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SLOW STEADY ROTATION OF A DISC WITH SLIP IN A VISCOUS FLUID

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The problem considered in this paper is concerned with the determination of the viscous couple on a uniformly rotating disc with angular velocity Ω about its axis in an infinite extent of viscous fluid. The problem is then formulated as the mixed boundary value problem and solved by finding the azimuthal component of velocity υ for the case of rotational symmetry, satisfying the mixed boundary conditions. To obtain the couple on the disc atempt here is first made for both large and small values of the slip parameter β through the method of perturbation. For small values of β the perturbation method fails. Therefore an approximate solution has been assumed which provides the couple not only for small β but also agrees qualitatively with the perturbation value for large β. It is seen that the effect of slip coefficient β is to decrease the couple on the disc.

Key Words : Viscous Fluid; Couple; Azimuthal Component; Dual Integral Equation

1. INTRODUCTION

The problem of the motion of viscous fluid about a disc which is translating or rotating about the axis of symmetry or about an axis normal to the symmetric axis is of considerable interest for many years. The earliest attack on it was by Jeffery (1915)[6], who considered the steady rotation of an oblate spheroid about its major axis and then degenerated it to a disc by suitable limiting process. He tried to analyse disc problems with the help of dual integral equations. The dual integral equations approach was afterwards used by many mathematicians as an efficient tool for solving the disc related problems.

By exploiting the above method, the problem concerned with disc and flat plate was subsequently solved by Gupta[4], Kanwal[8] and Moon *et al.*[11]

It is well known that a viscous fluid normally adhers to the solid surface in contact with, but it is also known[3] that slipping can take place at a surface made fluid repellent through treatment with certain chemicals. Slipping is also obsered in the case of flow of a highly rarefield gas[13], the slip velocity being proportional to the shear stress. The constant of proportionality is of the order of mean free path and therefore small.

In Tamada and Miura[12] studied the slip flow of viscous fluid at low Reynolds numbers past a flat plate of finite length and he also derived an integral equation for the singularities. Whereas Laurmann[9] did the same problem previously but the plate was here supposed to be of infinite length.

In Hocking[5] and more recently Miksis and Davis[10] studied flow over a rough surface. It was concluded that the no-slip condition may be replaced by an effective slip condition. The slip coefficient was found to be equal to the average roughness which is small.

Thus most of the slip flow problems involve small slip coefficient and may be investigated through simple perturbation about the no-slip situation. But since the stress becomes infinite at an edge, the simple expansion breaks down in such problems.

The present problem is concerned with the rotation of a finite disc in an infinite expanse of viscous fluid. We too are confronted with the same difficulty that the stress becomes infinite at an edge and a way out is suggested through the use of a simple approximation. Because the approximate solution is tenable for both small and large values of the slip parameter β whereas the perturbation method is valid only for large values of β.

2. Formulation of the Problem

Let us consider the flow induced by a disc of radius a rotating steadily with angular velocity Ω about its axis in an infinite expanse of viscous fluid of viscosity μ. In this axiar symmetric problem we shall use non-dimensional cylindrical polar coordinates (r, θ, z). Where the space coordinates have been non-dimensionalized by a; the disc is in the plane $z = 0$ with its axis along the z axis.

In the case of slow motion on account of rotational symmetry there will be only the azimuthal component of velocity v (non-dimensionalized by $a\Omega$) satisfying the equation

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}-\frac{1}{r^2}\right)v=0. \qquad \text{... (1)}$$

The above equation is to be solved under the conditions

$$\left.\begin{aligned} v-r&=\beta\frac{\partial v}{\partial z}, \quad r>1 \\ \frac{\partial v}{\partial z}&=0, \quad r>1 \end{aligned}\right\} z=0 \qquad \text{... (2)}$$

where β is the slip coefficient.

Substituting $v=r\,\Phi$, and invoking the symmetry about the plane $z = 0$, the above system reduces to the boundary value problem

$$\frac{\partial^2\Phi}{\partial r^2}+\frac{3}{r}\frac{\partial\Phi}{\partial r}+\frac{\partial^2\Phi}{\partial z^2}=0, \quad z\geq 0, \qquad \text{... (3)}$$

$$\left.\begin{aligned} \Phi-1&=\beta\frac{\partial\Phi}{\partial z}, \quad r<1 \\ \frac{\partial\Phi}{\partial z}&=0, \quad r>1 \end{aligned}\right\} z=0. \qquad \text{... (4)}$$

Now, eq. (3) is seen to be satisfied by

$$\Phi=\int_0^\infty A(u)\,e^{-uz}\frac{J_1(ur)}{ur}\,du, \qquad \text{... (5)}$$

where J_1 is Bessel's function.

Satisfaction of boundary condition (4) leads to the dual integral equations[15].

$$\int_0^\infty \frac{A(u)}{u}(1+\beta u)\,J_1(ur)\,du=r,\ r<1 \qquad \text{... (6)}$$

$$\int_0^\infty A(u) J_1(ur)\, du = 0, \quad r > 1, \qquad \dots (7)$$

Let us assume

$$\int_0^\infty A(u) J_1(ur)\, du = \chi(r), \quad r , 1$$
$$= 0, \quad r > 1 \qquad \dots (8)$$

and then using inverse Hankel's transform, eq. (8) is seen to provide

$$A(u) = u \int_0^1 r\chi(r) J_1(ru)\, dr. \qquad \dots (9)$$

Substituting the above value of $A(u)$ in (8) and formally effecting the change of order of integration, we obtain the following Fredholm integral eq. (15).

$$\int_0^1 x\chi(x) K(r, x)\, dx = r - \beta\chi(r), \quad 0 < r < 1 \qquad \dots (10)$$

where
$$K(r, x) = \int_0^\infty J_1(ru) J_1(xu)\, du, \qquad \dots (11)$$

and also in terms of hypergeometric function[1] it (11) can be written as

$$K(r, x) = \frac{r}{2x^2}\, {}_2F_1\left[\frac{3}{2}, \frac{1}{2}; 2; \frac{r^2}{x^2}\right], \quad r < x, \qquad \dots (12)$$

or $K(r, x)$ [14] may be written as

$$K(r, x) = \frac{1}{\pi}\int_0^\pi \frac{\cos\xi\, d\xi}{(x^2 + r^2 - 2xr\cos\xi)^{\frac{1}{2}}}. \qquad \dots (13)$$

Now, replacing ξ by 2α and substituting t for $\cos\alpha$ in (13), we integrate to obtain

$$K(r, x) = \frac{1}{\pi (xr)^{\frac{1}{2}}}\left\{\left(\frac{2}{c} - c\right)K(c) - \frac{2}{c}E(c)\right\}, \qquad \dots (14)$$

where $c^2 = 4xr(x + r)^{-2}$, $K(c)$ and $E(c)$ are the complete elliptic integrals[1] of the first and second kids respectively

$$K(c) = \int_0^1 (1 - t^2)^{-\frac{1}{2}}(1 - c^2 t^2)^{-\frac{1}{2}}\, dt, \qquad \dots (15)$$

$$E(c) = \int_0^1 (1-t^2)^{-\frac{1}{2}} (1-c^2 t^2)^{-\frac{1}{2}} dt. \qquad \dots (16)$$

Here, the elliptic integrals are finite everywhere except at $c = 1$ (when $x = r$) because in this case $K(c)$ has logarithmic singularity. Thus the kernel of the Fredholm integral eq. (10) is bounded except at $x = r$.

3. VISCOUS COUPLE

We now obtain an expression for viscous couple necessary for maintaining the rotation by integrating the stress moment over the disc. We thus find

$$M = -2a^3 \int_0^1 2\pi r^2 T_{z\theta} \, dr, \qquad \dots (17)$$

where stress $T_{z\theta} = \mu \Omega \left(\frac{\partial v}{\partial z} \right)$.

Substituting the value of $v = r\Phi$ at $z = 0$ from (5) in the above equation and using (9), we get

$$M = 4\mu\pi a^3 \Omega \int_0^1 r^2 \chi(r) \, dr, \qquad \dots (18)$$

The non-dimensional moment coefficient may now be expressed as

$$C_M = \frac{M}{\mu \Omega a^3} = 4\pi \int_0^1 r^2 \chi(r) \, dr. \qquad \dots (19)$$

Further, multiplying (10) by $\frac{r^2}{\sqrt{1-r^2}}$ and integrating them between [0, 1], we obtain

$$\int_0^1 x\chi(x)\, dx \int_0^\infty J_1(xu)\, du \int_0^1 r^2 \frac{J_1(ru)}{\sqrt{1-r^2}} dr$$

$$= \int_0^1 \frac{r^3}{\sqrt{1-r^2}} dr - \beta \int_0^1 r^2 \frac{\chi(r)}{\sqrt{1-r^2}} dr.$$

On integration, the above equation yields

$$\frac{\pi}{4} \int_0^1 x^2 \chi(x)\, dx = \frac{2}{3} - \beta \int_0^1 r^2 \frac{\chi(r)}{\sqrt{1-r^2}} dr. \qquad \dots (21)$$

Substituting the value of C_M from (19) in (21), we get

$$C_M = \frac{32}{3} - 16\,\beta \int_0^1 r^2 \frac{\chi(r)}{\sqrt{1-r^2}}\, dr. \qquad \ldots (22)$$

For the no-slip case $\beta = 0$, the solution of eq. (21) provides [see eq. (42)].

$$\chi_0(r) = \frac{4r}{\pi\sqrt{1-r^2}}. \qquad \ldots (23)$$

Further using eq. (19), we have

$$C_{M_0} = 4\,\pi \int_0^1 r^2 \chi_0(r)\, dr = \frac{32}{3}. \qquad \ldots (24)$$

Next, multiplying (10) by $r\,\chi_0(r)$ and integrating them between [0, 1], we obtain

$$\int_0^1 r\,\chi_0(r) \int_0^1 (x\,\chi(x)\, K(r,x)\, dx)\, dr$$

$$= \int_0^1 r^2 \chi_0(r)\, dr - \beta \int_0^1 r\,\chi_0(r)\,\chi(r)\, dr. \qquad \ldots (25)$$

Now substituting the value of $\frac{C_{M_0}}{4\pi}$ for $\int_0^1 r^2 \chi_0(r)\, dr$ in the above equation, we get

$$\int_0^1 r\,\chi_0(r) \int_0^1 (x\,\chi(x)\, K(r,x)\, dx)\, dr = \frac{C_{M_0}}{4\pi} - \beta \int_0^1 r\,\chi_0(r)\,\chi(r)\, dr. \qquad \ldots (26)$$

For $\beta = 0$, (10) takes the form

$$\int_0^1 x\,\chi_0(x)\, K(r,x)\, dx = r. \qquad \ldots (27)$$

To get the value of C_M, we multiply (27) by $r\,\chi(r)$ and integer r from 0 to 1 to obtain

$$\frac{C_M}{4\pi} = \int_0^1 r\,\chi(r) \int_0^1 (x\,\chi_0(x)\, K(r,x)\, dx\,)\, dr. \qquad \ldots (28)$$

Then, by using (26) and (28), we deduce the difference of moment coefficient for slip and no-slip case, as

$$C_M - C_{M_0} = -\,4\,\pi\,\beta \int_0^1 r\,\chi_0(r)\,\chi(r)\, dr. \qquad \ldots (29)$$

4. Solution for Large β

After arranging the terms of Fredholm integral eq. (10) and multiplying both sides of equation by $4\pi^2$, we obtain

$$4\pi^2 \beta r \chi(r) = 4\pi^2 r^2 - \int_0^1 \chi(x)\, m(r,x)\, dx, \qquad \text{... (30)}$$

where

$$\begin{aligned} m(r,x) &= 4\pi^2 rx\, K(r,x) \\ &= 4\pi\sqrt{rx}\left[\left(\frac{2}{c} - c\right) K(c) - \frac{2}{c} E(c)\right], \end{aligned} \qquad \text{... (31)}$$

which tallies with the expression (9) of Ashour's paper[2]. If we write

$$r^{\frac{1}{2}} \chi(r) = \Psi(r),$$

and $\beta = -\dfrac{1}{4\pi\lambda}$ the eq. (30) may be expressed as

$$\Psi(r) \;-4\pi\lambda r^{\frac{3}{2}} + 4\lambda \int_0^1 \Psi(x)\, k(r,x)\, dx, \qquad \text{... (32)}$$

where the new kernel

$$k(r,x) = \frac{m(r,x)}{4\pi\sqrt{rx}}. \qquad \text{... (33)}$$

The new kernel $k(r, x)$ will still have $x = r$ as a line singularity.

With minor change in notation the above eq. (32) is seen to be same as eq. (13) of Ashour[2] when $H = -4\pi$, and we can thus exploit the relevent result obtained by him.

Following Ashour[2], the integral eq. (32) can be solved by the method of continued substitution. It follows that

$$\chi(r) = r^{-\frac{1}{2}}\Psi(r) = \frac{1}{\beta} r^{-\frac{1}{2}} \sum_{m=0}^{\infty} \Phi_m(r) \left(-\frac{1}{4\pi\beta}\right)^m,$$

where

$$\Phi_0(r) = r^{\frac{3}{2}}, \quad \Phi_m(r) = 4\int_0^1 \Phi_{m-1}(r)\, K(x,r)\, dr. \qquad \text{... (35)}$$

Writing (34) in expanded form as

$$\chi(r) = \frac{1}{\beta}\eta_0(r) - \frac{1}{4\pi\beta^2}\eta_1(r) + \frac{1}{16\pi^2\beta^3}\eta_2(r) + \ldots, \qquad \text{... (36)}$$

where $\eta_m(r) = r^{-\frac{1}{2}}\Phi_m(r)$ and its values are presented in Table I of Ashour's paper[2].

5. Determination of Couple Coefficient (C_M) for Large β

In terms of $\chi(r)$ the formula for couple is given by eq. (19) as

$$C_M = 4\pi \int_0^1 r^2 \chi(r)\, dr. \quad \text{... (37)}$$

Using (36), we can write the values of $r^2 \chi(r)$ in increasing powers of β as

$$r^2 \chi(r) = \frac{1}{\beta} r^2 \eta_0(r) - \frac{1}{4\pi\beta^2} r^2 \eta_1(r) + \frac{1}{16\pi^2\beta^3} r^2 \eta_2(r) + ... \quad \text{... (38)}$$

To obtain the couple, we need the values of $r^2 \chi(r)$ which on using Table 1 of Ashour[2], are tabulated below:

TABLE I

n,r	0.05	0.15	0.25	0.35	0.45	0.55
0	0.0001	0.003	0.016	0.0429	0091	0.166
1	0.0008	0.021	0.095	0.251	0.519	0.906
2	0.004	0.108	0.490	1.295	2.585	4.391
3	0.019	0.525	2.361	6.196	12.188	19.885
4	0.089	2.445	10.956	28.612	55.805	92.281
5	0.406	11.197	50.094	130.512	253.854	419.205
6	1.843	50.857	227.319	591.565	1141.594	1894.074
7	8.338	229.995	1027.625	2672.583	5186.228	8541.995
8	37.635	1037.993	4636.563	12054.980	23382.675	38496.150

TABLE II

Values of $r^2 \eta_m(r)$

n,r	0.65	0.75	0.85	0.95
0	0.275	0.422	0.614	0.857
1	1.403	1.973	2.520	2.853
2	6.556	8.819	10.672	11.414
3	29.862	39.350	46.596	49.049
4	134.494	175.798	206.512	216.302
5	606.499	788.906	930.580	965.223
6	2310.61	3547.406	4145.199	4329.924
7	12312.920	15966.000	18642.670	19466.925
8	55448.900	71887.500	83918.380	87611.992

Now, the values of couple coefficient is given by

$$C_M = \frac{1}{\beta} C_0(r) - \frac{1}{4\pi\beta^2} C_1(r) + \frac{1}{16\pi^2\beta^3} C_2(r) + \ldots, \qquad \ldots (39)$$

where $$C_m(r) = \int_0^1 r^2 \eta_m(r)\, dr, \ m = 0, 1, 2, \ldots, \infty.$$

Using tapezoidal rule of numerical integration (with uniform mesh size $(h) = 0.1$), we get

$$C_M = \frac{1}{\beta}(0.206) - \frac{1}{4\pi\beta^2}(0.912) + \frac{1}{16\pi^2\beta^3}(2.995) - \frac{1}{64\pi^3\beta^4}(18.150) + \ldots, \qquad \ldots (40)$$

or $$C_M = \frac{0.206}{\beta} - \frac{0.073}{\beta^2} + \frac{0.019}{\beta^3} - \frac{0.009}{\beta^4} + \ldots . \qquad \ldots (41)$$

We see that for $\beta \to \infty$, the couple vanishes as it should. The same conclusion may be drawn from (22), if we use for large β the value of

$$\chi(r) = \frac{1}{\beta} r^2 \eta_0(r) = \frac{r^3}{\beta},$$

as given by (38).

6. Approximate Solutions

Following Williams[15], the solution of (10) for $\beta = 0$ may be obtained as

$$\chi_0(r) = \frac{4r}{\pi\sqrt{1 - r^2}}, \qquad \ldots (42)$$

and then using (19), we obtain the no-slip couple coefficient[7]

$$C_{M_0} = \frac{32}{3}. \qquad \ldots (43)$$

This result can also be obtained from (22) by putting $\beta = 0$.

But, for small β, if we attempt a perturbation solution in powers of β by the method of iteration, we will not succeed because of the singularity in $\chi_0(r)$ at $r = 1$. Thus, a straight forward perturbation will break down. Therefore for small β we proceed to find an approximate $\overline{\chi(r)}$ as follows.

Assume

$$\chi(r) \simeq \overline{\chi(r)} = \frac{4r}{\pi[\bar{\beta} + \sqrt{1 - r^2}]}, \qquad \ldots (44)$$

where $\bar{\beta}$ is some positive function of β vanishing at $\beta = 0$.

Substituting the value of $\chi(r)$ from (44) into the eq. (22), we get

$$-\overline{\beta}(1-\overline{\beta}^2)\log\left(\frac{1+\overline{\beta}}{\overline{\beta}}\right)+\frac{2}{3}+\frac{\overline{\beta}}{2}-\overline{\beta}^2$$

$$=\frac{2}{3}-\frac{4\beta}{\pi}\left[(1-\overline{\beta}^2)\log\left(\frac{1+\overline{\beta}}{\overline{\beta}}\right)+\overline{\beta}-\frac{1}{2}\right]. \qquad \text{... (45)}$$

Which is identically satisfied with $\beta=\frac{\pi}{4}\overline{\beta}$.

Hence with $\overline{\beta}=\frac{4}{\pi}\beta$, we have

$$\overline{\chi(r)}=\frac{4r}{\pi\left[\frac{4}{\pi}\beta+\sqrt{1-r^2}\right]} \qquad \text{... (46)}$$

The coefficient of couple $C_{\overline{M}}$ for $\overline{\chi(r)}$ is then evaluated by using (22), giving

$$C_{\overline{M}}=\frac{32}{3}-\frac{64\beta}{\pi}\left[\left(\frac{1-16\beta^2}{\pi^2}\right)\log\left(\frac{1+\frac{4\beta}{\pi}}{\frac{4\beta}{\pi}}\right)+\frac{4\beta}{\pi}-\frac{1}{2}\right]. \qquad \text{... (47)}$$

It should be further noted that if for small β we make the approximations of $\overline{\chi(r)}$, as

$$\overline{\chi(r)}\simeq\frac{4r}{\pi(1-r^2)^{1/2}}-\frac{16r\beta}{\pi^2(1-r^2)}+\frac{64\beta^2 r}{\pi^3(1-r^2)^{3/2}}-\ \dots\ . \qquad \text{... (48)}$$

For small β, the coefficient of couple may be approximated from the formula (22) as follows

$$\overline{C_M}\approx\frac{2}{3}-\frac{4\beta}{\pi}\int_0^{1-\eta\beta^2}\frac{r^3}{1-r^2}\,dr. \qquad \text{... (49)}$$

On integration, the above expression (49) yields

$$\overline{C_M}=\frac{32}{3}-\frac{64\beta}{\pi}\log\frac{1}{\beta}+O(\beta). \qquad \text{... (50)}$$

This is seen to correspond to the first two terms of expansion of (47) for small β.

Further for small β if we approximate $\chi(r)$ to $\chi_0(r)$ in eq. (29) and integrate from 0 to $1-\eta\beta^2$, we get

$$C_M-C_{M_0}\simeq\frac{64\beta}{\pi}\log\frac{1}{\beta}+O(\beta), \qquad \text{... (51)}$$

which matches with the earlier result given by eq. (50).

Thus, we confirm that the approximation $\overline{\chi(r)}$, as given by (44) conforms to the result for small values of β.

Next, for large β, the coefficient of couple $\overline{C_M}$ as obtained from (47) comes out as

$$\overline{C_M} = \frac{\pi}{\beta} + O\left(\frac{1}{\beta^2}\right), \qquad \text{... (52)}$$

which as in section 5 is seen to vanish for $\beta \to \infty$, but with a difference in the coefficient of $1/\beta$ from that occurring in (41).

Thus, the above analysis suggests that the proposed approximation (46) is a good approximation and may be conveniently used for small as well as large values of β. It is seen that the effect of slip coefficient β is to decrease the couple on the disc.

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