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PRE-INDUCED *L*-SUPRA TOPOLOGICAL SPACES

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The aim of this paper is to introduce and to study the concepts of "pre-induced *L*-supra topological spaces" and "pre-Scott continuity". Pre-Scott continuous functions turn out to be the natural tool for studying the pre-induced *L*-supra topological spaces. Finally, we discuss the connections between some separation and covering properties of an ordinary topological space and its corresponding pre-induced *L*-supra topological space.

Key Words : Fuzzy Supra Topology; Fuzzy Lattice; Prime Element; Pre-Open Set; Scott Continuity; Pre-Scott Continuity; Induced Fuzzy Topological Space; Pre-Induced *L*-Supra Toplogical Space; Completely Hausdorff Space

1. INTRODUCTION

Abd. El-Monsef and Ramadan[1], introduced the concept of fuzzy supra topology as follows : A family $F \subset I^X$ is called fuzzy supra topology on X if 0, $1 \in F$ and it is closed under arbitrary supremum. To standardize the terminology we use[6]. If a lattice L of membership values has been chosen then the corresponding Chang-Goguen spaces are "*L*-topological spaces". Which have "*L*-topology" and which with "*L*-continuous mapping" make the "fixed basis" Category "*L*-top". Thus we use *L*-topological space instead of "*L*-fuzzy topological spaces" and "*L*-supra topological spaces" rather than "*L*-fuzzy supra topological spaces". In section 2, we introduce a new class of functions from a topological space (X, T) to a fuzzy lattice L with its Scott topology, called pre-Scott continuous functions. Then we study some of their properties and characterizations. We prove that the set $P(T)$ of pre-Scott continuous functions from (X, T) to L is an *L*-supra topology. Pre-Scott continuous functions turn out to be the natural tool for studying pre-induced *L*-supra toplogical space (*PIL-ST* space). In section 3, we discuss the connections between several properties of an ordinary topological space (X, T) and its corresponding *PIL-ST* space $(X, P(T))$. For example, (X, T) is strongly compact iff the *PIL-ST* space $(X, P(T))$ is fuzzy supra *p*-compact. Throughout this work X and Y will be non-empty ordinary sets and $L = L(\leq, \vee \wedge, ')$ will denote a fuzzy lattice i.e. a complete completely distributive lattice with a smallest element 0 and a largest element 1 $(0 \neq 1)$ and with an order-reversing involution $a \to a'$ $(a \in L)$. L is therefore a continuous lattice, $I = [0, 1]$ will denote the unit closed interval and L^X will denote the lattice of *L*-fuzzy subsets of X. We will denote by 1_A the characteristic function of the ordinary subset A of X.

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We would like to mention the following definitions and results.

Definition 1.1[4] — Let (X, T) be a topological space. A subset A of X is called pre-open if $A \subset \text{Int. } ClA$.

Definition 1.2[2] — Let (X, T) be a topological space and $a \in X$. A function $f: (X, T) \to I$ is called a Scott continuous (or lower semi continuous) at $a \in X$ iff for every $\alpha \in [0, 1]$ with $\alpha < f(a)$ there is a neighbourhood U of 'a' such that $\alpha < f(x)$ for every $x \in U$. f is called Scott continuous (or lower semi-continuous) at every point of X.

Definition 1.3[8] — The set $\omega(T)$ of Scott continuous functions from a topological space (X, T) to L with its Scott-topology is an L-topology, called the induced L-topology (IL-T).

Definition 1.4[3] — An element p of L is called prime iff $p \neq 1$ and whenever $a, b \in L$ with $a \wedge b \leq p$ then $a \leq p$ or $b \leq p$. The set of all prime elements of L will be denoted by $pr(L)$.

In[9], Warner has determined the prime elements of the fuzzy lattice L^X. Here, $pr(L^X) = \{ x_p : x \in X \text{ and } p \in pr(L) \}$

where for each $x \in X$ and each $p \in pr(L)$, $x_p : X \to L$, is the fuzzy set defined by

$$x_p(y) = \begin{cases} p, & \text{if } y = x, \\ 1, & \text{otherwise.} \end{cases}$$

These x_p are called the L-fuzzy points of X and we have x_p is a member of an L-fuzzy set g and we write $x_p \in g$ iff $g(x) \not\leq p$.

Definition 1.5[3] — Let L be a complete Lattice and $x, y \in L$. We say that x is way below y, in symbols $x \ll y$ iff for every directed subset D of L with $y \leq \vee D$, there exists a $d \in D$ such that $x \leq d$.

Proposition 1.6[9] — The set of the form $\{l \in L : \delta \ll l\}$ are Scott open.

Result 1.7[10] — The sets of the form $\{l \in L : l \not\leq p\}$ where $p \in pr(L)$ generate the Scott topology of L.

2. Pre-Scott Continuous Functions and Pre-Induced L-Supra Topological Spaces

Considering a fuzzy lattice L with its Scott topology we introduce the concept of pre-Scott continuity. We obtain an L-supra topological space from a given ordinary topological space. Let (X, T) be a topological space and $f: (X, T) \to L$ be a function where L has its Scott topology. By the result 1.5 of section 1, we have f is Scott continuous iff for every $p \in pr(L), f^{-1}(\{ l \in L : l \not\leq p\}) \in T$.

In ordinary topological space (X, T) we define the pre-open neighbourhood as follows :

Let p be a point in (X, T). A subset N of X is a pre-open neighbourhood of p iff N is a superset of a pre-open set S containing p :

$p \in S \subset N$, where S is a pre-open set in (X, T).

Definition 2.1 — Let (X, T) be a topological space and $a \in X$. A function $f: (X, T) \to L$, where L has its Scott topology is said to be pre-Scott continuous $a \in X$ iff for every $p \in pr(L)$ with $f(a) \not\leq p$, there is a pre-open neighbourhood N of 'a' in (X, T) such that $f(a) \not\leq p$ for every $x \in N$, i.e. $N \subset f^{-1}(\{l \in L : l \not\leq p\})$. Then f is called pre-Scott continuous on X iff f is pre-Scott continuous at every point of X.

When $L = I$, the definition becomes $f: (X, T) \to I$ is pre-Scott continuous at $a \in X$ iff for every $p \in pr(L) = [0, 1]$ with $f(a) > p$ there is a pre-open neighbourhood N of 'a' if (X, T) such that $f(x) > p$ for every $x \in N$. From this definition, it is clear that every Scott continuous function is a pre-Scott continuous function.

Proposition 2.2 — The characteristic function of every pre-open set is pre-Scott continuous.

PROOF : Let A be a pre-open set in a topological space (X, T) and $a \in X, p \in pr(L)$ with $1_A(a) \not\leq p$. Then $a \in A$ and A is a pre-open neighbourhood of a. We also have $1_A(x) \not\leq p$ for every $x \in A$. Hence J_A is pre-Scott continuous at $a \in X$.

Proposition 2.3 — If $\{f_i : j \in \wedge\}$ is an arbitrary family of pre-Scott continuous functions from a topological space (X, T) to L, then $f = \bigvee_{j \in \wedge} f_j$ is also pre-Scott continuous.

PROOF : Let $p \in pr(L)$ and $a \in X$ with $f(a) = \bigvee_{j \in \wedge} f(x) \not\leq p$, then there is a $j \in \wedge$, such that $f_j(a) \not\leq p$. Since f_j is pre-Scott continuous at 'a', there is a pre-open neighbourhood N of 'a' such that $f_j(a) \not\leq p$ for all $x \in N$. Hence $f(x) = \bigvee_{j \in \wedge} f_j(x) \not\leq p$ for all $x \in N$. Thus f is pre-Scott continuous at $a \in X$.

Since the intersection of two pre-open sets may not be pre-open, we have the following :

Proposition 2.4 — Let (X, T) be a topological space. If $f, g : (X, T) \to L$ are pre-Scott continuous functions then $f \wedge g : (X, T) \to L$ is not pre-Scott continuous.

Proposition 2.5 — Let (X, T) be a topological space. $f: (X, T) \to L$ is pre-Scott continuous iff for every $p \in pr(L), f^{-1}(\{l \in L : l \not\leq p\})$ can be expressed as a union of some pre-open sets in (X, T).

PROOF : Let $p \in pr(L)$ and $x \in f^{-1}(\{l \in L : l \not\leq p\})$. Then $f(x) \not\leq p$. Here, f is pre-Scott continuous at x, thus there exists a pre-open set N_x in (X, T) such that $x \in N_x$ and $N_x \subset f^{-1}(\{l \in L : l \not\leq p\})$. Hence $f^{-1}(\{l \in L : l \not\leq p\}) = \bigcup N_x$ where N_x is pre-open.

On the other hand, let $a \in X$ and $p \in pr(L)$ with $f(a) \nleq p$. Then $a \in f^{-1}(\{l \in L : l \nleq p\})$. By the hypothesis there is a pre-open set N in (X, T) such that $a \in N$ and $N \subset f^{-1}(\{l \in L : l \nleq p\})$, which implies that f is pre-Scott continuous.

***Theorem* 2.6** — *For a topologial space (X, T), the set $P(T) = \{f \in L^X : f : (X, T) \to L$ is pre-Scott continuous$\}$ is an L-supra topology on X.*

PROOF : It follows immediately from Propositions 2.2, 2.3 and 2.4.

Definition 2.7 — The L-supra topology $P(T)$ obtained in Theorem 2.6 is called a pre-induced L-supra topology (PIL-ST) and the space $(X, P(T))$ is called a pre-induced L-supra topological space (PIL-ST space)

Lemma 2.8 — If A is pre-open in a topological space (X, T) then $I_A \in P(T)$.

PROOF : By the Proposition 2.2, the lemma follows immediately.

Remark : Since every Scott continuous function from a topological space (X, T) to a fuzzy lattice L is pre-Scott continuous, we have $\omega(T) \subset P(T)$, where $\omega(T)$ is the L-topology of Scott continuous functions from (X, T) to L.

Definition 2.9 — A function $f : (X, P(T_1)) \to (Y, P(T_2))$ from a PIL-ST space to another PIL-ST space is said to be fuzzy supra p-continuous if $f^{-1}(\lambda) \in P(T_1)$ for every $\lambda \in P(T_2)$.

In[4], a function $f : (X, T) \to (Y, G)$ from a topological space (X, T) to another topological space (Y, G) is said to be pre-iresolute if the inverse image of each pre-open subset in Y is pre-open in X.

Using the above definition we have the following Teorem :

***Theorem* 2.10** — *A function $f : (X, P(T_1)) \to (Y, P(T_2))$ is fuzzy supra p-continuous iff $f(X, T_1) \to (Y, T_2)$ is pre-irresolute.*

PROOF : Suppose that $f : (X, P(T_1)) \to (Y, P(T_2))$ is fuzzy supra p-continuous. Let A be a pre open set in (Y, T_2) and $I_A \in p(T_2)$. By fuzzy supra p-continuity $f^{-1}(I_A) = 1_{f^{-1}(A)} \in P(T_1)$. We shall show that $f^{-1}(A)$ is pre-open in (X, T_1). Let $p \in pr(L)$ and $x \in f^{-1}(A)$. Then $1_{f^{-1}(A)}(x) \nleq p$. Since $1_{f^{-1}(A)} \in p(T_1)$, there exists a pre-open set N_x in (X, T_1) such that $x \in N_x$ and $N_x \subset 1^{-1}_{f^{-1}(A)}(\{l \in L : l \nleq p\}) = f^{-1}(A)$. Thus we have for each $x \in f^{-1}(A)$, there exists a pre-open set N_x in (X, T_1) such that $x \in N_x \subset f^{-1}(A)$. This shows that $f^{-1}(A)$ in pre-open in (X, T_1).

Consequently, $f: (X, T_1) \to (Y, T_2)$ in pre-irresolute.

Conversely, suppose $f: (X, T_1) \to (Y, T_2)$ is pre-irresolute. Take $\alpha \in P(T_2)$, we shall show that $f^{-1}(\alpha) \in P(T_1)$ i,e, $f^{-1}(\alpha): (X, T) \to L$ is pre-Scott continuous. Let $a \in X$ and $p \in pr(L)$ with $f^{-1}(\alpha)(a) \not\leq P$. Then $\alpha(f(a)) \not\leq P$. Since $\alpha: (Y, T_2) \to L$ is pre-Scott continuous at $f(a) \in Y$, there exists a pre-open set N in (Y, T_2) such that $f(a) \in N$ and $\alpha(y) \not\leq p$ for all $y \in N$. Since N is pre-open in $(Y, T_2), f^{-1}(N)$ is also pre-open in (X, T_1). Now, we have $a \in f^{-1}(N)$, which implies there is a pre-open set B is (X, T_1) such that $a \in B \subset f^{-1}(N)$. Hence $f^{-1}(\alpha)(x) = \alpha(f(x)) \not\leq p$ for every $x \in B$. This shows that $f^{-1}(\alpha)$ is pre-Scott continuous. Consequently, $f: (X, P(T_1)) \to (Y, P(T_1))$ is fuzzy supra p-continuous.

3. Connections Between Properties of a Topological Space and its Pre-Induced L-Supra Topological Space

In this section, we study the connections between some separation and covering properties of an ordinary topological space and its corresponding pre-induced L-supra topological space.

Definition 3.1 — A *PIL-ST* space $(X, P(T))$ is said to be fuzzy supra p-compact iff for every prime element p of L and every collection $\{f_j : j \in \wedge\}$ of supra open L-fuzzy sets $\left(\underset{j \in \wedge}{\vee} f_j\right)(x) \not\leq p$ for all $x \in X$, there is a finite sub set $\wedge_0$ of $\wedge$ such that $\left(\underset{j \in \wedge_0}{\vee} f_j\right)(x) \not\leq p$ for all $x \in X$.

In[5], strongly compactness in a topological space was defined in the following way: A topological space (X, T) is strongly compact iff every pre-open cover of X admits a finite sub cover.

Now, we prove the following theorem:

***Theorem* 3.2** — *A PIL-ST space $(X, P(T))$ is fuzzy supra p-compact iff the corresponding topological space (X, T) is strongly compact.*

PROOF : Suppose that (X, T) is strongly compact. Let $p \in pr(L)$ and $\mathcal{B} = \{f_j : j \in \wedge\}$ be a family of supra pre-open L-fuzzy sets in $(X, P(T))$ with $\left(\underset{j \in \wedge_0}{\vee} f_j\right)(x) \not\leq p$ for all $x \in X$; where $f_j(x) = \alpha_j$ if $x \in A_j$ and $f_j(x) = 0$ otherwise these A_j are pre-open in (X, T) and $\alpha_j \in L$ for every $j \in \wedge$. Then for each $x \in X$, there is a $j \in \wedge$ such that $f_j(x) \not\leq p$, i.e. $\alpha_j \not\leq p$. Let $\gamma = A_j$: there is a $j \in \wedge$ such that $\alpha_j \not\leq p$ and $f_j \in \mathcal{B}\}$. Then γ is a family of pre-open sets in (X, T) covering X, i.e. γ is a pre-open cover of X. From the strong compactness of (X, T), there exists a finite sub-family of γ say γ_0 where $\gamma_0 = \{A_1, A_2, \ldots\ldots, A_n\}$ such that $X = \bigcup_{j=1}^{n} A_j$. Hence $\left(\bigcup_{j=1}^{n} f_j\right)(x) \not\leq p$ for all $x \in X$ and thus $(X, P(T))$ is fuzzy supra p-compact.

Conversely, suppose $(X, P(T))$ is fuzzy supra p-compact. Let $\{A_j : j \in \wedge\}$ be a pre-open covering of X. Then $1_{A_j} \in P(T)$ for every $j \in \wedge$ as A_j is pre-open in (X, T). Thus $\{1_{A_j} : j \in \wedge\}$ is a family of supra pre-open L-fuzzy sets in $(X, P(T))$ with $\left(\bigvee_{j \in \wedge_0} f_j\right)(x) \not\leq p$ for all $x \in X$. From the fuzzy supra p-compactness of $(X, P(T))$ there exists a finite subset $\wedge_0$ of $\wedge$ such that $\left(\bigvee_{j \in \wedge_0} 1_{A_j}\right)(x) \not\leq p$ for all $x \in X$. Hence $X = \bigcup_{j \in \wedge_0} A_j$. Thus (X, T) is strongly compact.

In general topology, a topological space (X, T) is called completely Hausdorff (or Urysohn space)[7] iff for any distinct points x, y of X there are open sets U and V such that $x \in U, y \in V$ and $Cl\,(U) \cap Cl\,(V) = \phi$.

Definition 3.3 — A topological space (X, T) is called pre-completely Hausdorff iff for any distinct points x, y of x there are pre-open sets A and B such that $x \in A, y \in B$ and $Cl\,(A) \cap Cl\,(B) = \phi$.

Definition 3.4 — A *PIL-ST* space $(X, P\,(T))$ is said to be fuzzy supra completely Hausdorff iff for every distinct points x, y of X and every $p, q \in pr\,(L)$ there exists supra open L-fuzzy sets f and g such that $x_p \in f, y_q \in g$ and $\forall\, z \in$, $Cl\,(f) \cap cl\,(g) = 0$.

***Theorem* 3.5** — *The topological space (X, T) is pre-completely Hausdorff iff the PIL-ST space (X, P (T)) is fuzzy supra completely Hausdorff.*

PROOF : Let $x, y \in X\,(x \neq y)$ and $p, q \in pr\,(L)$. By pre-complete Hausdorffness of (X, T), there exists two pre-open sets U and V in (X, T) such that $x \in U$ and $y \in V$ and $Cl\,(U) \cap Cl\,(V) = \phi$. Now $1_U, 1_V \in P\,(T)$ since U and V are pre-open sets (X, T). Also $1_U\,(x) \not\leq p$ and $1_v\,(x) \not\leq p$ and $\forall z \in X, Cl\,1_U\,(z) = 1_{clU}\,(z) = 0$ since $Cl\,(U) \cap Cl\,(V) = \phi$. Thus $Cl\,1_U\,(z) \wedge Cl\,1_V\,(z) = 0$. Hence $(X, P\,(T))$ is a fuzzy supra completely Hausdorff space.

Conversely, let $x, y \in X\,(x \neq y)$ and $p, q \in pr\,(L)$. From the fuzzy supra complete Hausdorffness of $(X, P(T))$ there exists L-fuzzy supra open sets f, g which are defined by

$f(z) = \alpha$, if $z \in U, f(z) = 0$ otherwise and $g(z) = \beta$, if $z \in V, g(z) = 0$ otherwise, where U and V are pre-open sets in (X, T) and $\alpha, \beta \in L$, such that $x \in f, y \in g$ and $\forall\, z \in X, Cl(f) \cap Cl(g) = 0$. Hence we have $x \in U$ and $y \in V$, where U and V are pre-open in (X, T) and $Cl(U) \cap Cl(V) = \phi$. Hence (X, T) is pre-completely Hausdorff.

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