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A FUZZY EXTENSION OF SIDDIQI *et al.*'s RESULTS FOR VECTOR VARIATIONAL-LIKE INEQUALITIES

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In this paper, we consider the existence of solutions of vector variational-like inequalities for fuzzy mappings, which is a fuzzy extension of Siddiqi *et al.*[31]

Key Words : Fuzzy Mappings; Closed Fuzzy Set; Vector Variational-Like Inequalities; Fan's Geometrical Lemma

1. INTRODUCTION AND PRELIMINARIES

In 1980, Giannessi[12] introduced a variational inequality for vector-valued mappings in Euclidean spaces. Since then, many authors[6-10, 16, 17, 19, 28, 30, 31] have intensively studied variational inequalities for vector-valued mappings or for multifunctions with vector values in abstract spaces.

Especially some of them[1, 19, 31] considered vector variational-like inequalities on locally convex Hausdorff topological vector spaces or Hausdorff topological vector spaces.

In 1989, Chang *et al.*[5] introduced the concept of variational inequalities for fuzzy mappings in locally convex Hausdorff topological vector spaces and investigated the existence theorems for some kinds of variational inequalities for fuzzy mappings, which were the fuzzy extensions of some variational inequality problems in[15, 29, 32, 34]. Recently, Chang[2] proved the coincidence theorems for fuzzy mappings and some existence theorems for more general variational inequalities for fuzzy mappings. Lee *et al.* [18, 20] obtained some existence theorems of certain variational inequalities for fuzzy mappings. Especially, in [18] they followed the approach of Chang and Zhu[5] and used the results of Kim and Tan[13]. Noor[27] suggested an iterative scheme for finding the approximate solutions for a variational inequality for fuzzy mappings, and proved that this approximate solution converges strongly to the exact solution to his inequality. Lee *et al.*[21] formulated a strongly quasivariational inequality for fuzzy mappings, which is a more generalized form than that of Noor[27], used the projection method to suggest an iterative algorithm for finding the approximate solutions for their inequality, and proved that this approximate solution converges strongly to the exact solution for the inequality. Lee *et al.*[23] considered vector variational inequalities for fuzzy mappings, which were the fuzzy extensions of vector variational inequalities studied by Chen and Yang[10], and obtained some existence theorems of solutions for their inequalities for fuzzy mappings. By using the Fan-Glicksberg-Kakutani fixed point theorem and the scalarization method of Luc *et al.*[25, 26], Chang *et al.*[3] obtained several kinds of existence theorems for vector quasi-variational inequalities of type (1) on locally convex Hausdorff topological vector spaces. They [4] also proved some other existence theorems for vector quasivariational inequalities of types (1) and (2) on Hausdorff topological vector spaces, using the Fan-Browder fixed point theorem, the selection theorem of Yannelis-Prabhakar[33] and the scalarization method of Luc *et al.*[25, 26]. Park *et al.*[28] considered the fuzzy extension of their existence theorem for a general vector-valued variational inequality considered in the non-compact setting and Lee *et al.*[22] introduced two kinds of vector variational inequalities for fuzzy mappings. Their first inequality is a more general form than the vector variational inequality for fuzzy mappings, which was studied by Lee *et al.*[23]. And they proved the existence theorems of solutions for their inequalities. Their results can be regarded as fuzzy extensions of corresponding ones by Lee *et al.*[24] and Siddiqi *et al.*[30]

Our motivation of this paper is to consider vector variational-like inequalities for fuzzy mappings on Hausdorff topological vector spaces. In proof of our main theorem, we use Fan's geometrical lemma[11], which has been applied to variational inequality problems, complementarity problems, game theory and so on.

Let E be a nonempty subset of a vector space X and D be a nonempty set. A function F from D into the collection $\mathcal{F}(E)$ of all fuzzy sets on E is called a fuzzy mapping. If $F : D \to \mathcal{F}(E)$ is a fuzzy mapping, then $F(x), x \in D$ (denoted by F_x in the sequel) is a fuzzy set in $\mathcal{F}(E)$ and $F_x(y), y \in E$ is the degree of membership of y in F_x.

Let $A \in \mathcal{F}(E)$ and $\alpha \in (0, 1]$. Then the set

$$(A)_\alpha = \{ x \in E : A(x) \geq \alpha\}$$

is called an α-cut set of A.

Now we give some definitions and preliminary results needed in the later section.

The following lemma is the geometrical lemma due to Fan[11].

Lemma 1.1 — Let K be a nonempty compact convex subset of a Hausdorff topological vector space X. Let A be a subset of $K \times K$ satisfying the following conditions;

(*i*) for each $x \in K$, $(x, x) \in A$,

(*ii*) for each fixed $x \in K$, the set $A_x = \{y \in K : (x, y) \in A\}$ is closed in K,

(*iii*) for each fixed $y \in K$, the set $A^y = \{x \in K : (x, y) \notin A\}$ is convex in K.

Then there exists an $x_0 \in K$ such that $K \times \{x_0\} \subset A$.

Definition 1.1[19] — Let X and Y be two topological spaces, and $T: X \to 2^Y$ be a multifunction. We say that

(1) T is upper semi-continuous (briefly, u.s.c) at $x_0 \in X$ if for any open set N containing $T(x_0)$, there exists a neighbourhood M of x_0 such that $T(M) \subset N$. T is u.s.c. if T is u.s.c. at every $x_0 \in X$.

(2) T is closed at $x \in X$ if for any net $\{x_\lambda\}$ in X such that $x_\lambda \to x$ and for any net $\{y_\lambda\}$ in Y such that $y_\lambda \to y$ and $y_\lambda \in T(x_\lambda)$ for any λ, we have $y \in T(x)$.

(3) T has a closed graph if the graph of T, Gr $(T) = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$.

Definition 1.2[22] — Let X and Y be two topological spaces, and $F : X \to \mathcal{F}(Y)$ be a fuzzy mapping. We say that F is a fuzzy mapping with closed fuzzy set-values if $F_x(y)$ is u.s.c. on $X \times Y$ as a real ordinary function.

Lemma 1.2 — If A is a closed subset of a topological space X, then the characteristic function χ_A of A is an u.s.c. real-valued function.

Lemma 1.3 — Let K be a nonempty closed convex subset of a real Hausdorff topological vector space X, E be a nonempty closed convex subset of a real Hausdorff topological vector space Y and $\alpha : X \to (0, 1]$ be a lower semi-continuous function. Let $F : K \to \mathcal{F}(E)$ be a fuzzy mapping with $(F_x)_{\alpha(x)} \neq 0$ for any $x \in X$. Let $\tilde{F} : K \to 2^E$ be a multifunction defined by $\tilde{F}(x) = (F_x)_{\alpha(x)}$. If F is a fuzzy mapping with closed fuzzy set-values, then $\tilde{F}$ is a closed multifunction.

PROOF : Let $\{x_\lambda\}$ be a net in K converging to x, $\{y_\lambda\}$ be a net in E converging to y and $y_\lambda \in \tilde{F}(x_\lambda) = (F_{x_\lambda})_{\alpha(x_\lambda)}$. Then $F_{x_\lambda}(y_\lambda) \geq \alpha(x_\lambda)$. Since $F_x(y)$ is u.s.c. on $X \times Y$ as a real ordinary function, $F_x(y) \geq \overline{\lim}\, F_{x_\lambda}(y_\lambda) \geq \underline{\lim}\, F_{x_\lambda}(y_\lambda) \geq \underline{\lim}\, \alpha(x_\lambda) \geq \alpha(x)$, which shows $y \in \tilde{F}(x)$.

2. Vector Variational-Like Inequalities for Fuzzy Mappings

Let X and Y be two Hausdorff topological vector spaces, and $L(X, Y)$ be the space of all linear continuous operators from X into Y. Let K be a nonempty compact convex subset of X and $\{C(x) : x \in K\}$ be a family of convex cones in Y such that for any $x \in K$, $C(x) \neq Y$ and *int* $C(x) \neq 0$, where *int* denotes the interior. Let $F : X \to \mathcal{F}(L(X, Y))$ be a fuzzy mapping on $L(X, Y)$ and $\alpha : X \to (0, 1]$ be a function. We define a partial order $\leq_{C(x)}$ in Y with the convex cone $C(x)$ as; for $y_1, y_2 \in Y$,

$$y_1 \leq_{C(x)} y_2 \text{ if and only if } y_2 - y_1 \in C(x).$$

Definition 2.1[14] — A mapping $f : K \to Y$ is convex if for any $x_1, x_2 \in K$ and $t \in [0, 1]$,

$$f(t x_1 + (1-t) x_2) \leq_{C(x)} t f(x_1) + (1-t) f(x_2),$$

that is, $$tf(x_1) + (1-t) f(x_2) - f(tx_1 + (1-t) x_2) \in C(x).$$

In this paper, first we consider the following vector variational-like inequality for fuzzy mappings:

(FVVLI) find $x_0 \in K$ such that for any $y \in K$, there exists $s_0 \in (F_{x_0})_{\alpha(x_0)}$ such that

$$\langle s_0, \theta(y, x_0) \rangle \notin -\,int\, C(x_0),$$

where $\theta : K \times K \to X$ is a function and $\langle s, x \rangle$ is the evaluation of $s \in L(X, Y)$ at $x \in X$.

Next we consider the following vector variational inequality for fuzzy mappings :

(FVVI) find $x_0 \in K$ such that for any $y \in K$, there exists $s_0 \in (F_{x_0})_{\alpha(x_0)}$ such that

$$\langle s_0, y - x_0 \rangle \notin -\, int\, C(x_0).$$

It is obvious that (FVVLI) is a generalized form of (FVVI).

Now we consider our main result for (FVVLI), where bilinear form $\langle \cdot, \cdot \rangle$ is supposed to be continuous for simplicity.

Theorem 2.1 — *Let K be a nonempty compact convex subset of X. Let $F : K \to \mathcal{F}(L(X, Y))$ be a fuzzy mapping with closed fuzzy set-values. Let a multifunction $W : K \to 2^Y$, defined by $W(x) = Y \setminus \{-\, int\, C(x)\}$, $x \in K$, have a closed graph, and $\theta : K \times K \to X$ be a continuous mapping. Suppose that*

(1) there exists a lower semi-continuous function $\alpha : X - (0, 1]$ such that for any $x \in K$, the cut set $(F_x)_{\alpha(x)}$ is nonempty. and $\bigcup_{x \in K} (F_x)_{\alpha(x)}$ is contained in some compact subet of $L(X, Y)$,

(2) for any $x \in K$, there exists $s \in (F_x)_{\alpha(x)}$ such that

$$\langle s, \theta(x, x) \rangle \notin -intC(x), \text{ and}$$

(3) the operator

$$x \to \langle s, \theta(x, y) \rangle$$

of K into Y is convex with respect to the convex cone $C(y)$ for any $y \in K$ and any $s \in (F_y)_{\alpha(y)}$.

Then (FVVLI) is solvable.

PROOF : Define a multifunction $\tilde{F} : K \to 2^{L(X, Y)}$ by $\tilde{F}(x) = (F_x)_{\alpha(x)}$. It follows from Lemma 1.3 and the condition (1) that $\tilde{F}$ is a nonempty closed multifunction such that $\tilde{F}(K)$ is contained in some compact subset of $L(X, Y)$. Let

$$A := \{(x, y) \in K \times K : \text{there exists } s \in \tilde{F}(y) \text{ such that } \langle s, \theta(x, y) \rangle \notin -\, int\, C(y)\}.$$

Now we will show that (*i*), (*ii*) and (*iii*) of Lemma 1.1 are satisfied. From the definition of A and the condition (2), we can deduce that $(x, x) \in A$.

Next we will show that for each fixed $x \in K$,

$$\begin{aligned} A_x &:= \{y \in K : (x, y) \in A\} \\ &= \{y \in K : \text{there exists } s \in \tilde{F}(y) \text{ such that } \langle s, \theta(x, y) \rangle \notin -\, int\, C(y)\} \end{aligned}$$

is closed in K. Indeed, let $\{y_\lambda\}$ be a net in A_x such that $y_\lambda \to y_0 \in K$. Since $y_\lambda \in A_x$, there exists $s_\lambda \in \tilde{F}(y_\lambda)$ such that $\langle s_\lambda, \theta(x, y_\lambda) \rangle \in W(y_\lambda)$.

By the condition (1), without loss of generality, we can assume that there exists $s_0 \in L(X, Y)$ such that $s_\lambda \to s_0$. From Lemma 1.3, $\tilde{F}$ is a nonempty closed multifunction, hence $s_0 \in \tilde{F}(y_0)$. Since W has a closed graph, $s_0 \in \tilde{F}(y_0)$ satisfies $\langle s_0, \theta(x, y_0) \rangle \in W(y_0)$.

Hence $y_0 \in A_x$ and thus A_x is closed in K.

It remains to show that for each $y \in K$,

$$A^y := \{x \in K : (x, y) \notin A\}$$

$$= \{x \in K : \text{for any } s \in \tilde{F}(y), \langle s, \theta(x, y) \rangle \in -\, int\, C(y)\}$$

is convex. In fact, let $x_1, x_2 \in A^y$ and $t \in [0, 1]$, by the condition (3), we have for any $y \in K$ and any $s \in \tilde{F}(y)$ from the convexity of the operator $x \mapsto \langle s, \theta(x, y) \rangle$ with respect to the convex cone $C(y)$,

$$\langle s, \theta(tx_1 + (1-t)x_2, y) \rangle \leq_{C(y)} t \langle s, \theta(x_1, y) \rangle + (1-t) \langle s, \theta(x_2, y) \rangle.$$

Thus

$$t \langle s, \theta(x_1, y) \rangle + (1-t) \langle s, \theta(x_2, y) \rangle - \langle s, \theta(tx_1 + (1-t)x_2, y) \rangle \in C(y).$$

Since $\langle s, \theta(x_1, y) \rangle \in -\, int\, C(y)$ and $\langle s, \theta(x_2, y) \rangle \in -\, int\, C(y)$, we have

$$\langle s, \theta(tx_1 + (1-t)x_2, y) \rangle \in -\, int\, C(y).$$

Thus $tx_1 + (1-t)x_2 \in A^y$ and hence A^y is convex.

By Lemma 1.1, there exists an $x_0 \in K$ such that $K \times \{x_0\} \subset A$. This implies that there exists an $x_0 \in K$ such that for any $y \in K$, there exists $s_0 \in \tilde{F}(x_0) = (F_{x_0})_{\alpha(x_0)}$ such that

$$\langle s_0\, \theta(y, x_0) \rangle \notin -\, int\, C(x_0).$$

This completes the proof.

From Theorem 2.1, we can obtain the following theorem for multi-valued mappings as a corollary.

Theorem 2.2 — *Let K be a nonempty compact convex subset of X. Let $T : K \to 2^{L(X, Y)}$ be a continuous mapping with nonempty closed set-values and $\theta : K \times K \to X$ be continuous mapping. Let a multifunction $W : K \to 2^Y$, defined by W(x) = $Y \setminus \{-\, int$ C (x)\}, $x \in K$, have a closed graph. Suppose that*

(1) there exists $s \in T(x)$ such that

$$\langle s, \theta(x, x) \rangle \notin -\, int\, C(x) \textit{ for any } x \in K, \textit{ and}$$

(2) the operator

$$x \mapsto \langle s, \theta(x, y) \rangle$$

of K into Y is convex for any $y \in K$ *and* $s \in T(y)$.

Then there exists an $x_0 \in K$ *such that for any* $y \in K$, *there exists* $s_0 \in T(x_0)$ *such that*

$$\langle s_0, \theta(y, x_0) \rangle \notin - int\, C(x_0).$$

PROOF : Define a fuzzy mapping $F : K \to \mathcal{F}(L(X, Y))$ by $F_x = \chi_{T(x)}$, the characteristic function of $T(x)$ for $x \in K$, then from Lemma 1.2, F is a fuzzy mapping with closed fuzzy set-values. On the other hand, $(F_x)_1 = (\chi_{T(x)})_1 = T(x)$ is nonempty and $\bigcup_{x \in K} (F_x)_1 = \bigcup_{x \in K} T(x) = T(K)$ is compact in $L(X, Y)$. By Theorem 2.1, we can obtain the conclusion of Theorem 2.2.

Now we can also obtain the following theorem as a corollary.

***Theorem* 2.3**[31] — *Let K be a nonempty compact convex subset of X. Let* $T : K \to L(X, Y)$ *and* $\theta : K \times K \to X$ *be continuous mappings. Let a multifunction* $W : K \to 2^Y$, *defined by* $W(x) = Y \setminus \{- int\, C(x)\}$, $x \in K$, *have a closed graph. Suppose that*

(1) for any $x \in K$,

$$\langle T(x), \theta(x, x) \rangle \notin - \text{int}\, C(x) \text{ and}$$

(2) the operatpr

$$x \mapsto \langle T(y), \theta(x, y) \rangle$$

of K into Y is convex for any $y \in K$.

Then there exists an $x_0 \in K$ *such that*

$$\langle T(x_0), \theta(y, x_0) \rangle \notin - \text{int}\, C(x_0),$$

for any $y \in K$.

When $\theta(x, y) = x - y$ *in Theorem 2.1, we can obtain the existence theorem of solutions to the following vector variational inequality for fuzzy mappings.*

***Theorem* 2.4** — *Let K be a nonempty compact convex subset of X. Let* $F : K \to \mathcal{F}(L(X, Y))$ *be a fuzzy mapping with closed fuzzy set-values, and a multifunction* $W : K \to 2^Y$, *defined by* $W(x) = Y \setminus \{- int\, C(x)\}$, $x \in K$, *have a closed graph. Suppose that*

(1) there exists a lower semi-continuous function $\alpha : X \to (0, 1]$ *such that for any* $x \in K$, *the cut set* $(F_x)_{\alpha(x)}$ *is nonempty and* $\bigcup_{x \in K} (F_x)_{\alpha(x)}$ *is contained in some compact subset of L(X, Y), and*

(2) the operator

$$x \mapsto \langle s, x - y \rangle$$

of K into Y is convex for any $y \in K$ *and any* $s \in T(y)$.

Then (FVVI) is solvable.

From Theorem 2.4, we can obtain the following theorem for multi-valued mappings as a corollary using the same method in the proof of Theorem 2.2.

***Theorem* 2.5** — *Let K be a nonmepty convex subset of X. Let* $T: K \to 2^{L(X,Y)}$ *and* $\theta: K \times K \to X$ *be continuous mappings. Let a multifunction* $W: K \to 2^Y$, *defined by* $W(x) = Y \setminus \{- \operatorname{int} C(x)\}$, $x \in K$, *have a closed graph. Suppose that the operator* $x \mapsto \langle s, x - y \rangle$ *of K into Y is convex for any* $y \in K$ *and* $s \in T(y)$.

Then there exists an $x_0 \in K$ *such that for any* $y \in K$ *there exists* $s_0 \in T(x_0)$ *such that*

$$\langle s_0, y - x_0 \rangle \notin - \operatorname{int} C(x_0).$$

Subsequently, we can also obtain the following theorem as a corollary.

***Theorem* 2.6** — *Let K be a nonempty compact convex subset of X. Let* $T: K \to L(X, Y)$ *and* $\theta: K \times K \to X$ *be continuous mappings. Let a multifunction* $W: K \to 2^Y$, *defined by* $W(x) = Y \setminus \{- \operatorname{int} C(x)\}$, $x \in K$, *have a closed graph. Suppose that the operator* $x \mapsto \langle T(y), x - y \rangle$ *of K into Y is convex for any* $y \in K$.

Then there exists an $x_0 \in K$ *such that*

$$\langle T(x_0), y - x_0 \rangle \notin - \operatorname{int} C(x_0),$$

for any $y \in K$.

REFERENCES

1. Q. H. Ansari, *Ann. Sci. Quebec*, **19** (1995), 131-37.
2. S. S. Chang, *Fuzzy sets and systems*, **61** (1994), 359-68.
3. S. S. Chang, G. M. Lee and B. S. Lee, *Fuzzy Sets and Systems*, **87** (1997), 307-15.
4. S. S. Chang, G. M. Lee and B. S. Lee, *Fuzzy sets and systems*, **102** (1999), 333-44.
5. S. S. Chang and Y. G. Zhu, *Fuzzy Sets and Systems*, **32** (1989), 359-67.
6. G. Y. Chen, *J. Optim. Th. Appl.*, **74**(3) (1992), 445-56.
7. G. Y. Chen and G. H. Cheng, *Vector variational inequality and vector optimization problem*; in Lecture Notes in Economics and Mathematical Systems, Vol. **258**, Spring-Verlag, 1987.
8. G. Y. Chen and B. D. Craven, *J. Austral. Math. Soc. (Series A)*, **47** (1989), 418-23.
9. G. Y. Chen and B. D. Craven, *Zeitscrift fur Operations Research*, **3** (1990), 1-12.
10. G. Y. Chen and X. Q. Yang, *J. Math. Anal. Appl.*, **153** (1990), 136-58.
11. K. Fan, *Math. Ann.*, **142** (1961), 305-10.
12. F. Giannessi, *Theorems of alternative quadratic programs and complementarity problems*; *In*: Variational Inequality and Complementarity Problems. (Edited by R. W. Cottle, F. Giannessi and J. L. Lions), Wiley, New York, 1980.

13. W. K. Kim and K. K. Tan, *Bull. Austral. Soc.*, **46** (1992), 139-48.
14. D. Kuroiwa, *Appl. Math. Lett.*, **9** (1996), 97-101.
15. M. Lassonde, *J. Math. Anal. Appl.*, **97** (1983), 151-201.
16. B. S. Lee and G. M. Lee, *Appl. Math. lett.*, **12** (1999), 43-50.
17. B. S. Lee and G. M. Lee, *Appl. Math. Lett.*, **12** (1999), 13-17.
18. B. S. Lee, G. M. Lee, S. J. Cho and D. S. Kim, *Proceedings of Fifth International Fuzzy Systems Association World Congress*, Seoul, (19943), 326-29.
19. B. S. Lee, G. M. Lee and D. S. Kim, *Indian J. pure appl. Math.*, **28** (1997), 33-41.
20. B. S. Lee, G. M. Lee and D. S. Kim, *J. Korean Math. Soc*, **33** (1996), 609-24.
21. G. M. Lee, D. S. Kim and B. S. Lee, *Fuzzy Sets and Systems*, **78** (1996), 381-86.
22. G. M. Lee, D. S. Kim and B. S. Lee, *Nonlinear Analysis Forum*, **4** (1999), 119-29.
23. G. M. Lee, D. S. Kim, B. S. Lee and S. J. Cho, *Appl. Math. Lett.*, **6** (1993), 47-51.
24. G. M. Lee, B. S. Lee, D. S. Kim and G. Y. Chen, *Indian J. pure appl. Math.*, **28** (1997), 633-39.
25. D. T. Luc, *Theory of Vector Optimization, Lecture Notes in Economics and Mathematical Systems*, Vol. **319**, Springer, Berlin, 1989.
26. D. T. Luc and C. Vargas, *Nonlinear Analysis T.M.A.*, **18** (1992), 1-7.
27. N. A. Noor, *Fuzzy Sets and Systems*, **55** (1993), 309-12.
28. S. Park, B. S. Lee and G. M. Lee, *J. Math. and Math. Sci.*, **21**(4) (1998), 637-42.
29. M. H. Shih and K. K. Tan, *J. Math. Anal. Appl.*, **108** (1985), 333-43.
30. A. H. Siddiqi, Q. H. Ansari and A. Khaliq, *Indian J. pure appl. Math.*, **84** (1995), 171-80.
31. A. H. Siddiqi, Q. H. Ansari and R. Ahmad, *Indian J. pure appl. Math.*, **28**(8) (1997), 1009-16.
32. W. Takahashi, *J. Math. Soc. Japan*, **28** (1976), 142-50.
33. N. C. Yannelis and N. D. Prabhakar, *J. Math. Econom.*, **12** (1983), 233-45.
34. C. L. Yen, *Pacific J. Math.*, **97** (1981), 142-50.