dimension n.*

PROOF: If we show that Q_n satisfies P_1, P_2, P_3 and P_4 then the rest follows from Theorem 2.1. Suppose $V(Q_n) = \mathcal{P}(X)$ where X is a set with $|X| = n$. Clearly Q_n satisfies P_1. Since $I \in V(Q_n)$ is of level m if and only if $|I| = n - m$, we have that the total number of levels possible is $n + 1$. Since the number of distinct subsets I of X with $|I| = n - m$, that can be frmed is ${}^nC_{n-m} = {}^nC_m$, we have that the number of nodes at level m is ${}^nC_m$ for all $1 \le m \le n$. Hence P_2 Let $1\,(I) = m$. Then $|I| = n - m$. If IJ then $|J| = n - m - 1$. Since number of distinct sets J with $|J| = n - m - 1$ that can be formed from the elements of I is ${}^{(n-m)}C_{n-m-1} = n - m$, we have that there exists $n - m$ arcs which start at I and end at subsets J of I with $|J| = |I| - 1$. Therefore Q_n satisfies one part of P_3. To show the other part, suppose $J \in V(Q_n)$ with $1\,(J) = m$. Since IJ means $I \supseteq J, |I| = |J| + 1$; and $|J| = n - m$, there exists m such subsets I such that $IJ \in E(Q_n)$. Therefore there are m arcs ends at J. Hence P_3 holds. It is clear that if $I \in V(Q_n)$ then I is determined by $\{x_i\}$, $1 \le i \le k$ where $I = \{x_i/1 \le i \le k\}$. Now P_4 follows from the definition of Q_n.

By combining Theorems 2.1 and 2.2, we get the following main theorem.

***Theorem* 2.3** — *A directed graph is directed hypercube of dimension n if and only if it satisfies P_1, P_2, P_3 and P_4.*

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