Indian J. pure appl. Math., **34**(10) : 1479-1485, October 2003

WAVE PROPAGATION IN AN ANISOTROPIC GENERALIZED THERMOELASTIC SOLID

BALJEET SINGH

Department of Mathematics, Government College, Sector 11, Chandigarh, India,
e-mail: baljeet_gill@hotmail.com

(*Received* 13 *January* 2003; *accepted* 23 *April* 2003)

The plane wave propagation in a homogeneous transversely isotropic thermally conducting elastic solid is studied with two thermal relaxation times. Three types of plane waves, quasi-P, quasi-S and thermal waves, are shown to exist. The analytical expressions for their velocities of propagation are obtained. The velocities of these waves are found to depend on the angle of propagation and frequency. This dependence of velocities on the direction of propagation and frequency is shown graphically. Effects of thermal parameters and anisotropy upon these velocities are observed.

Key Words : Anisotropic Generalized Thermoelastic Solid; Plane Waves; Relaxation Time

1. INTRODUCTION

The dynamical theory of thermoelasticity is the study of dynamical interaction between thermal and mechanical fields in solid bodies and is of much importance in various engineering fields such as earthquake engineering, soil dynamics, aeronautics, astronautics, nuclear reactors, high energy particle accelerators, etc. The theories on generalized thermoelasticity[1-2] have become the center of recent research due to their applications in many modern technological problems. Various problems on wave propagation in an isotropic generalized thermoelastic solid are studied by some researchers. Notable among them are Norwood and Warren[3], Sinha and Sinha[4], Singh and Kumar[5] and Singh[6-7].

There are reasonable grounds for assuming anisotropy in the continents. Anisotropy in the earth crust's and upper mantle affects the wave characteristics considerably. Banerjee and Pao[8] investigated the propagation of plane harmonic thermoelastic waves in infinitely extended anisotropic medium after taking into account the thermal relaxation. Dhaliwal and Sherief[9] derived the governing equations of generalized thermoelasticity for anisotropic media. Singh and Sharma[10] and Sharma[11, 12], investigated generalized thermoelastic waves in transversely isotropic media after taking into account one relaxation time and obtained a cubic equation which gives the non-dimensional velocities of various plane waves.

The present research work is an attempt to study the propagation of plane waves in transversely isotropic generalized thermoelastic solid with two thermal relaxations. The analytical expressions for velocities of these plane waves are derived by an approach used by Sidhu and Singh[13]. The computational work is performed to obtain the numerical values of the velocities of the plane waves for a particular material as model for the generalized anisotropic thermoelastic solid. The graphical presentation of velocities of the plane waves with the direction of propagation are exhibited for two different theories of generalized thermoelasticity[1, 2]

2. FORMULATION OF THE PROBLEM

Consider a homogeneous transversely isotropic thermally conducting elastic medium at uniform temperature T_0. The medium is assumed transversely isotropic in such a way that the planes of isotropy are perpendicular to z-axis. The origin is taken on the thermally insulated and stress free plane surface and z-axis normally into the half-space which is represented by $z \geq 0$. For two dimensional motion in x-z plane, the governing field equations of generalized thermoelasticity in absence of body forces and heat sources are[1, 2, 12]

$$c_{11}\, u_{,xx} + c_{44}\, u_{,zz} + (c_{13} + c_{44})\, w_{,xz} - \beta_1\, (T + \tau_1 \dot{T})_{,x} = \rho\, \ddot{u} \qquad \ldots (1)$$

$$c_{44}\, w_{,xx} + c_{33}\, w_{,zz} + (c_{13} + c_{44})\, u_{,xz} - \beta_3\, (T + \tau_1 \dot{T})_{,z} = \rho\, \ddot{w} \qquad \ldots (2)$$

$$K_1\, T_{,xx} + K_3\, T_{,zz} - \rho\, C_e\, (\dot{T} + \tau_0\, \ddot{T})$$

$$= T_0\, [\beta_1\, (\dot{u}_{,x} + \tau_0\, \Omega\, \ddot{u}_{,x}) + \beta_3\, (\dot{w}_{,z} + \tau_0\, \Omega\, \ddot{w}_{,z})] \qquad \ldots (3)$$

where $$\beta_1 = (c_{11} + c_{12})\, \alpha_1 + c_{13}\, \alpha_3, \;\; \beta_3 = 2\, c_{13}\, \alpha_1 + c_{33}\, \alpha_3; \qquad \ldots (4)$$

c_{ij} are the isothermal elasticities, ρ and C_e are respectively the density and specific heat at constant strain; τ_0, τ_1 are thermal relaxation times; K_3, K_1 and α_3, α_1 are thermal conductivities and the coefficients of linear thermal expansion along and perpendicular to the axis of symmetry respectively. The coma notation is used for spatial derivatives and dot notation for time differentiation. The use of symbol Ω, in eq. (3) makes these fundamental equations possible for the two different theories of the generalized thermoelasticity. For the L-S (Lord-Shulman) theory $\tau_1 = 0, \Omega = 1$ and for *G-L* (Green-Lindsay) theory $\tau_1 > 0$ and $\Omega = 0$. The thermal relaxations τ_0 and τ_1 satisfy the inequality $\tau_1 \geq \tau_0 \geq 0$ for the *G-L* theory only.

3. Propagation of Plane Waves

For plane waves of circular frequency ω, wave number k, and phase velocity c, incident at the free boundary $z = 0$ at an angle θ with the z-axis, we may assume

$$u = A\, \exp\, (i\, P_1), \;\; w = B \exp\, (iP_1), \;\; T = C \exp\, (iP_1), \qquad \ldots (5)$$

where A, B, C are the amplitude factors and

$$P_1 = \omega\, t - k\, (x \sin\theta - z \cos\theta), \qquad \ldots (6)$$

is the phase factor.

For waves reflected at $z = 0$, we assume

$$u = A\, \exp\, (i\, P_2), \;\; w = B \exp\, (iP_2), \;\; T = C \exp\, (iP_2), \qquad \ldots (7)$$

where $$P_2 = \omega\, t - k\, (x \sin\theta + z \cos\theta), \qquad \ldots (8)$$

is the phase factor associated with reflected waves. Making use of eq. (5) or (7), in eqs. (1) to (4), we obtain

$$-(D_1 - \rho\, c^2)\, A \pm (c_{13} + c_{44}) \sin\theta \cos\theta\, B + (i/k)\, \tau'\, \beta_1 \sin\theta\, C = 0, \qquad \ldots (9)$$

$$\pm (c_{13} + c_{44}) \sin\theta \cos\theta\, A - (D_2 - \rho\, c^2)\, B \mp (i/k)\, \tau'\, \beta_3 \cos\theta\, C = 0, \qquad \ldots (10)$$

$$\tau T_0 c^2 \beta_1 \sin\theta A \mp \tau T_0 c^2 \beta_3 \cos\theta B - (i/k)(D_3 - \rho c^2 \tau^* C_e) C = 0, \quad \text{... (11)}$$

where the upper sign corresponds to the incident waves [eq. (5)] and the lower sign corresponds to the reflected waves (eq. (7)). D_1, D_2, D_3 are given by

$$D_1(\theta) = c_{11}\sin^2\theta + c_{44}\cos^2\theta,$$

$$D_2(\theta) = c_{33}\sin^2\theta + c_{44}\sin^2\theta,$$

$$D_3(\theta) = K_3\cos^2\theta + K_1\sin^2\theta, \quad \text{... (12)}$$

and $\tau^* = \tau_0 - (i/kc), \quad \tau = \tau_0\Omega - (i/kc), \quad \tau = (1 + i k c \tau_1).$

Eqs. (9) to (11) in A, B, C can have a nontrivial solution only if the determinant of their coefficients vanishes, i.e.,

$$A_0\zeta^3 + A_1\zeta^2 + A_2\zeta + A_3 = 0, \quad \text{... (13)}$$

where

$$A_0 = \tau^*,$$

$$A_1 = -(D_1\tau^* + D_2\tau^* + D_4 + \bar{\beta}^2\varepsilon_1\tau' v_1^2\tau\cos^2\theta + \varepsilon_1 \cdot \tau' v_1^2\tau\sin^2\theta),$$

$$A_2 = D_1D_2\tau^* + D_1D_4 + D_2D_4 + D_1\bar{\beta}^2\varepsilon_1\tau' v_1^2\tau\cos^2\theta + D_2\varepsilon_1\tau' v_1^2\tau\sin^2\theta$$

$$-\tau^*(c_{13} + c_{44})^2\sin^2\theta\cos^2\theta - 2(c_{13} + c_{44})\bar{\beta}\tau\varepsilon_1\tau' v_1^2\sin^2\theta\cos^2\theta,$$

$$A_3 = -D_1D_2D_4(c_{13} + c_{44})^2\sin^2\theta\cos^2\theta. \quad \text{... (14)}$$

and $\zeta = \rho c^2, \; D_4 = D_3/C_e, \; \varepsilon_1 = \beta_1^2 T_0/\rho C_e v_1^2, \; v_1^2 = c_{11}/\rho, \; \bar{\beta} = \beta_3/\beta_1$

Using Cardan's method to solve eq. (13), we obtain

$$\xi^3 + 3H\xi + G = 0, \quad \text{... (15)}$$

where $\xi = \tau^*\zeta + (A_1/3), \; H = (3\tau^* A_2 - A_1^2)/9,$

$$G = (27\tau^{*2}A_3 - 9\tau^* A_1A_2 + 2A_1^3)/27. \quad \text{... (16)}$$

The three roots of eq. (15) can be written as

$$\xi_1 = h_1 + h_2, \quad \xi_2 = h_1 g + h_2 g^2, \quad \xi_3 = h_1 g^2 + h_2 g. \quad \text{... (17)}$$

where $h_1^3 = [-G + \{G^2 + 4H^3\}^{1/2}]/2, \; h_2^3 = [-G - \{G^2 + 4H^3\}^{1/3}]/2,$

$g = (-1 \pm \sqrt{-3})/2$, a cube root of unity.

Therefore, the three roots of eq. (13) are

$$\zeta_1 = [\xi_1 - (A_1/3)]/\tau^*, \ \zeta_2 = [\xi_2 - (A_1/3)]/\tau^*, \quad \zeta_3 = [\xi_3 - (A_1/3)]/\tau^*. \quad \text{... (18)}$$

which give expressions for the velocities of propagation of quasi-P, thermal and quasi-S waves respectively. It may be noted that whether we take the upper sign or the lower sign in eqs. (9) to (11), we get the same three values of ζ given by eq. (13). Thus, in general, in two dimensional transversely isotropic generalized thermoelastic media with two relaxation times, there are three types of plane waves, whose phase velocities vary with the direction of propagation and frequency (ω).

Particular case — For isotropic elastic solid, we take

$$c_{11} = c_{33} = \lambda + 2\mu, \quad c_{44} = \mu, \quad c_{13} = \lambda,$$

$$\beta_1 = \beta_3 = \beta \to 0, \ K_1 = K_3 = K \to 0, \quad \tau_0 = \tau_1 = 0,$$

then $\quad D_4 \to 0, \quad \varepsilon_1 \to 0, \quad A_3 \to 0,$

which reduce theeq. (13) to a quadratic equation in ζ as

$$\zeta^2 + B_1 \zeta + B_2 = 0, \quad \text{... (19)}$$

where $\quad B_1 = -(D_1' + D_2'),$

$$B_2 = D_1' D_2' - (\lambda + \mu)^2 \sin^2\theta \cos^2\theta,$$

$$D_1' = (\lambda + 2\mu)\sin^2\theta + \mu\cos^2\theta,$$

$$D_2' = (\lambda + 2\mu)\cos^2\theta + \mu\sin^2\theta,$$

The solution of eq. (19) gives the expressions for velocities of propagation of P and SV waves in two-dimensional model for isotropic elastic media.

It may be noted that if we put $c_{11} = c_{33} = \lambda + 2\mu$, $c_{44} = \mu$, $c_{13} = \lambda$, and neglect the thermal parameters in cubic eq. (13) given by Sharma[12], it reduces to

$$(\zeta - i\omega)(\zeta - \lambda_1)(\zeta - \lambda_2) = 0, \quad \text{... (20)}$$

which gives three values of ζ as $i\omega$, 1 and $\mu/(\lambda + 2\mu)$, i.e., the velocities of propagation for P and SV waves as 1 and $[\mu/(\lambda + 2\mu)]^{1/2}$, which are non-dimensional quantities.

Moreover, if we take $\tau_1 = 0, \Omega = 1$, eq. (13) of the present problem will provide the dimensional velocities of plane waves in model considered by Singh and Sharma[10] and Sharma[11,12]

4. Numerical Results and Discussion

To study in greater detail the dependence of velocities of propagation of plane waves on their direction of propagation, we consider single crystal of zinc as an anisotropic generalized thermoelastic solid for which the basic data are[14].

$c_{11} = 1.628 \times 10^{11}\ Nm^{-2},\ c_{33} = 1.562 \times 10^{11}\ Nm^{-2},$

$c_{13} = 0.508 \times 10^{11}\ Nm^{-2},\ c_{44} = 0.385 \times 10^{1}\ Nm^{-2},$

$\beta_1 = 5.75 \times 10^{6}\ Nm^{-2}\ deg^{-1},\ \beta_3 = 5.17 \times 10^{6}\ Nm^{-2}\ deg^{-1},$

$\rho = 7.14 \times 10^{3}\ kg\ m^{-3},\ C_e = 3.9 \times 10^{2}\ J\ kg^{-1}\ deg^{-1},$

$K_1 = 1.24 \times 10^{2}\ Wm^{-1}\ deg^{-1},\ K_3 = 1.24 \times 10^{2}\ Wm^{-1}\ deg^{-1},$

$T_0 = 296^{o}\ K,\ \varepsilon_1 = .053,\ \tau_0 = 0.05,\ \tau_1 = 0.1.$

The variations of the velocities of quasi-P, thermal and quasi-S waves with the angle of propagation (θ) are shown in Fig. 1 for L-S and G-L theories when ω = 2. The velocities of propagation of these plane waves are also compared with those for isotropic elastic case. The numerical values of the velocities of propagation of quasi-P, thermal and quasi-S waves are calculated for the frequency range $0 < \omega \leq 30$ when direction of propagation makes 45^{0} with vertical axis. The

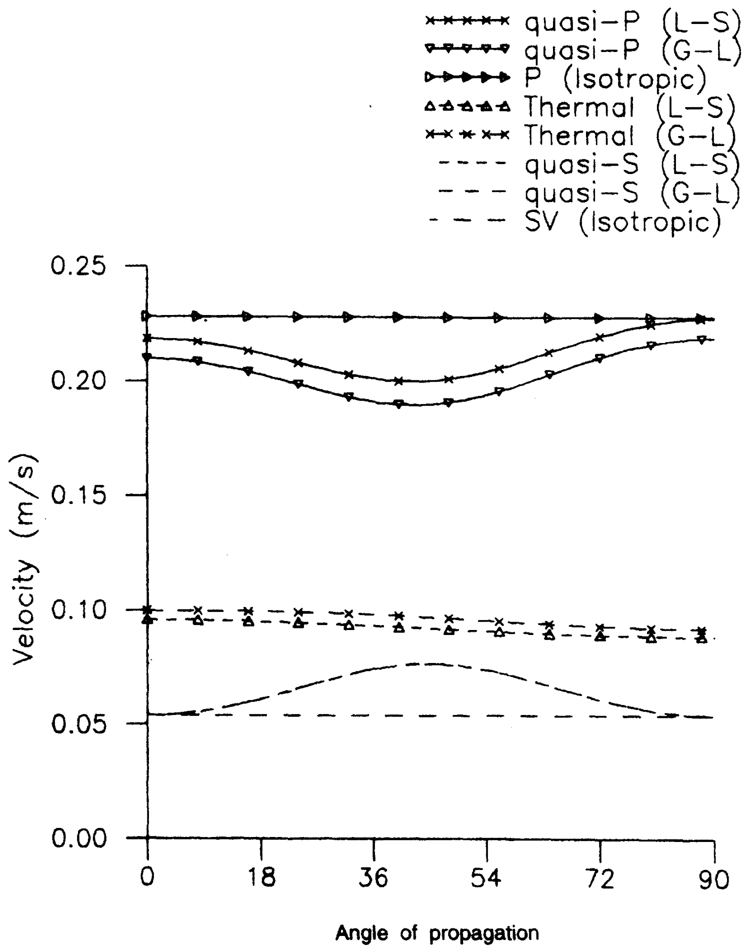

FIG. 1. Velocity as a function of angle between direction of propagation and vertical axis

variations of these velocities with frequency are shown in Fig. 2. From Figures 1 and 2, we observe the following :

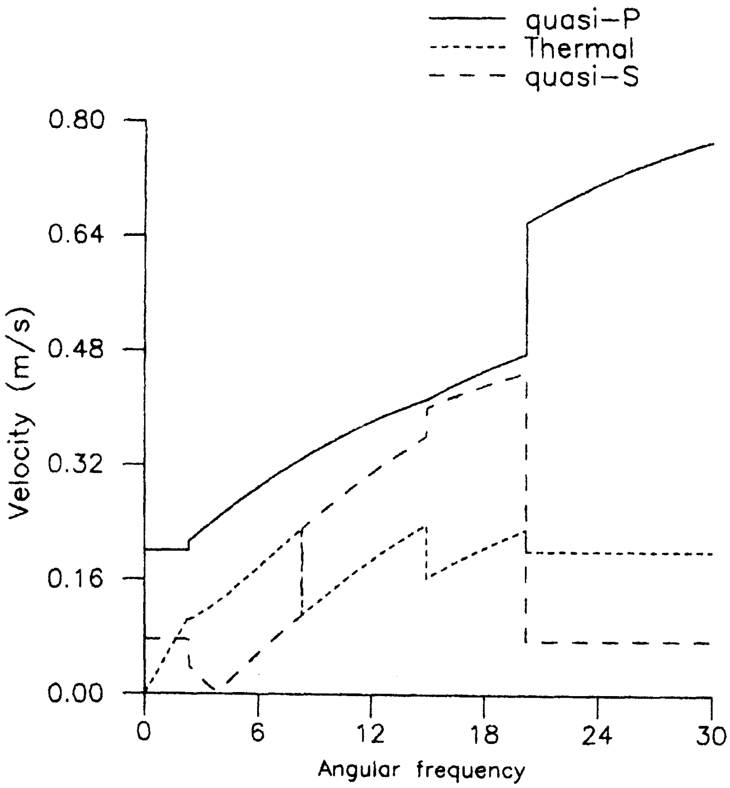

FIG. 2. Velocity as a function of angular frequency

1. The velocities of propagation of plane waves depend on angle of propagation. For quasi-P waves, its velocity first decreases and then increases as angle of propagation varies from 0^{0} to 90^{0}. The velocity for quasi-S wave first increases and then decreases with the increase in angle of propagation. The velocity for thermal wave at each angle of propagation is observed uniform though varies slightly.

2. The comparison of the numerical values of velocities of plane waves for L-S and G-L cases reveals the effect of second thermal relaxation time on the velocity of each wave. The effect of second thermal relaxation time is observed minimum on the velocity of quasi-S wave.

3. The effects of thermal parameters and anisotropy are observed on velocities when compared with isotropic case.

4. For the frequency range $0 < \omega \leq 20$, the velocities of plane waves change arbitrarily. Beyond $\omega > 20$, the velocity for quasi-P wave increases slightly, whereas the velocities for thermal and quasi-S waves remain almost constant.

It is concluded that anisotropy in generalized thermoelastic media has significant effect on the velocities of propagation of plane waves. This research work is supposed to be useful in further studies, both theoretical and observational, of wave propagation in the more realistic models of the thermoelastic solids present in the earth's interior.

REFERENCES

1. H. W. Lord and Y. Shulman, *J. Mech. Phys. Solids*, **15**(5) (1967), 299-309.
2. A. E. Green and K. A. Lindsay, *J. Elasticity*, **2**(1) (1972), 1-7.
3. F. R. Norwood and W. E. Warren, *Q. J. Mech. Appl. Math.*, **15** (1969), 283-90.
4. A. N. Sinha and S. B. Sinha, *J. Phys. Earth*, **22** (1974), 237.
5. B. Singh and R. Kumar, *Int. J. Engng. Sci.*, **36** (1998), 865-912.
6. B. Singh, *Journal of Sound and Vibration*, **235** (2000), 685-96.
7. B. Singh, *Int. J. Engng. Sci.*, **39** (2001), 583-98.
8. S. Banerjee and Y. H. Pao, *J. Acoust. Soc. Am.*, **56** (1974), 1444-53.
9. R. S. Dhaliwal and H. H. Sherief, *Q. Appl. Math.*, **38** (1980), 1-8.
10. H. Singh and J. N. Sharma, *J. Acoust. Soc. Am.*, **77** (1985), 1046.
11. J. N. Sharma, *Arch. Mech.*, **38** (1986), 665.
12. J. N. Sharma, *Indian J. pure appl. Math.*, **19**(3) (1988), 294-304.
13. R. S. Sidhu and S. J. Singh, *J. Acoust. Soc. Am.*, **76** (1984), 594.
14. P. Chadwick and L. T. C. Seet, *Mathematica*, **17** (1970), 255-74.