Indian J. pure appl. Math., **34**(10) : 1469-1478, October 2003

OSCILLATORY PROPERTIES OF SOLUTIONS OF NONLINEAR PARABOLIC EQUATIONS WITH FUNCTIONAL ARGUMENTS

YUTAKA SHOUKAKU* AND NORIO YOSHIDA**

**Toyama National College of Maritime Technology, Toyama, 933-0293 Japan*

***Department of Mathematics, Faculty of Science, Toyama University, Toyama, 930-8555, Japan*

(*Received* 4 *October* 2002; *accepted* 22 *April* 2003)

The oscillations of nonlinear parabolic equations are studied, and it is shown that every solution of certain boundary value problems is oscillatory or satisfies some limit condition. The approach used is to reduce the multi-dimensional problem to a one-dimensional oscillation problem by using some integral means of solutions.

Key Words : Oscillation; Functional Argument; Parabolic Equation

1. INTRODUCTION

Beginning with the work of Bykov and Kultaev[1] there has been an increasing interest in studying the oscillatory properties of solutions of nonlinear parabolic equations with functional arguments. In 1992 Mishev[4] has investigated the oscillatory character of the nonlinear parabolic equation

$$\frac{\partial}{\partial t}(u(x,t)+\lambda u(x,t-\tau))-a(t)\,\Delta u+q(x,t)f(u(x,t-\tau))=0$$

which is a generalization of that studied by Yoshida[6]. The purpose of this paper is to establish oscillation criteria for more general parabolic equation. We are concerned with the oscillatory behaviour of solutions of

$$\frac{\partial}{\partial t}\left(u(x,t)+\sum_{i=1}^{l} h_i(t)\,u(x,\tau_i(t))\right)-a(t)\,\Delta u$$
$$+q(x,t)f(u(x,\sigma(t)))=0,\quad (x,t)\in G\times \boldsymbol{R}_+, \qquad \text{... (E)}$$

where Δ is the Laplacian in $\boldsymbol{R}^n$, $\boldsymbol{R}_+=[0,\infty)$ and G is a bounded domain of $\boldsymbol{R}^n$ with piecewise smooth boundary ∂G. We assume throughout this paper that :

(A-I) $a(t)$ is a nonnegative continuous function on $\boldsymbol{R}_+$, $q(x,t)$ is a nonnegative continuous function on $\overline{G}\times\boldsymbol{R}_+$ and $h_i(t)\ (i=1,2,\dots l)\in C^1(\boldsymbol{R}_+)$ are nonnegative on $\boldsymbol{R}_+$;

(A-II) $f(s)$ is continuous on $\boldsymbol{R}$, $f(s)$ is positive and convex in $(0,\infty)$, and $f(-s)=-f(s)$ for $s\geq 0$;

(A-III) $\tau_i(t)$ $(i = 1, 2, \ldots, l)$ and $\sigma(t)$ are continuous functions on $\boldsymbol{R}_+$ such that $\lim_{t\to\infty} \tau_i(t) = \infty$ and $\lim_{t\to\infty} \sigma(t) = \infty$.

2. Oscillation Criteria for Equation (*E*)

We consider two kinds of boundary conditions :

$$u = 0 \text{ on } \partial G \times \boldsymbol{R}_+, \qquad \ldots (1)$$

$$\frac{\partial u}{\partial \nu} + \mu u = 0 \text{ on } \partial G \times \boldsymbol{R}_+, \qquad \ldots (2)$$

where ν denotes the unit exterior normal vector to ∂G and μ is a nonnegative continuous function on $\partial G \times \boldsymbol{R}_+$.

By a solution of eq. (E) we mean a function $u \in C^2(\overline{G} \times [0, \infty)) \cap C^1(\overline{G} \times [\hat{t}_{-1}, \infty)) \cap C(\overline{G} \times [\tilde{t}_{-1}, \infty))$ which satisfies (*E*), where

$$\hat{t}_{-1} = \min\left\{0, \min_{1 \le i \le l}\left\{\inf_{t \ge 0} \tau_i(t)\right\}\right\},$$

$$\tilde{t}_{-1} = \min\left\{0, \inf_{t \ge 0} \sigma(t)\right\}.$$

It is well known that the first eigenvalue λ_1 of the eigenvalue problem

$$\Delta \upsilon + \lambda \upsilon = 0 \text{ in } G,$$

$$\upsilon = 0 \text{ on } \partial G$$

is positive, and the corresponding eigenfunction $\Phi(x)$ can be chosen so that $\Phi(x) > 0$ in G.

Associated with every function $u \in C(\overline{G} \times \boldsymbol{R}_+)$, we define

$$U(t) \equiv \frac{1}{\int_G \Phi(x)\, dx} \int_G u(x, t)\, \Phi(x)\, dx,$$

$$\tilde{U}(t) \equiv \frac{1}{|G|} \int_G u(x, t)\, dx,$$

where $|G|$ denotes the volume of G, i.e. $|G| = \int_G dx$.

The following notation will be used :

$$F(t) = \int_0^t a(s)\left(1 - \sum_{i=1}^{l} h_i(s)\right) ds,$$

$$Q(t) = \min\left\{q(x, t); x \in \overline{G}\right\}.$$

***Theorem* 1** — *Assume that (A-I)-(A-III) hold, and that*

(A-IV) $f(s_1 s_2) \geq f_1(s_1) f_2(s_2)$ *for* $s_1 \geq 0, s_2 > 0$, *where* $f_1(s_1) \geq 0, f_2(s_2) > 0$ *and* $f_2(s_2)$ *is nondecreasing for* $s_2 > 0$;

(A-V) $\sum_{i=1}^{l} h_i(t) \leq 1,\ \tau_i(t) \geq t\ (i = 1, 2, ..., l).$

If every eventually positive solution y(t) of the differential inequality

$$y'(t) + H(t) f_2(y(\sigma(t))) \leq 0 \qquad \text{... (3)}$$

satisfies $\lim_{t \to \infty} y(t) = 0$, *then every solution u of the problem (E), (1) is oscillatory in* $G \times \boldsymbol{R}_+$ *or satisfies*

$$\lim_{t \to \infty} \exp(\lambda_1 F(t))\, U(t) = 0, \qquad \text{... (4)}$$

where
$$H(t) = Q(t) \exp(\lambda_1 F(t)) f_1\left(\exp(-\lambda_1 F(\sigma(t)))\left(1 - \sum_{i=1}^{l} h_i(\sigma(t))\right)\right),$$

$$U_1(t) = U(t) + \sum_{i=1}^{l} h_i(t)\, U(\tau_i(t)). \qquad \text{... (5)}$$

PROOF : Suppose to the contrary that there is a nonoscillatory solution u which does not satisfy (4). Without loss of generality we may assume that $u > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$. Since $\lim_{t \to \infty} \sigma(t) = \infty$, we see that $u(x, \sigma(t)) > 0$ in $G \times [t_1, \infty)$ for some $t_1 \geq t_0$. Multiplying (E) by $\Phi(x)\left(\int_G \Phi(x)\,dx\right)^{-1}$ and integrating over G, we obtain

$$\frac{d}{dt}\left(U(t) + \sum_{i=1}^{l} h_i(t)\, U(\tau_i(t))\right)$$
$$- a(t) L_\Phi \int_G (\Delta u)\, \Phi(x)\,dx + L_\Phi \int_G q(x,t) f(u(x, \sigma(t)))\, \Phi(x)\,dx = 0, \qquad \text{... (6)}$$

where $L_\Phi = \left(\int_G \Phi(x)\,dx\right)^{-1}$. From Green's formula it follows that

$$a(t) L_\Phi \int_G (\Delta u)\, \Phi(x)\,dx = -\lambda_1 a(t)\, U(t). \qquad \text{... (7)}$$

An application of Jensen's inequality shows that

$$L_{\Phi} \int_{G} q(x,t) f(u(x,\sigma(t))) \Phi(x)\, dx \geq Q(t) f(U(\sigma(t))), \quad t \geq t_1. \qquad \dots (8)$$

Combining (6)-(8) yields

$$\frac{d}{dt}\left(U(t) + \sum_{i=1}^{l} h_i(t) U(\tau_i(t)) \right) + \lambda_1 a(t) U(t) + Q(t) f(U(\sigma(t))) \leq 0, \quad t \geq t_1. \qquad \dots (9)$$

From (5) and (9) we see that

$$U'_1(t) \leq -\lambda_1 a(t) U(t) - Q(t) f(U(\sigma(t))) \leq 0, \quad t \geq t_1.$$

Therefore $U_1(t)$ is nonincreasing. Since $\tau_i(t) \geq t_1$ on $[t_2, \infty)$ for some $t_2 \geq t_1$ by (A-III), we have $U_1(t) \geq U_1(\tau_i(t))$ on $[t_2, \infty)$. In view of the inequality $U(t) \leq U_1(t)$, we obtain

$$U(t) = U_1(t) - \sum_{i=1}^{l} h_i(t) U(\tau_i(t))$$

$$\geq U_1(t) - \sum_{i=1}^{l} h_i(t) U_1(\tau_i(t))$$

$$\geq \left(1 - \sum_{i=1}^{l} h_i(t) \right) U_1(t), \; t \geq t_2. \qquad \dots (10)$$

We observe, using (10), that (9) can be reduced to the following

$$U'_1(t) + \lambda_1 a(t) \left(1 - \sum_{i=1}^{l} h_i(t) \right) U_1(t)$$

$$+ Q(t) f\left(\left(1 - \sum_{i=1}^{l} h_i(\sigma(t)) \right) U_1(\sigma(t)) \right) \leq 0, \quad t \geq t_2,$$

which is equivalent to

$$\frac{d}{dt}(\exp(\lambda_1 F(t)) U_1(t))$$

$$+ Q(t) \exp(\lambda_1 F(t)) f\left(\left(1 - \sum_{i=1}^{l} h_i(\sigma(t)) \right) U_1(\sigma(t)) \right) \leq 0, \quad t \geq t_2. \qquad \dots (11)$$

The assumption (A-IV) implies

$$f\left(\left(1-\sum_{i=1}^{l} h_i(\sigma(t))\right)U_1(\sigma(t))\right)$$

$$= f\left(\exp(-\lambda_1 F(\sigma(t)))\exp(\lambda_1 F(\sigma(t)))\left(1-\sum_{i=1}^{l} h_i(\sigma(t))\right)U_1(\sigma(t))\right)$$

$$\geq f_1\left(\exp(-\lambda_1 F(\sigma(t)))\left(1-\sum_{i=1}^{l} h_i(\sigma(t))\right)\right)f_2(\exp(\lambda_1 F(\sigma(t)))\,U_1(\sigma(t))). \quad \dots (12)$$

From (12) we see that (11) can be rewritten as

$$\frac{d}{dt}(\exp(\lambda_1 F(t))\,U_1(t))$$

$$+\ Q(t)\exp(\lambda_1 F(t))f_1\left(\exp(-\lambda_1 F(\sigma(t)))\left(1-\sum_{i=1}^{l} h_i(\sigma(t))\right)\right)$$

$$\times f_2(\exp(\lambda_1 F(\sigma(t)))\,U_1(\sigma(t))) \leq 0.$$

Hence, $\exp(\lambda_1 F(t))\,U_1(t)$ is a positive solution of (3) on $[t_2, \infty)$ which does not satisfy (4). This contradicts the hypothesis and completes the proof.

***Theorem* 2** — *Assume that (A-I)-(A-V) hold. If every eventually positive solution y(t) of the differential inequality*

$$y'(t) + Q(t)f_1\left(1-\sum_{i=1}^{l} h_i(\sigma(t))\right)f_2(y(\sigma(t))) \leq 0 \quad \dots (13)$$

satisfies $\lim_{t\to\infty} y(t) = 0$, *then every solution u of the problem (E), (2) is oscillatory in* $G \times \boldsymbol{R}_+$ *or satisfies*

$$\lim_{t\to\infty} U_2(t) = 0, \quad \dots (14)$$

where

$$U_2(t) = \tilde{U}(t) + \sum_{i=1}^{l} h_i(t)\,\tilde{U}(\tau_i(t)). \quad \dots (15)$$

PROOF : Suppose that there is a nonoscillatory solution u of the problem (E), (2) which does not satisfy (14). We may assume that $u > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$. By (A-III) we find that $u(x, \tau_i(t)) > 0$, $u(x, \sigma(t)) > 0$ in $G \times [t_1, \infty)$ for some $t_1 \geq t_0$. Dividing (E) by $|G|$ and integrating over G, we obtain

$$\frac{d}{dt}\left(\tilde{U}(t) + \sum_{i=1}^{l} h_i(t)\,\tilde{U}(\tau_i(t))\right) - \frac{a(t)}{|G|}\int_G \Delta u\, dx$$

$$' + \frac{1}{|G|} \int_G q(x,t) f(u(x,\sigma(t)))\, dx = 0. \qquad \text{... (16)}$$

From Green's formula it follows that

$$\frac{a(t)}{|G|} \int_G \Delta u\, dx = \frac{a(t)}{|G|} \int_{\partial G} (-\mu u)\, dS \le 0, \quad t \ge t_1. \qquad \text{... (17)}$$

Applying of Jensen's inequality, we obtain

$$\frac{1}{|G|} \int_G q(x,t) f(u(x,\sigma(t)))\, dx \ge Q(t) f(\tilde{U}(\sigma(t))), \; t \ge t_1. \qquad \text{... (18)}$$

Combining (16)-(18) yields

$$\frac{d}{dt}\left(\tilde{U}(t) + \sum_{i=1}^{l} h_i(t)\, \tilde{U}(\tau_i(t)) \right) + Q(t) f(\tilde{U}(\sigma(t))) \le 0, \quad t \ge t_1 \qquad \text{... (19)}$$

Proceeding as in the proof of Theorem 1, we have

$$\tilde{U}(t) \ge \left(1 - \sum_{i=1}^{l} h_i(t) \right) U_2(t), \quad t \ge t_2 \qquad \text{... (20)}$$

for some $t_2 \ge t_1$. From (15), (19) and (20) we obtain

$$U_2'(t) + Q(t) f\left(\left(1 - \sum_{i=1}^{l} h_i(\sigma(t)) \right) U_2(\sigma(t)) \right) \le 0, \; t \ge t_2.$$

From (A-IV) we see that

$$U_2'(t) + Q(t) f_1 \left(1 - \sum_{i=1}^{l} h_i(\sigma(t)) \right) f_2(U_2(\sigma(t)) \le 0, \; t \ge t_2.$$

Hence $U_2(t)$ is a positive solution of (13) which does not satisfy (14). This contradicts the hypothesis and completes the proof.

Remark 1 : If $U(t)$ is eventually positive, then the inequality

$$0 \le U(t) \le U_1(t) \le \exp(\lambda_1 F(t))\, U_1(t)$$

holds. Therefore, we can replace (4) of Theorem 1 with

$$\lim_{t \to \infty} U(t) = 0.$$

In a similar way we can replace (14) of Theorem 2 with

$$\lim_{t \to \infty} \tilde{U}(t) = 0.$$

***Theorem* 3** — *(Linear case) Assume that (A-I)-(A-III) and (A-V) hold. If the differential inequality*

$$y'(t) + Q(t)\exp(\lambda_1 (F(t) - F(\sigma(t))))\left(1 - \sum_{i=1}^{l} h_i(\sigma(t))\right) y(\sigma(t)) \le 0$$

has no eventually positive solution, then every solution u of the linear parabolic equation

$$\frac{\partial}{\partial t}\left(u(x,t) + \sum_{i=1}^{l} h_i(t)\, u(x, \tau_i(t))\right) - a(t)\,\Delta u + q(x,t)\, u(x, \sigma(t)) = 0 \qquad \dots (21)$$

satisfying (1) is oscillatory in $G \times \boldsymbol{R}_+$.

***Theorem* 4** — *(Linear case) Assume that (A-I)-(A-III) and (A-V) hold. If the differential inequality*

$$y'(t) + Q(t)\left(1 - \sum_{i=1}^{l} h_i(\sigma(t))\right) y(\sigma(t)) \le 0$$

has no eventually positive solution, then every solution u of (21) satisfying (2) is oscillatory in $G \times \boldsymbol{R}_+$.

Combining Theorems 1, 2 and a result of Kitamura and Kusano[2], *we obtain the following corollaries.*

Corollary 1 — Assume that (A-I)-(A-V) hold. Every solution u of the problem (E), (1) is oscillatory in $G \times \boldsymbol{R}_+$ or satisfies (4), if

$$\int_{R[\sigma]} Q(t)\exp(\lambda_1 F(t)) f_1\left(\exp(-\lambda_1 F(\sigma(t)))\left(1 - \sum_{i=1}^{l} h_i(\sigma(t))\right)\right) dt = \infty,$$

where $R[\sigma] = \left\{ t \in \boldsymbol{R}_+ ;\ 0 \le \sigma(t) \le t \right\}$.

Corollary 2 — Assume that (A-I)-(A-V) hold. Every solution u of the problem (E), (2) is oscillatory in $G \times \boldsymbol{R}_+$ or satisfies (14), if

$$\int_{R[\sigma]} Q(t) f_1\left(1 - \sum_{i=1}^{l} h_i(\sigma(t))\right) dt = \infty.$$

Combining Theorems 3, 4 and a result of Koplatadze and Čanturija[3] yields the following results.

Corollary 3— (*Linear case*) Assume that (A-I), (A-III), (A-V) and the following

(A-VI) $\sigma(t) \le t$ and $\sigma(t)$ is nondecreasing on $[t_0, \infty)$ for some $t_0 > 0$.

Every solution u of the problem (1), (21) is oscillatory in $G \times \boldsymbol{R}_+$ if

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} Q(s)\exp(\lambda_1 (F(s) - F(\sigma(s))))\left(1 - \sum_{i=1}^{l} h_i(\sigma(s))\right)ds > \frac{1}{e}. \quad \text{... (22)}$$

Corollary 4 — (*Linear case*) Assume that (A-I), (A-III), (A-V) and (A-VI) hold. Every solution u of the problem (2), (21) is oscillatory in $G \times \boldsymbol{R}_+$ if

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} Q(s)\left(1 - \sum_{i=1}^{l} h_i(\sigma(s))\right)ds > \frac{1}{e}.$$

Remark 2 : In case $\sigma(t) \le t\ (t \in \boldsymbol{R}_+)$, it is obvious that $R[\sigma] = [T, \infty)$ for some $T > 0$. A special case of the problem (E), (1) is the following :

$$\frac{\partial}{\partial t}(u(x,t) + hu(x, t+\tau)) - \left(\frac{L}{\pi}\right)^2 u_{xx} + q(x,t)(u(x, t-\sigma))^{\gamma} = 0,$$

$$(x,t) \in (0, L) \times \boldsymbol{R}_+, \quad \text{... (23)}$$

$$u(0, t) = u(L, t) = 0, \ t > 0, \quad \text{... (24)}$$

where $h\ (< 1)$, τ, σ are positive constants and $\gamma (\ge 1)$ is the quotient of odd integers.

Corollary 5 — If

$$\int^{\infty} Q(t)\exp((1-h)(1-\gamma)t)\,dt = \infty \quad \text{... (25)}$$

then every solution u of the problem (23), (24) is oscillatory in $(0, L) \times \boldsymbol{R}_+$ or satisfies

$$\lim_{t\to\infty} \exp((1-h)t)\, U_1(t) = 0, \quad \text{... (26)}$$

where
$$U_1(t) = \left(\frac{\pi}{2L}\right)\int_0^L (u(x,t) + hu(x, t+\tau))\sin\left(\frac{\pi}{L}\right)x\,dx.$$

PROOF : In the case where $G = (0, L) \subset \boldsymbol{R}$, we observe that $\lambda_1 = (\pi/L)^2$ and $\Phi(x) = \sin(\pi/L)\,x$. Since $f(s) = s^{\gamma}$, we may choose $f_1(s) = f_2(s) = s^{\gamma}$. It is easy to see that condition (25) implies

$$\int_{t_0}^{\infty} Q(t)\exp\left(\left(\frac{\pi}{L}\right)^2\left(\frac{L}{\pi}\right)^2(1-h)t\right)$$

$$\times\left(\exp\left(-\left(\frac{\pi}{L}\right)^2\left(\frac{L}{\pi}\right)^2(1-h)(t-\sigma)\right)(1-h)\right)^{\gamma} dt$$

$$= (1-h)^{\gamma}\exp((1-h)\gamma\sigma)\int_{t_0}^{\infty} Q(t)\exp((1-h)(1-\gamma)t)\,dt = \infty.$$

Hence, the conclusion follows from Corollary 1.

Example 1 — We consider the parabolic equation

$$\frac{\partial}{\partial t}(u(x,t)+hu(x,t+\tau))-\left(\frac{L}{\pi}\right)^2 u_{xx}+h\exp(-\sigma-\tau)\,u(x,t-\sigma)=0,$$

$$(x,t)\in(0,L)\times\boldsymbol{R}_+, \qquad \ldots (27)$$

Where h, τ, σ are constants with $0<h<1$, $\tau>0$, $\sigma>0$. Here $n = 1$, $l = 1$, $a(t)=(L/\pi)^2$, $h_1(t) = h$, $\tau_1(t) = t+\tau$, $\sigma(t)=t-\sigma$, $\gamma=1$ and $q(x,t)=Q(t)=h\exp(-\sigma-\tau)$. It is readily seen that

$$\int_{t-\sigma}^{t} h\exp(-\sigma-\tau)\exp\left(\left(\frac{\pi}{L}\right)^2\left(\frac{L}{\pi}\right)^2((1-h)s-(1-h)(s-\sigma))\right)(1-h)\,ds$$

$$= h\,\sigma(1-h)\exp(-\tau-h\,\sigma)$$

$$\le \frac{h\,\sigma}{e^{(h\,\sigma)}}\le\frac{1}{e},$$

and therefore (22) does not hold. Hence, Corollary 3 is not applicable to (27). Since $\int^{\infty} h\exp(-\sigma-\tau)\,dt=\infty$, from Corollary 5 it follows that every nonoscillatory solution u of (24), (27) satisfies (26). In fact,

$$u(x,t)=\exp(-t)\sin\left(\frac{\pi}{L}\right)x$$

is a nonoscillatory solution which satisfies (26).

Example 2 — Consider the boundary value problem

$$\frac{\partial}{\partial t}\left(u(x,t)+\frac{e}{3}u(x,t+1)\right)-a(t)\,u_{xx}+\frac{4}{3}e^{2t-3}(u(x,t-1))^3=0,$$

$$(x,t)\in(0,L)\times\boldsymbol{R}_+, \qquad \ldots (28)$$

$$-u_x(0,t)=u_x(L,t)=0,\ t>0. \qquad \ldots (29)$$

Here $n = 1$, $l = 1$, $\gamma=3$, $h_1(t)=\frac{e}{3}$, $\tau_1(t)=t+1$, $\sigma(t)=t-1$ and $q(x,t)=Q(t)=\frac{4}{3}e^{2t-3}$.

Since

$$\int^{\infty}\frac{4}{3}e^{2t-3}\left(1-\frac{e}{3}\right)^3 dt=\infty.$$

Corollary 2 implies that every nonoscillatory solution u of (28), (29) satisfies

$$\lim_{t \to \infty} \frac{1}{L} \int_0^L \left(u(x,t) + \frac{e}{3} u(x, t+1) \right) dx = 0 .$$

In fact, one such solution is $u(x,t) = e^{-t}$.

Example 3 — Consider the problem

$$\frac{\partial}{\partial t}\left(u(x,t) + \frac{e}{3} u(x,t+1) \right) - \frac{1}{3} u_{xx} + e^{-2} u(x, t-2) = 0, \ (x,t) \in (0, \pi) \times \boldsymbol{R}_+, \qquad \text{... (30)}$$

$$-u_x(0,t) = u_x(\pi, t) = 0, \ t > 0. \qquad \text{... (31)}$$

Here $n = 1$, $l = 1$, $a(t) = \frac{1}{3}$, $h_1(t) = \frac{e}{3}$, $\tau_1(t) = t + 1$, $L = \pi$, $\sigma(t) = t - 2$, $\gamma = 1$ and $q(x,t) = Q(t) = e^{-2}$. Since

$$\int_{t-2}^{t} e^{-2}\left(1 - \frac{e}{3} \right) ds = 2e^{-2}\left(1 - \frac{e}{3} \right) \le \frac{1}{e},$$

$$\int^{\infty} e^{-2}\left(1 - \frac{e}{3} \right) ds = \infty.$$

Corollary 4 does not apply but Corollary 2 does. Therefore every solution u of (30), (31) is oscillatory in $(0, \pi) \times \boldsymbol{R}_+$ or satisfies

$$\lim_{t \to \infty} \frac{1}{\pi} \int_0^{\pi} \left(u(x,t) + \frac{e}{3} u(x,t+1) \right) dx = 0.$$

For example $u(x,t) = e^{-t} \cos x$ is such a solution.

REFERENCES

1. Ya. V. Bykov and T. Ch. Kultaev, *Izv. Akad. Nauk Kirgiz. SSR*, **6** (1983), 3-9 (Russian).
2. Y. Kitamura and T. Kusano, *Proc. Amer. Math. Soc.*, **78** (1980), 64-68.
3. R. G. Koplatadze and T. A. Čanturija, *Differentialnye Uravnenija*, **18** (1982), 1463-65, (Russian).
4. D. P. Mishev, *Houston J. Math.*, **18** (1992), 259-69.
5. S. Tanaka and N. Yoshida, *Tamkang J. Math.*, **18** (1997), 169-81.
6. N. Yoshida, *Hiroshima Math. J.*, **16** (1986), 304-14.