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ON EXISTENCE AND UNIQUENESS OF SOLUTIONS OF FUZZY INTEGRODIFFERENTIAL EQUATIONS

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The present paper is concerned with the existence and uniqueness of solutions of fuzzy integrodifferential equations of the form

$$\frac{dX(t)}{dt} = f(t, X(t), \int_{t_0}^{t} k(t, s, X(s))\, ds)$$

under some suitable conditions.

Key Words : Fuzzy Integrodifferential Equations; Mean Square Calculus; Fuzzy Random Variable; Level Set; Fuzzy Number

1. INTRODUCTION

In this paper, we study the more general form of fuzzy integro-differential equations which are of great theoretical interest and are of importance in many branches of science, engineering and technology. In section 2, we first recall some basic results on fuzzy number space (E^n, D) and the second order fuzzy random variable space (L_2, ρ). Second 3 is devoted to the study of the existance and uniqueness of solutions for fuzzy integro-differential equations of the form

$$\frac{dX(t)}{dt} = f(t, X(t), \int_{t_0}^{t} k(t, s, X(s))\, ds), \quad t \in T = [t_0, b],$$

$$X(t_0) = X_0,$$

where $f : T \times L_2 \times L_2 \to L_2, k : T^2 \times L_2 \to L_2$ are m.s. continuous fuzzy mappings with respect to $t, s, t_0, b \in R$, and $X_0 \in L_2$.

This paper is motivated by the recent paper of Feng[1,2]. In[1] Feng has developed the concepts of mean square Riemann integral and differential associated with a class of fuzzy processes and studied their properties. And Feng[2] studied the existence and uniqueness of a solution, the continuity of the solution with respect to the initial value and the stability of fuzzy differential equations

$$x'(t) = f(t, x(t)), \quad t \in T = [t_0, b],$$

when f satisfies the Lipschitz condition and $t_0, b \in R$.

2. PRELIMINARIES

The symbol $P_C(R^n)$ denotes the family of all nonempty compact convex subsets of R^n. Denote the addition and scalar multiplication in $P_C(R^n)$ as usual. Denote $E^n = \{u : R^n \to [0, 10]\, u$ satisfied (i)-$(iv)\}$, where

(i) u is normal, i.e., there exists an $x_0 \in R^n$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, i.e., $u(\tau x + (1-\tau)y) \geq \min\,(u(x), u(y)), x, y \in R^n,\ \tau \in [0, 1]$;

(iii) u is upper semicontinuous;

(iv) $[u]^0 = \{\overline{x \in R^n \mid u(x) > 0}\}$ is compact.

Let $u, \upsilon \in E^n$, and set

$$D(u, v) = \sup_{0 \leq \tau \leq 1} d([u]^\tau, [\upsilon]^\tau),$$

where $[u]^\tau = \{x \in R^n \mid u(x) \geq \tau\}$, $0 < \tau \leq 1$, is the r-level set of u, d is the hausdorff metric defined in $P_C(R^n)$, i.e.,

$$d(A, B) = \max \{ \sup_{a \in A} \inf_{b \in B} |a-b|, \sup_{a \in B} \inf_{b \in A} |a-b|\},$$

for all $A, B \in P_C(R^n)$, where $|\cdot|$ denotes the usual Euclidean norm in R^n. The norm $\|u\|$ of a fuzzy number $u \in E^n$ is defined by

$$\|u\| = D(u, \hat{0}) = \|[u]^0\| = \sup_{a \in [u]^0} |a|,$$

where $|\cdot|$ denotes the usual Euclidean norm in R^n and $\hat{0}$ is the fuzzy number in E^n which membership function equal 1 at 0 and zero elsewhere. (E^n, D) is a complete metric space.

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space. A fuzzy random variable (f.r.v. for short) is a Borel measurable function $X : (\Omega, \mathcal{A}) \to (E^n, D)$. Let

$$L_2(\Omega, \mathcal{A}, P) = \{X \mid X \text{ is a f.r.v. with } E\|X\|^2 < \infty\}.$$

Two f.r.v.'s X and Y over $(\Omega, \mathcal{A}, P)$ are called equivalent if $P(X \neq Y) = 0$. The all equivalent elements in L_2 are identified. Define

$$\varphi(X, Y) = \left(\int_\Omega (D(X, Y))^2\, dP \right)^{1/2}, \quad X, Y \in L_2.$$

The norm $\|X\|_2$ of an element $X \in L_2$ is defined by

$$\|X\|_2 = \varphi(X, \hat{0}) = \left(\int_\Omega (D(X, \hat{0}))^2\, dP \right)^{1/2}.$$

Then (L_2, φ) is a complete metric space [1, Corollary 2.2] and φ satisfies that

$$\varphi(X+Z, Y+Z) = \varphi(X, Y), \quad \varphi(\lambda X, \lambda Y) = |\lambda| \varphi(X, Y),$$

$$\varphi(\lambda X, kX) \le |\lambda - k| \| X \|_2,$$

for any $X, Y, Z \in L_2$ and $\lambda, k \in R$.

Let $(X_n)_{n \ge 1}$ be a sequence in L_2. We call that X_n converges in mean square or m.s. converges to X as $n \to \infty$ if $\varphi(X_n, X) \to 0$.

3. Fuzzy Integro-Differential Equations

We consider the fuzzy integro-differential equations of the form

$$\frac{dX(t)}{dt} = f(t, X(t), \int_{t_0}^{t} k(t, s, X(s))\, ds), \quad t \in T = [t_0, b],$$

$$X(t_0) = X_0, \qquad \text{... (3.1)}$$

where $f: T \times L_2 \times L_2 \to L_2$, $k: T^2 \times L_2 \to L_2$ are m.s. continuous fuzzy mappings with respect to $t, s, t_0, b \in R$, and $X_0 \in L_2$.

A mapping $X: T \to L_2$ is said to solve the L_2-problem if and only if the following conditions are satisfied:

(*i*) $X(t)$ is continuous in the mean square sense;

(*ii*) $X(t_0) = X_0$;

(*iii*) $f(t, X(t), \int_{t_0}^{t} k(t, s, X(s))\, ds)$ is mean square integrable on T;

(*iv*) The mean square derivative of $X(t)$ exists for every $t \in T$ and satisfies

$$X'(t) = f(t, X(t), \int_{t_0}^{t} k(t, s, X(s))\, ds).$$

By [1], conditions (*i*)-(*iv*) hold if and only if for all $t \in T$

$$X(t) = X_0 + \int_{t_0}^{t} f(s, X(s), \int_{t_0}^{s} k(s, \tau, X(\tau))\, d\tau)\, ds; \qquad \text{... (3.2)}$$

the integral in (3.2) is the mean square integral.

Lemma 3.1 ([3, p. 758]) — Let $u(t)$, $p(t)$ and $q(t)$ be real valued non-negative continuous functions defined on R^+, for which the inequality

$$u(t) \leq u_0 + \int_0^t p(s)\, u(s) ds + \int_0^t p(s) \left[\int_0^s q(\tau)\, u(\tau)\, d\tau \right] ds$$

holds for all $t \in R^+$, where u_0 is a non-negative constant. Then

$$u(t) \leq u_0 \left[1 + \int_0^t p(s) \exp \left(\int_0^s (p(s) + q(\tau))\, d\tau \right) ds \right]$$

for all $t \in R^+$.

Theorem 3.1 — *Suppose the mappings* $k: T^2 \times L_2 \to L_2$ *and* $f: T \times L_2 \times L_2 \to L_2$ *satisfy the following conditions :*

$$(i)\ \rho\, (f(t, X_1, Y_1), f(t, X_2, Y_2)) \leq \lambda\, (\rho\, (X_1, X_2) + \rho\, (Y_1, Y_2))$$

for all $t \in T$ and for all $X_1, X_2, Y_1, Y_2 \in L_2$,

$$(ii)\ \rho\, (k(t, s, X_1), k(t, s, X_2)) \leq q \rho\, (X_1, X_2)$$

for all $t, s \in T$ and for all $X_1, X_2 \in L_2$,

$$(iii)\ \rho \left(\int_{t_0}^t f(s, X_0, \int_{t_0}^s k(s, \tau, X_0)\, d\tau) ds, \hat{0} \right) < M$$

for all $t \in T$ and $X_0 \in L_2$, where $\lambda, q, M > 0$ are constants.

Then eq. (3.1) has a unique solution.

PROOF : Let $C(I, L_2) = \{X : X : I \to L_2, X(t)$ is m.s. continuous$\}$, where I is a bounded interval in R. Define $H(X, Y) = \sup_{t \in I} \rho\, (X(t), Y(t))$, $X, Y \in C(I, L_2)$. Since (L_2, ρ) is a complete metric space, a standard proof applies to show that also $C(I, L_2)$ is complete.

Define a sequence of fuzzy random variables as follows :

$$X(t_0) = X_0,$$

$$X_1(t) = X_0 + \int_{t_0}^t f(s, X_0, \int_{t_0}^s k(s, \tau, X_0)\, d\tau) ds,$$

...

$$X_n(t) = X_0 + \int_{t_0}^t f(s, X_{n-1}(s), \int_{t_0}^s k(s, \tau, X_{n-1}(\tau))\, d\tau) ds. \quad \text{... (3.3)}$$

The assumptions on the mappings k and f as indicated in the mean square sense imply the existence of the integrals in the definition of fuzzy random variables $X_n(t), n = 1, 2, \ldots$. From (i), (ii), (iii) and (3.3), we have

$$\rho\,(X_{n+1}(t)\, X_n(t))$$

$$= \rho\left(\int_{t_0}^{t} f(s, X_n(s), \int_{t_0}^{s} k(s, \tau, X_n(\tau))d\tau \right) ds,$$

$$\int_{t_0}^{t} f\left(s, X_{n-1}(s), \int_{t_0}^{s} k(s, \tau, X_{n-1}(\tau))\, d\tau \right) ds$$

$$\leq \int_{t_0}^{t} \lambda \left[\rho\,(X_n(s), X_{n-1}(s)) + \int_{t_0}^{s} q\rho\,(X_n(\tau), X_{n-1}(\tau))\, d\tau \right] ds$$

$$\leq \int_{t_0}^{t} \lambda\rho\,(X_n(s), X_{n-1}(s))ds$$

$$+ \int_{t_0}^{t} \lambda \left(\int_{t_0}^{s} q\rho(X_n(\tau), X_{n-1}(\tau))d\tau \right) ds$$

$$\leq M\lambda^n \frac{(t-t_0)^n}{n!} \left[1 + \sum_{i=1}^{n} \binom{n}{i} \frac{(t-t_0)^i q^i}{(n+i)! - n!} \right]$$

$$\leq M\lambda^n \frac{(t-t_0)^n}{n!} [1 + (t-t_0)q]^n$$

$$= M \frac{[\lambda(t-t_0)(1+(t-t_0)q)]^n}{n!}.$$

Thus we have

$$H(X_{n+1}, X_n) = \sup_{t \in T} \rho(X_{n+1}(t), X_n(t)) \to 0 \quad \text{as} \quad n \to \infty.$$

Hence the sequence $\{X_n(t)\}$ converges uniformly on T to some fuzzy random variable $X(t)$. Now we shall show that $X(t)$ satisfies the equation

$$X(t) = X_0 + \int_{t_0}^{t} f(s, X(s), \int_{t_0}^{t} k(s, \tau, X(\tau)d\tau)ds$$

for all $t \in T$. Since $X_n(t_0) = X_0$, this is obvious for $t = t_0$. Now for all $t \in T$, we have

$$\rho\,(X(t), X_0 + \int_{t_0}^{t} f(s, X(s), \int_{t_0}^{s} k(s, \tau, X(\tau))d\tau)ds)$$

$$= \rho\left(\lim_{n\to\infty} X_n(t), X_0 + \int_{t_0}^{t} f(s, X(s), \int_{t_0}^{s} k(s, \tau, X(\tau))d\tau)ds \right)$$

$$= \left(\lim_{n\to\infty} X_n(t_0), \int_{t_0}^{t} f(s, X_n(s), \int_{t_0}^{s} k(s, \tau, X_n(\tau))d\tau)ds \right),$$

$$X_0 + \int_{t_0}^{t} f\left(s, X(s), \int_{t_0}^{s} k(s, \tau, X(\tau))d\tau)ds \right)$$

$$\leq \lim_{n\to\infty} \int_{t_0}^{t} \lambda \left[\rho(X_n(s), X(s)) + \int_{t_0}^{s} q\rho\,(X_n(\tau), X(\tau))dt \right] ds$$

$$= 0,$$

so that $X(t)$ is the desired solution of eq. (3.1) in the mean square sense. The uniqueness of the solution $X(t)$ of the eq. (3.1) is shown by using the inequality established by Pachpatte [3, p. 758]. Let $X(t)$ and $Y(t)$ be the two solutions of the eq. (3.1). Then

$$\rho\,(X(t), Y(t)) \leq \int_{t_0}^{t} \rho\,(f(s, D(s), \int_{t_0}^{s} k(s, \tau, X(\tau))\, d\tau,$$

$$f(s, Y(s), \int_{t_0}^{s} k(s, \tau, Y(\tau))\, d\tau))\, ds.$$

Using (*i*) and (*ii*) we have

$$\rho\,(X(t), Y(t)) \leq \int_{t_0}^{t} \lambda\rho\,(X(s), Y(s))ds$$

$$+ \int_{t_0}^{t} \lambda \left(\int_{t_0}^{s} q\rho\,(X(\tau), Y(\tau))d\tau \right) ds$$

$$\leq \varepsilon + \int_{t_0}^{t} \lambda\rho\,(X(s), Y(s))\, ds$$

$$+ \int_{t_0}^{t} \lambda \left(\int_{t_0}^{s} q\rho\,(X(\tau), Y(\tau))\, d\tau \right) ds$$

for all $t \in T$ and for every $\varepsilon > 0$. By using Lemma 3.1 with $u(t) = \rho\,(X(t), Y(t))$, we get

$$\rho\,(X(t), Y(t)) \leq \varepsilon\, [1 + \lambda\, exp((\lambda + q)\,(t - t_0))].$$

Since $\varepsilon > 0$ is arbitrary,

$$H(X, Y) = \sup_{t \in T} \rho\,(X(t), Y(t)) = 0.$$

This is the uniqueness of the solution $X(t)$ of eq. (3.1) in the mean square sense.

Theorem 3.2 — *Set* $\mathcal{S}(r) = \{X(t) : X(t) \in L_2$ *and* $\rho\,(X(t), \hat{0}) \leq r\}$ *for some* $r > 0$. *Suppose the following conditions are satisfied* :

$$(i)\ \rho(f(t, X(t), \int_{t_0}^{t} k(t, s, X(s))ds), f(t, Y(t), \int_{t_0}^{t} k(t, s, Y(s))\, ds))$$

$$\leq \lambda \left[\rho\,(X(t), Y(t)) + \rho \left(\int_{t_0}^{t} k(t, s, X(s))ds, \int_{t_0}^{t} k(t, s, Y(s))\, ds \right) \right]$$

for all $t, s \in T$ *and for all* $X(t), Y(t) \in \mathcal{S}(\tau)$,

$$(ii)\ \rho(k(t, s, X(s)), k(t, s, Y(s))) \leq \mu\rho\,(X(s), Y(s))$$

for all $t, s \in T = [t_0, b]$, *where* $X(t), Y(t) \in \mathcal{S}(\tau)$, $t_0, b \in R$, *and* λ, μ *are positive constants.*

Then there exists a unique solution of the eq. (3.1) provided that

$$\lambda(b - t_0)\, [1 + \mu(b - t_0)] < 1$$

and

$$\rho\,(X_0, \hat{0}) + \int_{t_0}^{t} \rho(f(s, \hat{0}, \int_{t_0}^{s} k(s, \tau, \hat{0})d\tau), \hat{0})ds$$

$$\leq r[1 - \lambda(1 + \mu(t - t_0))\,(t - t_0)].$$

PROOF : We define the fuzzy random operator K from the set $\mathcal{S}(\tau)$ into L_2 as follows :

$$KX(t) = X_0 + \int_{t_0}^{t} f(s, X(s), \int_{t_0}^{s} k(s, \tau, X(\tau))\, d\tau)ds.$$

Then from the conditions of the theorem

$$
\begin{aligned}
&\rho(KX(t), \hat{0}) \\
&\le \rho(X_0, \hat{0}) + \int_{t_0}^{t} \rho\left(f(s, X(s), \int_{t_0}^{s} k(s, \tau, X(\tau))d\tau), \hat{0} \right) ds \\
&\le \rho(X_0, \hat{0}) \\
&+ \int_{t_0}^{t} \rho\left(f(s, X(s), \int_{t_0}^{s} k(s, \tau, X(\tau))\, d\tau), f(s, \hat{0}, \int_{t_0}^{s} k(s, \tau, \hat{0})\, d\tau) \right) ds \\
&+ \int_{t_0}^{t} \rho\left(f(s, \hat{0}, \int_{t_0}^{s} k(s, \tau, \hat{0})\, d\tau)ds), \hat{0} \right) ds \\
&\le \rho(X_0, \hat{0}) \\
&+ \int_{t_0}^{t} \lambda \left[\rho(X(s), \hat{0}) + \int_{t_0}^{s} \rho(k(s, \tau, X(\tau)), k(s, \tau, \hat{0}))\, d\tau \right] ds \\
&+ \int_{t_0}^{t} \rho\left(f(s, \hat{0}, \int_{t_0}^{s} k(s, \tau, \hat{0})d\tau), \hat{0} \right) ds \\
&\le \rho(X_0, \hat{0}) \\
&+ \int_{t_0}^{t} \lambda \left[\rho(X(s), \hat{0}) + \int_{t_0}^{s} \mu\rho(X(\tau), \hat{0})d\tau \right] ds \\
&+ \int_{t_0}^{t} \rho\left(f(s, \hat{0}, \int_{t_0}^{s} k(s, \tau, \hat{0})d\tau), \hat{0} \right) ds \\
&\le \rho(X_0, \hat{0}) + \int_{t_0}^{t} \lambda \left[r + \int_{t_0}^{s} \mu r d\tau \right] ds \\
&+ \int_{t_0}^{t} \rho\left(f(s, \hat{0}, \int_{t_0}^{s} k(s, \tau, \hat{0})d\tau), \hat{0} \right) ds \\
&\le \rho(X_0, \hat{0}) + \lambda r\, [1 + \mu(t - t_0)]\, (t - t_0)
\end{aligned}
$$

$$+ \int_{t_0}^{t} \rho\left(f(s, \hat{0}, \int_{t_0}^{s} k(s, \tau, \hat{0}) d\tau), \hat{0} \right) ds.$$

Again from the conditions of the theorem, we have

$$H(KX, \hat{0}) = \sup_{t \in T} \rho(KX(t), \hat{0})$$

$$\leq \sup_{t \in T} (r[1 - \lambda(1 + \mu(t - t_0)) (t - t_0)]$$

$$+ \lambda r [1 + \mu(t - t_0)] (t - t_0))$$

$$= r.$$

Hence

$$KX(t) \in \mathcal{S}(r).$$

Let $X(t), Y(t) \in \mathcal{S}(r)$. Then we have

$$H(KX, KY)$$

$$= \sup_{t \in T} \rho(KX(t), KY(t))$$

$$= \sup_{t \in T} \rho\left(\int_{t_0}^{t} f(s, X(s), \int_{t_0}^{s} k(s, \tau, X(\tau)) d\tau \right) ds,$$

$$\int_{t_0}^{t} f\left(s, Y(s), \int_{t_0}^{s} k(s, \tau, U(\tau)) d\tau) ds \right)$$

$$\leq \sup_{t \in T} \int_{t_0}^{t} \rho\left(f(s, X(s), \int_{t_0}^{s} k(s, \tau, X(\tau)) d\tau \right),$$

$$f\left(s, Y(s), \int_{t_0}^{s} k(s, \tau, Y(\tau)) d\tau \right) ds$$

$$\leq \sup_{t \in T} \int_{t_0}^{t} \lambda\left[\rho(X(s), Y(s)) + \rho\left(\int_{t_0}^{t} k(s, \tau, X(\tau)) d\tau, \int_{t_0}^{s} k(s, \tau, Y(\tau)) d\tau \right) \right] ds$$

$$\leq \sup_{t \in T} \int_{t_0}^{t} \lambda \rho(X(s), Y(s)) ds$$

$$+ \sup_{t \in T} \int_{t_0}^{t} \lambda \int_{t_0}^{s} \mu\rho(X(\tau), Y(\tau))\, d\tau\, ds$$

$$\leq \lambda(b - t_0)\, H(X, Y) + \lambda\mu(b - t_0)^2\, H(X, Y)$$

$$= \lambda(b - t_0)\, [1 + \mu(b - t_0)]\, H(X, Y).$$

The condition $\lambda(b - t_0)\,[1 + \mu(b - t_0) < 1$ implies that K is a contraction fuzzy random operator. Therefore, by applying the fixed point theorem, there exists a unique solution $X(t) \in \mathcal{S}(r)$ of eq. (3.1). This completes the proof of the Theorem.

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