Indian J. pure appl. Math., **34**(9) : 1287-1298, September 2003

B. Y. CHEN'S INEQUALITY FOR *S*-SPACE-FORMS: APPLICATIONS TO SLANT IMMERSIONS

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(*Received* 18 *March* 2002; *after revision* 16 *November* 2002; *accepted* 23 *December* 2002)

A version of Chen's inequality for a submanifold of an *S*-space-form, tangent to the structure vector fields of the ambient space, is established and some applications to the case of slant immersions are obtained from it. Proper slant submanifolds of minimum dimension satisfying the equality case are also characterized.

Key Words : Chen Inequality; Space Forms; Slant immersion

1. INTRODUCTION

Recently, Chen[3, 4, 5] introduced, for a Riemannian manifold M, a well defined Riemannian invariant δ_M, given by

$$\delta_M(p) = \tau(p) - (\inf K)(p),$$

for any $p \in M$, where τ is the scalar curvature and

$$(\inf K)(p) = \inf \{K(\pi) : \text{plane sections } \pi \subset T_p(M)\},$$

$K(\pi)$ denoting the sectional curvature of M associated with the plane section π. Moreover, for submanifolds M in a real-space form of constant sectional curvature c, Chen gave the following basic inequality involving the intrinsic invariant δ_M and the squared mean curvature of the immersion,

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)} |H|^2 + \frac{(n+1)(n-2)}{2} c, \qquad \text{... (1.1)}$$

where n denotes the dimension of M and H is the mean curvature vector. In[2], Carriazo establishes a contact version of Chen's inequality for a submanifold tangent to the structure vector field of a Sasakian-space-form. In fact, he proves that if a θ-slant immersion of a Riemannian $(n+1)$- manifold M into a Sasakian-space-form $\tilde{M}^{2m+1}(c)$ with metric g and f-structure ϕ is considered, then, for any point $p \in M$ and any plane section π of the contact distribution $\mathcal{D}$ at p,

$$\tau - K(\pi) \leq \frac{(n+1)^2(n-1)}{2n} |H|^2 + \frac{1}{2}(n+1)(n-2)\frac{c+3}{4} + n\cos^2\theta + \frac{3(c-1)}{4}\left(\frac{n}{2}\cos^2\theta - \Phi^2(\pi)\right).$$

*The authors are partially supported by the PAI project (Junta de Andalucia, Spain, 2002).

where $\Phi^2(\pi) = g^2(e_1, \phi e_2)$ is a real number in [0, 1] which is independent on the choice of the orthonormal basis $\{e_1, e_2\}$ of π. In particular, if one defines

$$(\inf_{\mathcal{D}} K)(p) = \inf \{K(\pi) : \text{plane sections } \pi \subset \mathcal{D}_p\}$$

for each point $p \in M$ and

$$\delta_M^{\mathcal{D}}(p) = \tau(p) - (\inf_{\mathcal{D}} K)(p),$$

then, for every 3-dimensional θ-slant submanifold M, the inequality

$$\delta_M^{\mathcal{D}} \leq \frac{9}{4} |H|^2 + 2\cos^2\theta$$

is obtained, where the equality holds if and only if M is minimal.

The purpose of the present paper is to establish a general inequality, similar to that of [2], for submanifolds tangent to the structure vector fields of an S-space-form (see, for references, [1,7]). Thus, in Section 2, we review basic formulas and definition for metric f-manifolds and their submanifolds, for later use. In Section 3, we establish the mentioned inequality. Finally, we present some applications in Section 4, by paying special attention to slant immersions. For example, we characterize $(2 + s)$-dimensional slant submanifolds satisfying our equality case.

2. PRELIMINARIES

Let $(\tilde{M}, g)$ be a Riemannian manifold and denote by $T\tilde{M}$ the Lie algebra of vector fields in $\tilde{M}$. $\tilde{M}$ is said to be a metric f-manifold if there exist a (1, 1) tensor field f, s global unit vector fields $\xi_1, \ldots, \xi_s$ (called *structure vector fields*) and s 1-forms $\eta_1, \ldots, \eta_s$ on $\tilde{M}$ such that $f^2 X = -X + \Sigma_\alpha \eta_\alpha(X)\xi_\alpha$, $g(X, \xi_\alpha) = \eta_\alpha(X)$ and $g(fX, fY) = g(X, Y) - \Sigma_\alpha \eta_\alpha(X)\eta_\alpha(Y)$, for any $X, Y \in T\tilde{M}$ and $\alpha \in \{1, \ldots, s\}$.

Let F denote the *fundamental 2-form* in $\tilde{M}$ given by $F(X, Y) = g(X, fY)$, for any $X, Y \in T\tilde{M}$. The f-structure f is said to be *normal* if $[f, f] + 2\Sigma_\alpha \xi_\alpha \otimes d\eta_\alpha = 0$, where $[f, f]$ is the Nijenhuis torsion of f. $\tilde{M}$ is called an *S-manifold* if the structure is normal and $F = d\eta_\alpha$, for any $\alpha = 1, \ldots, s$.

Given an S-manifold $\tilde{M}$, a plane section π in $T_p M$ is called an f-section if it is spanned by X and fX, where X is a unit tangent vector field orthogonal to the distribution $\mathcal{M}$ spanned by the structure vector fields. The sectional curvature $K(\pi)$ of an f-section π is called *f-sectional curvature*. An S-manifold is said to be an *S-space-form* if it has constant f-sectional curvature c and then, it is denoted by $\tilde{M}(c)$.

Now, let M be a submanifold immersed in an S-manifold $\tilde{M}$. The induced Riemannian metric on M is also denoted by g. Let TM be the Lie algebra of vector fields in M and $T^\perp M$ the set of all vector fields normal to M. We denote by σ the second fundamental form of M and by A_V the shape operator associated with any $V \in T^{\perp}M$. We put $\sigma_{ij}^{r} = g(\sigma(e_i, e_j), e_r)$, for any $e_i\, e_j \in TM$ and $e_r \in TM$. The mean curvature vector H is defined by $H = (1/\dim(M))$ trace (σ) and M is said to be *minimal* if H vanishes identically.

The curvature tensor field R of a submanifold M of an S-space-form $\tilde{M}(c)$ satisfies

$$
\begin{aligned}
R(X, Y, Z, W) &= g(\sigma(X, W), \sigma(Y, Z)) - g(\sigma(X, Z), \sigma(Y, W)) \\
&\sum_{\alpha, \beta} (g(fX, fW)\, \eta_\alpha(Y)\, \eta_\beta(Z) - g(fX, fZ)\, \eta_\alpha(Y)\, \eta_\beta(W) \\
&+ g(fY, fZ)\, \eta_\alpha(X)\, \eta_\beta(W) - g(fY, fW)\, \eta_\alpha(X)\, \eta_\beta(Z)) \\
&+ \frac{c+3s}{4}(g(fX, fW)_g (fY, fZ) - g(fX, fZ)\, g(fY, fW)) \\
&+ \frac{c-s}{4}(F(X, W)\, F(Y, Z) - F(X, Z)\, F(Y, W) - 2F(X, Y)\, F(Z, W)), \qquad \text{... (2.1)}
\end{aligned}
$$

for any $X, Y, Z, W \in T\tilde{M}$ (see, for references, [1, 7]).

From now on, we suppose that the structure vector fields are tangent to M and we denote by $n + s$ (resp. m) the dimension of M (resp. $\tilde{M}$). We consider $n \geq 2$. Hence, if we denote by $\mathcal{L}$ the orthogonal distribution to $\mathcal{M}$ in TM, we can write the orthogonal direct decomposition $TM = \mathcal{L} \oplus \mathcal{M}$.

For any $X \in TM$, we put $fX = TX + NX$, where TX (resp. NX) is the tangential (resp. normal) component of $f\,X$. It is well-known that

$$\sigma(X, \xi_\alpha) = -NX, \qquad \text{... (2.2)}$$

for any $X \in TM$ and any $\alpha = 1, ..., s$. Given a local orthonormal frame $\{e_1, ..., e_n\}$ of $\mathcal{L}$, we can define the squared norms fo T and N by

$$|T|^2 = \sum_{i,j=1}^{n} g^2(e_i, Te_j), \quad |N|^2 = \sum_{i=1}^{n} |Ne_i|^2, \qquad \text{... (2.3)}$$

respectively, $|T|^2$ and $|N|^2$ being independent of the choice of the above orthonormal frame.

The submanifold M is said to be *invariant* if N is identically zero, that is, if $fX \in TM$, for any $X \in TM$ and it is said to be *anti-invariant* if T is identically zero, that is, if $fX \in T^{\perp}M$, for any $X \in TM$. Moreover, if for each nonzero vector $X \in T_pM - \mathcal{M}_p$, we consider the angle $\theta(X)$ between fX and T_pM, then the submanifold is said to be θ-slant[6] if such angle is a constant, which is independent on the choice of $p \in M$ and $X \in T_pM - \mathcal{M}_p$. The angle θ of a slant immersion is called the *slant angle* of the immersion. Invariant and anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant immersion which is not invariant nor anti-invariant is called a *proper* slant immersion.

In[6], Hans-Uber has proved that a θ-slant submanifold M of a metric f-manifold $\tilde{M}$ satisfies

$$g(TX, TY) = \cos^2\theta \left(g(X, Y) - \sum_{\alpha} \eta_\alpha(X)\eta_\alpha(Y)\right), \quad \text{.... (2.4)}$$

$$g(NX, NY) = \sin^2\theta\, g(X, Y) - \sum_{\alpha} \eta_\alpha(X)\eta_\alpha(Y)), \quad \text{... (2.5)}$$

for any $X, Y \in TM$. On the other and, if $\{e_1, ..., e_n, \xi_1, ..., \xi_s\}$ is a local orthonormal frame of TM, it can be proved that

$$\sum_{j=1}^{n} g^2(e_i, fe_j) = \cos^2\theta \quad \text{... (2.6)}$$

for any $i = 1, ..., s$ [6].

For an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{n+s}\}$ of $T_pM, p \in M$, the scalar curvature τ at p is defined by

$$\tau = \sum_{i<j} K(e_i \wedge e_j), \quad \text{... (2.7)}$$

where $K(e_i \wedge e_j)$ denotes the sectional curvature of M associated with the plane section spanned by e_i, e_j. In particular, if we put $e_{n+\alpha} = \xi_\alpha$, for $\alpha = 1, ..., s$, then (2.7) implies :

$$2\tau = \sum_{i \neq j} K(e_i \wedge e_j) + 2\sum_{i=1}^{n}\sum_{\alpha=1}^{s} K(e_i \wedge \xi_\alpha). \quad \text{... (2.8)}$$

From (2.1), (2.3) and (2.8), we obtain the following relation between the scalar curvature and the mean curvature of M,

$$2\tau = (n+s)^2 |H|^2 - |\sigma|^2 + n(n-1)\frac{c+3s}{4} + 2ns + \frac{3(c-s)}{4}|T|^2, \quad \text{... (2.9)}$$

where $|\sigma|$ denotes the norm of the second fundamental form σ.

3. Chen's Inequality in S-Space Forms

Let M^{n+s} be a submanifold of $\tilde{M}^m(c)$, tangent to the structure vector fields $\xi_1, ..., \xi_s$ and $\pi \subset \mathcal{L}_p$ a plane section at $p \in M$, orthogonal to $\mathcal{M}_p$. Then,

$$F^2(\pi) = g^2(e_1, fe_2) \quad \text{... (3.1)}$$

is a real number in [0, 1] which is independent on the choice of the orthonormal basis $\{e_1, e_2\}$ of π. Denote by τ and $K(\pi)$ the scalar curvature of M and the sectional curvature of M associated with π, respectively.

First, we recall an algebraic lemma from[5] :

Lemma 3.1 — Let $a_1, ..., a_k, c$ be $k+1$ $(k \geq 2)$ real numbers such that :

$$\left(\sum_{i=1}^{k} a_i \right)^2 = (k-1) \left(\sum_{i=1}^{k} a_i^2 + c \right).$$

Then, $2\, a_1\, a_2 \geq c$, with the equality holding if and only if $a_1 + a_2 = a_3 = \ldots = a_k$.

Now, we can prove the following version, for the geometry of *f*-manifolds, of Theorem 3 of [3]:

***Theorem* 3.1** — *Let* $\varphi : M^{n+s} \to \tilde{M}^m(c)$ *be an isometric immersion from a Riemannian (n + s)-dimensional manifold into an S-space-form* $\tilde{M}^m(c)$, *such that the structure vector fields are tangent to M. Then, for any point* $p \in M$ *and any plane section* $\pi \subset \mathcal{L}_p$, *we have :*

$$\tau - K(\pi) \leq \frac{(n+s)^2 (n+s-2)}{2(n+s-1)} |H|^2 + \frac{1}{2}(n+1)(n-2)\frac{c+3s}{4} + ns +$$

$$+ \frac{3}{2} |T|^2 \frac{c-s}{4} - 3F^2(\pi)\frac{c-s}{4}. \qquad \ldots (3.2)$$

The equality in (3.2) holds at $p \in M$ if and only if there exists an orthonormal basis $\{e_1, ..., e_{n+s}\}$ of $T_p M$ and an orthonormal basis $\{e_{n+s+1}, ..., e_m\}$ of $T_p^{\perp} M$ such that $(a)\; e_{n+j} = (\xi_j)_p$, for $j = 1, \ldots, s$, (b) π is spanned by e_1, e_2 and (c) the shape operators $A_r = A_{e_r}$, $r = n+s+1, ..., m$, take the following forms :

$$A_{n+s+1} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \\ & & n+s-2 \end{pmatrix}, \qquad \ldots (3.3)$$

$$A_r = \begin{pmatrix} \sigma_{11}^r & \sigma_{12}^r & 0 \\ \sigma_{12}^r & -\sigma_{11}^r & 0 \\ 0 & 0 & 0 \\ & & n+s-2 \end{pmatrix}, \; r = n+s+2, ..., m. \qquad \ldots (3.4)$$

PROOF : Put

$$\varepsilon = 2\tau - \frac{(n+s)^2 (n+s-2)}{n+s-1} |H|^2 - (n+1)(n-2)\frac{c+3s}{4} -$$

$$- 2ns - \frac{3(c-s)}{4} |T|^2. \qquad \ldots (3.5)$$

Hence, (2.9) and (3.5) imply:

$$(n+s)^2 |H|^2 = (n+s-1)|\sigma|^2 + (n+s-1)\left(\varepsilon - \frac{2(c+3s)}{4}\right). \qquad \text{... (3.6)}$$

Let $\pi \subset \mathcal{L}_p$ be a plane section and choose an orthonormal frame $\{e_1, ..., e_{n+s}\}$ of $T_p M$ and an orthonormal basis $\{e_{n+s+1}, ..., e_m\}$ of $T_p^{\perp} M$ such that $e_{n+j} = \xi_j$, for $j = 1, ..., s$; π is spanned by e_1, e_2 and e_{n+s+1} is in the direction of the mean curvature vector H. Then, (3.6) gives

$$\left(\sum_{i=1}^{n+s} \sigma_{ii}^{n+s+1}\right)^2 = (n+s-1)\left[\sum_{i=1}^{n+s} (\sigma_{ii}^{n+s+1})^2 + \sum_{i \neq j} (\sigma_{ij}^{n+s+1})^2 \right.$$
$$\left. + \sum_{r=n+s+2}^{m} \sum_{i,j} (\sigma_{ij}^r)^2 + \varepsilon - \frac{2(c+3s)}{4}\right],$$

and so, by applying Lemma 3.1, we get :

$$2\sigma_{11}^{n+s+1}\sigma_{22}^{n+s+1} \geq \sum_{i \neq j} \left(\sigma_{ij}^{n+s+1}\right)^2$$
$$+ \sum_{r=n+s+2}^{m} \sum_{i,j} (\sigma_{ij}^r)^2 + \varepsilon - \frac{2(c+3s)}{4}. \qquad \text{... (3.7)}$$

Now, from (2.1) it follows that

$$K(\pi) = \sigma_{11}^{n+s+1}\sigma_{22}^{n+s+1} - \left(\sigma_{12}^{n+s+1}\right)^2 + \sum_{r=n+2+s}^{m} (\sigma_{11}^r \sigma_{22}^r - (\sigma_{12}^r)^2)$$
$$+ \frac{c+3s}{4} + \frac{3(c-s)}{4} g^2(e_1, fe_2). \qquad \text{... (3.8)}$$

Thus, from (3.7) and (3.8), we obtain :

$$K(\pi) \geq \sum_{r=n+s+1}^{m} \sum_{j>2} \left\{(\sigma_{1j}^r)^2 + (\sigma_{2j}^r)^2\right\} + \frac{1}{2} \sum_{i \neq j > 2} (\sigma_{ij}^{n+s+1})^2 +$$
$$+ \frac{1}{2} \sum_{r=n+s+2}^{m} \sum_{i,j>2} (\sigma_{ij}^r)^2 + \frac{1}{2} \sum_{r=n+s+2}^{m} (\sigma_{11}^2 + \sigma_{22}^r)^2$$
$$+ \frac{\varepsilon}{2} + \frac{3(c-s)}{4} g^2(e_1, fe_2) \geq \frac{\varepsilon}{2} + \frac{3(c-s)}{4} F^2(\pi). \qquad \text{... (3.9)}$$

Next, combining (3.1), (3.5) and (3.9), we get (3.2). If the equality (3.2) holds, then the inequalities in (3.7) and (3.9) become equalities. So, we have :

$$\sigma_{1j}^{n+s+1} = \sigma_{2j}^{n+s+1} = \sigma_{ij}^{n+s+1} = 0, \ i \neq j > 2;$$

$$\sigma_{1j}^{r} = \sigma_{2j}^{r} = \sigma_{ij}^{r} = 0,\ r = n+s+2, \dots, m;\ i, j = 3, \dots, n+s;$$

$$\sigma_{11}^{n+s+2} + \sigma_{22}^{n+s+2} = \dots = \sigma_{11}^{m} + \sigma_{22}^{m} = 0.$$

Moreover, we may choose e_1, e_2 such that $\sigma_{12}^{n+s+1} = 0$ and, by applying Lemma 3.1 and (2.2), we also get :

$$\sigma_{11}^{n+s+1} + \sigma_{22}^{n+s+1} = \sigma_{33}^{n+s+1} = \dots = \sigma_{n+s,n+s}^{n+s+1} = 0$$

Consequently, with respect to the chosen orthonormal basis $\{e_1, \dots, e_m\}$, the shape operators of M take the forms (3.3) and (3.4).

The converse follows from a direct calculation. □

Now, we can define :

$$(\inf_{\mathcal{L}} K)(p) = \inf \ \{K(\pi) \ : \ \text{plane sections } \pi \subset \mathcal{L}_p\}.$$

Then, $\inf_{\mathcal{L}} K$ is a well-defined function on M. Let $\delta_M^{\mathcal{L}}$ denote the difference between the scalar curvature and $\inf_{\mathcal{L}} K$, that is :

$$\delta_M^{\mathcal{L}}(p) = \tau(p) - \inf_{\mathcal{L}} K(p).$$

It is clear that $\delta_M^{\mathcal{L}} \leq \delta_M$. Then, if $c = s$, from (3.2) we get directly the following result :

Corollary 3.1 — Let $\varphi : M^{n+s} \to \tilde{M}^m(s)$ be an isometric immersion from a Riemannian $(n + s)$-manifold into an S-space-form with constant f-sectional curvature s, such that $\xi_1, \dots, \xi_s \in TM$. Then, we have:

$$\delta_M^{\mathcal{L}} \leq \frac{(n+s)^2 (n+s-2)}{2(n+s-1)} |H|^2 + \frac{(n-1)(n+2)}{2} s. \quad \dots (3.10)$$

Note that, if $s = 1$, (3.10) becomes the inequality obtained by Carriazo in[2] for a Sasakian-space-form of constant ϕ-sectional curvature 1. This seems to point out that (3.2)may be a natural version for the geometry of f-manifolds of Chen's inequality (1.1) (it is necessary to recall that the non-existence of S-manifolds of constant sectional curvature with $s > 1$ is a well-known fact[1]).

Corollary 3.2 — If equality holds in (3.2) at any point $p \in M$, then φ is an invariant immersion.

PROOF : Suppose the equality in (3.2). Since, from (3.3), $A_{n+s+1} e_{n+j} = 0$, for any $j = 1, \dots, s$, then

$$g(A_{n+s+1} e_{n+j}, e_i) = g(-Ne_i, e_{n+s+1}) = 0,$$

for any $i = 1, \dots, n$, where we have used (2.2). Furthermore, from (3.4),

$$g(A_r e_{n+j}, e_i) = g(-N e_i, e_r) = 0,$$

for any $r = n+s+2, \ldots, m$ and any $i = 1, \ldots, n$. Consequently, $N e_i = 0$, $i = 1, \ldots, n$, that is, $N \equiv 0$ and φ is an invariant immersion. □

Next, we are going to modify (3.2) in order to consider non-invariant submanifolds (for example, proper slant submanifolds) satisfying a similar equality. We can prove the following theorem:

***Theorem* 3.2** — *Let $\varphi : M^{n+s} \to \tilde{M}^m(c)$ be an isometric immersion from a Riemannian $(n+s)$-manifold into an S-space-form $\tilde{M}^m(c)$, tangent to the structure vector fields $\xi_1, \ldots, \xi_s$. Then, for any point $p \in M$ and any plane section $\pi \subset \mathcal{L}_p$, we have :*

$$\tau - K(\pi) \le \frac{(n+s)^2 (n+s-2)}{2(n+s-1)} |H|^2 + \frac{1}{2}(n+1)(n-2)\frac{c+3s}{4}$$

$$+ ns + \frac{3}{2}|T|^2 \frac{c-s}{4} - 3F^2(\pi)\frac{c-s}{4} - s|N|^2. \qquad \ldots (3.11)$$

Equality in (3.11) holds at $p \in M$ if and only if there exist an orthonormal basis $\{e_1, \ldots, e_{n+s}\}$ of T_pM and an orthonormal basis $\{e_{n+s+1}, \ldots, e_m\}$ of $T_p^{\perp} M$ such that (a) $e_{n+j} = (\xi_j)_p$, for any $j = 1, \ldots, s$, (b) π is spanned by e_1, e_2 and (c) the shape operators $A_r = A_{e_r}$, $r = n+s+1, \ldots, m$, take the forms

$$A_{n+s+1} = \begin{pmatrix} a & 0 & 0 & \cdots & \mu^{n+s+1}_{1,n+1} & \cdots & \mu^{n+s+1}_{1,n+s} \\ 0 & -a & 0 & \cdots & \mu^{n+s+1}_{2,n+1} & \cdots & \mu^{n+s+1}_{2,n+s} \\ 0 & 0 & 0 & \cdots & \vdots & & \vdots \\ \vdots & \vdots & 0 & \ddots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \mu^{n+s+1}_{n,n+1} & \cdots & \mu^{n+s+1}_{n,n+s} \\ \mu^{n+s+1}_{1,n+1} & \cdots & & \mu^{n+s+1}_{n,n+1} & 0 & \cdots & 0 \\ \vdots & & & \vdots & 0 & \ddots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ \mu^{n+s+1}_{1,n+s} & \cdots & & \mu^{n+s+1}_{n,n+s} & 0 & \cdots & 0 \end{pmatrix} \qquad \ldots (3.12)$$

and, for any $r = n+s+2, \ldots, m$,

$$A_r = \begin{pmatrix} \sigma^r_{11} & \sigma^r_{12} & 0 & \cdots & \mu^{n+s+1}_{1,n+1} & \cdots & \mu^{n+s+1}_{1,n+s} \\ \sigma^r_{12} & -\sigma^r_{11} & 0 & \cdots & \mu^{n+s+1}_{2,n+1} & \cdots & \mu^{n+s+1}_{2,n+s} \\ 0 & 0 & 0 & \cdots & \vdots & & \vdots \\ \vdots & \vdots & 0 & \ddots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \mu^{n+s+1}_{n,n+1} & \cdots & \mu^{n+s+1}_{n,n+s} \\ \mu^{n+s+1}_{1,n+1} & \cdots & & \mu^{n+s+1}_{n,n+1} & 0 & \cdots & 0 \\ \vdots & & & \vdots & 0 & \ddots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ \mu^{n+s+1}_{1,n+s} & \cdots & & \mu^{n+s+1}_{n,n+s} & 0 & \vdots & 0 \end{pmatrix}, \qquad \ldots (3.13)$$

where $\mu_i^r = g(fe_i, e_r),\ i = 1, \ldots, n,\ r = n+s+1, \ldots, m.$

PROOF : Following the first steps of the proof of Theorem 3.1, we state equations (3.5)-(3.9). Then, we can write inequality (3.9) as :

$$K(\pi) \geq \sum_{r=n+s+1}^{m} \sum_{j>2} \left\{ (\sigma_{1j}^r)^2 + (\sigma_{2j}^r)^2 \right\} + \frac{1}{2} \sum_{i \neq j > 2}^{n} (\sigma_{ij}^{n+s+1})^2$$

$$+ \frac{1}{2} \sum_{r=n+s+2}^{m} \sum_{i,j=3}^{n} (\sigma_{ij}^r)^2 + \frac{1}{2} \sum_{r=n+s+2}^{m} (\sigma_{11}^r + \sigma_{22}^r)^2 + \frac{\varepsilon}{2} + \frac{3(c-s)}{4} g^2(e_1, fe_2)$$

$$+ \sum_{r=n+s+1}^{m} \sum_{i=1}^{n} \sum_{j=n+1}^{n+s} (\sigma_{ij}^r)^2 \geq \frac{\varepsilon}{2} + \frac{3(c-s)}{4} g^2(e_1, fe_2)$$

$$+ \sum_{r=n+s+1}^{m} \sum_{i=1}^{n} \sum_{j=n+1}^{n+s} (\sigma_{ij}^r)^2. \qquad \ldots (3.14)$$

But, from (2.2) and (2.3) we find :

$$\sum_{r=n+s+1}^{m} \sum_{j=n+1}^{n+s} \sum_{j=n+1}^{n+s} (\sigma_{ij}^r)^2 = s\,|N|^2. \qquad \ldots (3.15)$$

Hence, by combining (3.1), (3.5), (3.14) and (3.15), we get (3.11). If the equality holds, then the inequalities in (3.7) and (3.14) become equalities. From this fact, (2.2) and Lemma 3.1, we obtain :

$$\sigma_{1j}^{n+s+1} = \sigma_{2j}^{n+s+1} = \sigma_{ij}^{n+s+1} = 0,\ 2 < i \neq j \leq n;$$

$$\sigma_{1j}^r = \sigma_{2j}^r = \sigma_{ij}^r = 0,\ r = n+s+2, \ldots, m;\ i, = 3, \ldots, n+s;$$

$$\sigma_{11}^{n+s+2} + \sigma_{22}^{n+s+2} = \ldots = \sigma_{11}^m + \sigma_{22}^m = 0;$$

$$\sigma_{11}^{n+s+1} + \sigma_{22}^{n+s+1} = \sigma_{33}^{n+s+1} = \ldots = \sigma_{n+s,n+s}^{n+s+1} = 0$$

So, if we also choose e_1, e_2 such that $\sigma_{12}^{n+s+1} = 0$, then we get (3.12) and (3.13). As in the proof of Theorem 3.1, the converse can be verified by a straight-forward computation. □

It is clear that (3.2) follows from (3.11), since $|N|^2 \geq 0$. On the other hand, it is also obvious that, if φ is an anti-invariant immersion, then $|T|^2 = 0, |N|^2 = n$ and $F^2(\pi) = 0$, for any plane section in $\mathcal{L}$. Consequently, from (3.11) we obtain :

Corollary 3.3 — Let M^{n+s} be an anti-invariant submanifold of an S-space-form $\tilde{M}^m(c)$, tangent to the structure vector fields $\xi_1, ..., \xi_s$. Then, we have :

$$\delta_M^{\mathcal{L}} \leq \frac{(n+s)^2(n+s-2)}{2(n+s-1)} |H|^2 + \frac{1}{2}(n+1)(n-2)\frac{c+3s}{4}.$$

By using Theorem 3.2 and following the same steps as in the proofs of Theorem 3 and 4 of[3], respectively, we can find some general pinching results for $\delta_M^{\mathcal{L}}$ if either $c > s$ or $c < s$.

***Theorem* 3.3** — *Let $\varphi : M^{n+s} \to \tilde{M}^m(c)$ be an isometric immersion from a Riemannian $(n + s)$-dimensional manifold $(n > 2)$ into an S-space-form $\tilde{M}^m(c)$, with $c > s$, tangent to the structure vector fields $\xi_1, ..., \xi_s$. Then :*

$$\delta_M^{\mathcal{L}} \leq \frac{(n+s)^2(n+s-2)}{2(n+s-1)} |H|^2 + \frac{1}{2}(n^2+2n-2)\frac{c+3s}{4} - \frac{ns}{2}.$$

Equality holds identically if and only if n is even and M^{n+s} is immersed as an invariant, totally geodesic submanifold of $\tilde{M}^m(c)$.

***Theorem* 3.4** — Let $\varphi : M^{n+s} \to \tilde{M}^m(c)$ *be an isometric immersion from a Riemannian $(n + s)$-dimensional manifold $(n > 2)$ into an S-space-form $\tilde{M}^m(c)$, with $c < s$, tangent to the structure vector fields $\xi_1, ..., \xi_s$. Then :*

$$\delta_M^{\mathcal{L}} \leq \frac{(n+s)^2(n+s-2)}{2(n+s-1)} |H|^2 + \frac{1}{2}(n+1)(n-2)\frac{c+3s}{4} + ns - |N|^2.$$

Equality holds at a point $p \in M$ if and only if there exist an orthonormal basis $\{e_1, ..., e_{n+s}\}$ of T_pM and an orthonormal basis $\{e_{n+s+1}, ..., e_m\}$ of $T_p^{\perp}M$ such that (a) $e_{n+j} = (\xi_j)_p$, for any $j = 1, ..., s$, (b) the subspace spanned by $e_3, ..., e_{n+s}$ is anti-invariant, (c) $K(e_1 \wedge e_2) = \inf_{\mathcal{L}} K$ at p and (d) the shape operators $A_r = A_{e_r}$, $r = n+s+1, ..., m$, take the forms (3.12) and (3.13).

4. Some Applications to Slant Immersions

We are going to study inequality (3.11) when M is a slant submanifold. First, we observe that, if M^{n+s} is a θ-slant submanifold of a metric f-manifold, then, from (2.3), (2.5) and (2.6) :

$$|T|^2 = n\cos^2\theta, \ |N|^2 = n\sin^2\theta. \qquad \text{... (4.1)}$$

Now, by using (3.11) and 4.1), we obtain :

***Theorem* 4.1** — *Let $\varphi : M^{n+s} \to \tilde{M}^m(c)$ be a θ-slant immersion of a Riemannian $(n + s)$-dimensional manifold into an S-space-form $\tilde{M}^m(c)$. Then, for any point $p \in M$ and any plane* *section* $\pi \subset \mathcal{L}_p$, *we have* :

$$\tau - K(\pi) \leq \frac{(n+s)^2 (n+s-2)}{2(n+s-1)} |H|^2 + \frac{1}{2}(n+1)(n-2)\frac{c+3s}{4} +$$

$$+ ns\cos^2\theta + \frac{3}{4}(c-s)\left(\frac{n}{2}\cos^2\theta - F^2(\pi)\right). \qquad \text{... (4.2)}$$

Hans-Uber has proved in[6] that there are no proper slant submanifolds of S-manifolds of dimension lower than $2 + s$. For $(2 + s)$-dimensional slant submanifolds, we can state the following result :

Corollary 4.1 — Let M^{2+s} be a θ-slant submanifold of an S-space-form $\tilde{M}^m(c)$. Then, in the above conditions :

$$\delta_M^{\mathcal{L}} \leq \frac{(2+s)^2 s}{2(s+1)} |H|^2 + 2s\cos^2\theta. \qquad \text{... (4.3)}$$

Equality holds if and only if M is minimal.

PROOF : Since $n = 2$, then it is clear that

$$\delta_M^{\mathcal{L}} = \tau - K(\mathcal{L}) \qquad \text{... (4.4)}$$

and $F^2(\mathcal{L}) = \cos^2\theta$. Thus, (4.3) follows directly from (4.2). On the other hand, it is easy to show that

$$\tau - K(\mathcal{L}) = 2s\cos^2\theta, \qquad \text{... (4.5)}$$

due to M being a submanifold of dimension $2 + s$ of an S-manifold. Hence, (4.4) and (4.5) imply the condition for the equality case in (4.3). □

Note that, if we consider inequality (3.2) as our starting point, then, by following the same steps as in Theorem 4.1 and Corollary 4.1, we obtain, for slant $(2 + s)$-dimensional submanifolds, the inequality

$$\delta_M^{\mathcal{L}} \leq \frac{(2+s)^2 s}{2(s+1)} + |H|^2 + 2s,$$

with equality holding if and only if the submanifold is invariant. So, the converse of Corollary 3.2 holds for $(2 + s)$-dimensional slant submanifolds. These results must be compared with those ones obtained by Carriazo[2] in the case $s = 1$.

Finally, we can restrict our study to some special plane sections, orthogonal to the structure vector fields. Let M^{n+s} be a submanifold of an S-space-form $\tilde{M}^m(c)$, such that $\xi_1, \dots, \xi_s \in TM$. Given a point $p \in M$, we say that a plane section $\pi \subset T_pM$ is a *T-section* if there exists a tangent vector $X \in \mathcal{L}_p$, such that π is spanned by X and TX. Now, for each point $p \in M$, we can define $(\inf_T K)(p) = \inf\{K(\pi) : T\text{-section } \pi\}$ and $\delta_M^T(p) = \tau(p) - (\inf_T) K(p)$. Since every T-section is orthogonal to the structure vector fields, it is clear that $\overset{\mathcal{T}}{\delta}_M \leq \overset{\mathcal{L}}{\delta}_M$. In the case of slant submanifolds, we can get the following inequality for $\overset{\mathcal{T}}{\delta}_M$:

***Theorem* 4.2** — *Let* $\varphi : M^{n+s} \to \widetilde{M}^m(c)$ *be a non-anti-invariant θ-slant immersion of a Riemannian $(n + s)$-dimensional manifold into an S-space-form $\widetilde{M}^m(c)$. Then :*

$$\overset{\mathcal{T}}{\delta}_M \leq \frac{(n+s)^2(n+s-2)}{2(n+s-1)} |H|^2 + \frac{1}{2}(n+1)(n-2)\frac{c+3s}{4} + $$

$$+ \; ns\cos^2\theta + \frac{1}{2}(n-2)\frac{3(c-s)}{4}\cos^2\theta.$$

PROOF : Given a $\mathcal{T}$-section π, we can choose two tangent vectors e_1, e_2 such that π is spanned by e_1 and e_2, being $e_2 = \sec\theta\, Te_1$. Then, (2.4) implies $F^2(\pi) = \cos^2\theta$. The proof ends by applying (4.2). □

Observe that, if $n = 2$, then $\overset{\mathcal{T}}{\delta}_M = \overset{\mathcal{L}}{\delta}_M$ and so, Corollary 4.1 also follows from Theorem 4.2.

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