Indian J. pure appl. Math., **34**(10) : 1513-1520, October 2003

A FIXED POINT THEOREM FOR UNBOUNDED *D*-METRIC SPACES

B. E. RHOADES

Department of Mathematics, Indiana University, Bloomington, IN 47405-7106, *U.S.A.*
e-mail : rhoades@indiana.edu

(*Received* 18 *July* 2002; *after final revision* 3 *November* 2002; *accepted* 29 *April* 2003)

We prove a fixed point theorem for a very general contractive condition on unbounded *D*-metric spaces.

Key Words : *D*-metric Spaces; Fixed Point Theorem; Unbounded

In 1984, Dhage [1] defined *D*-metric spaces and established the basic topological properties of such spaces. Since then he has written a number of papers in this area. See, e.g., [2], [3], [4], [5], [6] and [7]. In all cases he assumed the boundedness of the space. In a recent paper[9] the author and Dhage were able to remove the boundedness condition for certain classes of maps. It is the purpose of this paper to show that the boundedness condition can be removed for an even more general class of maps.

A nonempty set X, together with a *D*-metric function $\rho : X \times X \times X \to [0, \infty)$ is called a *D*-metric space with *D*-metric ρ, denoted by (X, ρ), if it satisfies the following properties:

(*i*) $\rho(x, y, z) = 0 \Leftrightarrow x = y = z$ (coincidence),

(*ii*) $\rho(x, y, z) = \rho(p\{x, y, z\})$ (symmetry)
where p is a permutation of $\{x, y, z\}$, and

(*iii*) $\rho(x, y, z) \le \rho(x, y, a) + \rho(x, a, z) + \rho(a, y, z)$

for each $x, y, z, a \in X$. (tetrahedral inequality)

Let

$$\Phi := \{\phi : \mathbb{R}^+ \to \mathbb{R}^+ : \phi \text{ is continuous, nondecreasing and satisfies}$$

$$\phi(t) < t \text{ for each } t > 0\}.$$

Note that, if $\phi \in \Phi$, then $\phi(0) = 0$. For, suppose that $\phi(0) = a > 0$. Since ϕ is nondecreasing, we have $a = \phi(0) \le \phi(a) < a$, a contradiction.

Let X be a *D*-metric space, f a self-map of X. We shall be interested in those maps f which satisfy the following inequality:

$$\rho(fx, fy, z) \le \phi(\max\{\rho(x, y, z), \rho(x, fx, z), \rho(y, fy, z)\ \rho(x, fy, z), \rho(y, fx, z)\}). \qquad \text{... (1)}$$

This class of maps has already been studied. See, e.g.[8]. Let $x \in X, f$ any self map of X. We define the *n*th orbit of a point x as $O(x, n) = \{x, fx, f^2x, ..., f^nx\}$. The diameter δ of $O(x, n)$, expressed in terms of the *D*-metric is defined by

$$\delta(O(x, n)) = \max_{0 \le r \le q \le p \le n} \rho(f^r x, f^q x, f^p x).$$

Lemma 1 — Let f satisfy (1). Then there exists an integer $p > 0$ such that

$$\delta(O(x, n)) = \rho(x, f^p x, x). \qquad \text{... (2)}$$

PROOF : If $\delta(O(x, n)) = 0$, then (2) is trivially true. Assume that $\delta(O(x, n)) > 0$. From the definition of $\delta(O(x, n))$, there exist integers r, q, p satisfying $0 \le r \le q \le p \le n$ such that

$$\delta(O(x, n)) = \rho(f^r x, f^q x, f^p x).$$

Suppose that $r > 0$. Then, from (1),

$$\begin{aligned}\delta(O(x, n)) &= \rho(f^r x, f^q x, f^p x)\\ &\le \phi\,(\max\,\{\rho(f^{r-1}x, f^{q-1}x, f^p x),\ \rho(f^{r-1}x, f^r x, f^p x),\ \rho(f^{q-1}x, f^q x, f^p x),\\ &\quad \rho(f^{r-1}x, f^q x, f^p x),\ \rho(f^{q-1}x, f^r x, f^p x)\})\\ &\le \phi\,(\delta(O(x, n)) < \delta(O(x, n)),\end{aligned}$$

a contradiction. Therefore $r = 0$.

Suppose now that $q > 0$. Then, from (1),

$$\begin{aligned}\delta(O(x, n)) &= \rho(f^q x, f^p x, x)\\ &\le \phi\,(\max\,\{\rho(f^{q-1}x, f^{p-1}x, x),\ \rho(f^{q-1}x, f^q x, x),\ \rho(f^{p-1}x, f^p x, x),\\ &\quad \rho(f^{q-1}x, f^p x, x),\ \rho(f^{p-1}x, f^q x, x)\})\\ &\le \phi\,(\delta(O(x, n)) < \delta(O(x, n)),\end{aligned}$$

a contradiction. Therefore $q = 0$. □

Theorem 1 — *Let X be a complete D-metric space, f a selfmap of X satisfying (1) for each $x, y, z \in X$, and some $\phi \in \Phi$. Then f has a unique fixed point u and f is continuous at u.*

PROOF : Without loss of generality we may assume that $f^n x \ne f^{n+1} x$, for each n. For, if there exists an integer n for which $f^n x \ne f^{n+1} x$, then f has a fixed point. We shall need the following lemmas, in order to verify that the orbits of f are bounded.

Lemma 2 — Let f satisfy (1), $x \in X$ such that $f^n x \ne f^{n+1} x$ for each n. Then, for each natural number p,

$$\rho(f^p x, f^{p+1} x, x) \le \phi^p\,(\delta(O(x, p+1)). \qquad \text{... (3)}$$

PROOF : The proof is by induction. For simplicity of notation we shall denote $\delta(O(x, n))$ by d_n. From (1) and (2),

$$\begin{aligned}\rho(fx, f^2x, x) \le \phi\ (\max\{&\rho(x, fx, x), \rho(fx, f^2x, x)\\ &\rho(x, f^2x, x), \rho(fx, fx, x)\}).\end{aligned}$$

$$\leq \phi(d_2).$$

Assume the induction hypothesis.

Using (1),

$$\rho(f^p x, f^{p+1}x, x) \leq \phi\,(\max\{\rho(f^{p-1}x, f^p x, x),\ \rho(f^p x, f^{p+1}x, x),$$

$$\rho(f^{p-1}x, f^{p+1}x, x), \rho(f^p x, f^p x, x)\}). \quad \ldots (4)$$

Using (1) and the induction hypothesis,

$$\rho(f^p x, f^p x, x) \leq \phi\,(\max\{\rho(f^{p-1}x, f^{p-1}x, x), \rho(f^{p-1}x, f^p x, x)\})$$

$$\leq \phi\,(\max\{\rho(f^{p-1}x, f^{p-1}x, x), \phi^{p-1}(d_p)))\}). \quad \ldots (5)$$

and (5) is now a recursion inequality for $\rho(f^p x, f^p x, x)$.

Since ϕ is increasing,

$$\rho(f^p x, f^p x, x) \leq \phi\,(\max\{\phi(\max\{\rho(f^{p-2}x, f^{p-2}x, x),$$

$$\phi^{p-2}(d_p)\},\ \phi^{p-1}(d_p))\})$$

$$\leq \phi\,(\max\{\max\{\phi(\rho(f^{p-2}x, f^{p-2}x, x)),$$

$$\phi^{p-1}(d_p)\},\ \phi^{p-1}(d_p))\})$$

$$= \phi\,(\max\{\phi\,(\rho(f^{p-2}x, f^{p-2}x, x), \phi^{p-1}(d_p))\})$$

$$\leq (\max\{\phi^2(\rho(f^{p-2}x, f^{p-2}x, x)), \phi^p(d_p))\}$$

$$\leq (\max\{\phi^p(\phi\,(\max\{\rho(f^{p-3}x, f^{p-3}x, x),$$

$$\phi^{p-3}((d_p)\}),\ \phi^p(d_{p-2})\}$$

$$\leq \max\{\phi^2(\max\{\phi(\rho(f^{p-3}x, f^{p-3}x, x)),$$

$$\phi^{p-2}(d_{p-2})\},\ \phi^p(d_p))\}$$

$$\leq \max\{\max\{\phi^3(\rho(f^{p-3}x, f^{p-3}x, x)),$$

$$\phi^p(d_{p-2})\}\ \phi^p(d_p)\}$$

$$= \max\{\phi^3(\rho(f^{p-3}x, f^{p-3}x, x)), \phi^p(d_p)\}$$

$$\leq$$

$$\vdots$$

$$\leq \max\{\phi^p(\rho(x, x, x)), \phi^p(d_p)\} = \phi^p(d_p). \quad \ldots (6)$$

From (1),

$$\rho(f^{p-1}x, f^{p+1}x, x) \le \phi\,(\max\{\rho(f^{p-2}x, f^px, x), \rho(f^{p-2}x, f^{p-1}x, x),$$
$$\rho(f^px, f^{p+1}x, x), \rho(f^{p-2}x, f^{p+1}x, x),$$
$$\rho(f^px, f^{p-1}x, x)\}).$$

Again using (1),

$$\rho(f^{p-2}x, f^{p+1}x, x) \le \phi\,(\max\{\rho(f^{p-3}x, f^px, x), \rho(f^{p-3}x, f^{p-2}x, x),$$
$$\rho(f^px, f^{p+1}x, x), \rho(f^{p-3}x, f^{p+1}x, x),$$
$$\rho(f^px, f^{p-2}x, x)\}).$$

In a similar manner one can use (1) to obtain inequalities for

$$\rho(f^{p-3}x, f^{p+1}x, x), \rho(f^{p-4}x, f^{p+1}x, x), \ldots, \rho(fx, f^{p+1}x, x).$$

Lemma 3 — Let f satisfy (1) and let $x \in X$ such that $f^nx \ne f^{n+1}x$ for each n. Then, for every pair of natural numbers p, q with $p < q$.

$$\rho(f^p x, f^q x, x) \le \phi^p\,(d_q). \qquad \ldots (7)$$

PROOF : The proof is by double induction. For $p = 1$ and any $q > 1$, from (1), with $z = x$,

$$\rho(fx, f^q x, x) \le \phi\,(d_q).$$

Assume the complete induction hypothesis on p and q; i.e., that

$$\rho(f^s x, f^t x, x) \le \phi^s\,(d_t) \qquad \text{for} \qquad 1 \le s \le p, 2 \le t \le q. \qquad \ldots (8)$$

Then, from (1), using (8), Lemma 2, and the fact the p is increasing,

$$\rho(f^p x, f^{q+1} x, x) \le \phi\,(\max\{\rho(f^{p-1} x, f^q x, x), \rho(f^{p-1} x, f^p x, x),$$
$$\rho(f^q x, f^{q+1} x, x), \rho(f^{p-1} x, f^{q+1} x, x), \rho(f^q x, f^p x, x)\})$$
$$\le \phi\,(\max\{\phi^{p-1}\,(d_q), \phi^{p-1}\,(d_p), \phi^q\,(d_{q+1}),$$
$$\rho(f^{p-1} x, f^{q+1} x, x), \phi^p\,(d_q)\})$$
$$\le \phi\,(\max\{\phi^{p-1}\,(d_{q+1}), \rho(f^{p-1} x, f^{q+1} x, x)\}).$$

Using (1), the induction hypothesis, and Lemma 2,

$$\rho(f^{p-1} x, f^{q+1} x, x) \le \phi\,(\max\{\rho(f^{p-2} x, f^q x, x), \rho(f^{p-2} x, f^{p-1} x, x),$$
$$\rho(f^q x, f^{q+1} x, x), \rho(f^{p-2} x, f^{q+1} x, x), \rho(f^q x, f^{p-1} x, x)\})$$

$$\leq \phi\ (\max\{\phi^{p-2}(d_q), \phi^{p-2}(d_{p-1}), \phi^{q}(d_{q+1}),$$

$$\rho(f^{p-2}x, f^{q+1}x, x), \phi^{p-1}(d_q)\})$$

$$\leq \phi\ (\max\{\phi^{p-2}(d_{q+1}), \rho(f^{p-2}x, f^{q+1}x, x)\}).$$

Continuing in this manner we obtain inequalities for

$$\rho(f^{p-3}x, f^{q+1}x, x), \ldots, \rho(fx, f^{q+1}x, x)$$

Using (1),

$$\rho(fx, f^{q+1}x, x) \leq \phi\ (d^{q+1}).$$

Then

$$\rho(f^{2}x, f^{q+1}x, x) \leq \phi\ (\max\{\phi\ (d_{q+1}), \rho(fx, f^{q+1}x, x)\})$$

$$\leq \phi\ (\max\{\phi(d_{q+1}), \phi(d_{q+1})\})$$

$$= \phi^{2}(d_{q+1}).$$

$$\rho(f^{3}x, f^{q+1}x, x) \leq \phi\ (\max\{\phi^{2}(d_{q+1}), \rho(f^{2}x, f^{q+1}x, x)\})$$

$$\leq \phi\ (\max\{\phi^{2}(d_{q+1}), \phi^{2}(d_{q+1})\})$$

$$= \phi^{3}(d_{q+1}).$$

Continuing in this manner.

$$\rho(f^{p}x, f^{q+1}x, x) \leq \phi\ (\max\{\phi^{p-1}(d_{q+1}), \rho(f^{p-1}x, f^{q+1}x, x)\})$$

$$\leq \phi\ (\max\{\phi^{p-1}(d_{q+1}), \phi^{p-1}(d_{q+1})\})$$

$$= \phi^{p}(d_{q+1}).$$

$$\rho(f^{p+1}x, f^{q+1}x, x) \leq \phi\ (\max\{\rho\ (f^{p}x, f^{q}x, x), \rho(f^{p}x, f^{p+1}x, x),$$

$$\rho(f^{q}x, f^{q+1}x, x), \rho(f^{p}x, f^{p+1}x, x), \rho(f^{p+1}x, f^{q}x, x)\})$$

$$\leq \phi\ (\max\{\phi^{p}(d_{q+1}), \rho(f^{p+1}x, f^{q}x, x)\}).$$

Inequality (9) is a recursion formula in q. Therefore, since ϕ is increasing,

$$\rho(f^{p+1}x, f^{q+1}x, x) \leq \phi\ (\max\{\phi^{p}(d_{q+1})\ \phi\ (\max\{\phi^{p}(d_q), \rho(f^{p+1}x, f^{q-1}x, x)\})$$

$$\leq\ \phi\ (\max\{\phi^{p}(d_{q+1}), \phi\ (\rho(f^{p+1}x, f^{q-1}x, x)\})$$

$$\leq \max\{\phi^{p+1}(d_{q+1}), \phi^2(\rho(f^{p+1}x, f^{q-1}x, x))\})$$

$$\leq \max\{\phi^{p+1}(d_{q+1})\ \phi^3$$

$$(\max\{\phi^p(d_{q-1}), \rho(f^{p+1}x, f^{q-2}x, x)\}))\}$$

$$= \max\{\phi^{p+1}(d_{q+1}), \rho(f^{p+1}x, f^{q-2}x, x)\}$$

$$\leq$$

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$$\leq \max \|[\ \phi^{p+1}(d_{q+1}), \phi^{q-(p+1)}(\rho(f^{p+1}x, f^{p+2}x, x))\}$$

$$\leq \max\{\phi^{p+1}(d_{q+1}), \phi^{q-(p+1)}(\phi^{p+1}(d_{p+2}))\}$$

$$= \phi^{p+1}(d_{q+1}).$$

Lemma 4 — Under the hypotheses of Theorem 1, for each $x, z \in X$, $d_n(x)$ and $d_n(z)$ are bounded.

PROOF : From Lemma 1 there exists an integer p satisfying $0 < p \leq n$ such that

$$d_n(x) = \rho(x, f^p x, x).$$

Since $d_1(x), \leq d_2(x) \leq \ldots \leq d_n(x), p = p(n)$ is nondecreasing in n. If $p(n)$ remains bounded as $n \to \infty$, there is nothing to prove.

If not, then choose N to be the smallest integer such that $p_0 = p_0(N)$ satisfies $\phi^{p_0}(d_n(x)) < d_n(x)/4$. This can be done since, for each $x \geq 0$, $\lim_n \phi^n(x) = 0$.

Then, for any $n > N$, from property (*iii*) for ρ,

$$d_n(x) = \rho(x, f^p x, x)$$

$$\leq \rho(x, f^p x, f^{p_0} x) + \rho(x, f^{p_0} x, x) + \rho(f^{p_0} x, f^p x, x)$$

$$= 2\rho(f^{p_0} x, f^p x, x) + \rho(x, f^{p_0} x, x)$$

$$\leq 2\phi^{p_0}(d_n(x)) + \rho(x, f^{p_0} x, x)$$

$$< \frac{d_n(x)}{2} + \rho(x, f^{p_0} x, x),$$

or $d_n(x) < 2\rho(x, f^{p_0} x, x)$, and $\{d_n(x)\}$ is bounded.

A similar argument applies to $d_n(z)$.

We are now ready to complete the proof of the theorem.

Let x, $z \in X$, r, s integers with $1 \leq r < s$. Then, from property (*iii*) for ρ, with $x_n := f^n x_0$,

$$\rho(x_n, x_{n+r}, x_{n+s}) \leq \rho(x_n, x_{n+r}, z) + \rho(x_n, z, x_{n+s}) + \rho(z, x_{n+r}, x_{n+s})$$

$$\leq \phi^n (d_{n+r}(z)) + \phi^n (d_{n+s}(z)) + \phi^{n+r} (d_{n+s}(z))$$

$$\leq 3\phi^n (M),$$

where M is an upper bound for $\{d_n(z)\}$. Therefore, $\{x_n\}$ is D-Cauchy, hence convergent to a point $u \in X$.

Using (1) with $z = fu$,

$$\rho(x_n, x_{n+1}, fu) \leq \phi \,(\max\{\rho(x_{n-1}, x_n, fu),\ \rho(x_n, x_{n+1}, fu),$$

$$\rho(x_{n-1}, x_{n+1}, fu), \rho(x_n, x_n, fu)\}).$$

Taking the limit as $n \to \infty$ yields

$$\rho(u, u, fu) \leq f(\rho(u,u, fu))$$

which, by property (*i*) for ρ, implies that $u = fu$.

To show uniqueness, suppose that u is also a fixed point of f.

From (1),

$$\rho(u, v, u) = \rho(fu, fv, u)$$

$$\leq \phi \,(\max\{\rho(u, v, u), \rho(u, fu, u),$$

$$\rho(v, fu, u), \rho(u, fv, u), \rho(v, fu, u)\})$$

$$= \phi \,(\max\{\rho(u, v, u), \rho(v, v, u)\})$$

$$= \phi(\rho(u, v, u)). \quad \text{... (10)}$$

Again from (1),

$$\rho(u, v, u) = \rho(fu, fv, u)$$

$$\leq \phi \,(\max\{\rho(u, v, u), \rho(u, fu, v), \rho(v, fv, v),$$

$$\rho(u, fv, v), \rho(v, fuv)\})$$

$$= \phi \,(\max\{\rho(u, v, v), (u, u, v)\})$$

$$= \phi(\rho(u, u, v)). \quad \text{... (11)}$$

Combining (10) and (11) gives the desired result.

To show that f is continuous at u, let $\{y_n\} \subset X$ with $\lim y_n = u$.

Then, using (1) with $x = z = u, y = y_n$,

$$\rho(u, fy_n, u) = \rho(fu, fy_n, u)$$
$$\leq \phi\ (\max\{\rho(u, y_n, u), \rho(u, fu, u), \rho(y_n, fy_n, u),$$
$$\rho(u, fy_n, u), \rho(y_n, fu, u)\}). \quad \text{... (12)}$$

Taking the lim sup of (1) yields

$$\limsup \rho(u, fy_n, u) \leq \phi\ (\max\{0, 0, \limsup \rho(u, fy_n, u)\}),$$

which implies that $\lim fy_n = u = fu$, and f is continuous at u. □

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