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WEYL'S THEOREM FOR OPERATORS WITH TACKED SPECTRA

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In this paper we study Weyl's theorem and some related materials for operators with tacked spectra which are introduced by K. B. Laursen and M. Mbekhta.

Key Words : Local Spectral Theory; Weyl's Theorem; Operator With Tacked Spectrum

1. INTRODUCTION

Let X be an infinite dimensional complex Banach space and let $B(X)$ be the set of all bounded linear operators acting on a Banach space X. Recall[6] that an operator $T \in B(X)$ is called *semi-Fredholm* if the range of T, denoted by $R(T)$, is closed and either the kernel of T, denoted by $N(T)$ or $X/R(T)$ is finite dimensional. Also, an operator $T \in B(X)$ is called *Fredholm* if $R(T)$ is closed and both $N(T)$ and $X/R(T)$ are finite dimensional. If T is semi-Fredholm, then the index of T is defined by

$$\text{ind}\,(T) = \dim N(T) - \dim X/R(T).$$

Denote the $\Phi(X)$ and $\Phi_0(X)$ sets of all Fredholm and all Fredholm operators with index zero respectively. For an operator $T \in B(X)$, we shall denote $\sigma(T), \sigma_p(T), \pi_0(T)$, iso$\sigma(T)$, and $\pi_{00}(T)$ by the spectrum of T, the set of all eigenvalues of T, the set of all eigenvalues of finite multiplicity of T, the set of all isolated points of $\sigma(T)$, and the set of all isolated points of $\sigma(T)$, and the set of all isolated eigenvalues of finite multiplicity of T, respectively. Let $\mathbf{C}$ denote the

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complex plain, and then the following definitions are well-known: the Fredholm spectrum of T is $\sigma_e(T) = \{\lambda \in \mathbf{C}; T-\lambda \notin \Phi(X)\}$, the Weyl spectrum of T is $\sigma_w(T) = \{\lambda \in \mathbf{C} : T-\lambda \notin \Phi_0(X)\}$ and the Browder spectrum of T is $\sigma_b(T) = \bigcap \{\sigma(T+K) : TK = KT, K \in K(X)\}$.

Following[2], we say that *Weyl's theorem holds for T* if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T).$$

We say that $T \in B(X)$ has the *single valued extension property* (say, SVEP) if, for an open set $V \subset \mathbf{C}, f = 0$ is the only analytic function $f : V \to X$ satsfying $(T-\lambda)f(\lambda) = 0$. For a closed set $F \subset \mathbf{C}$ we define a spectral manifolds $X_T(F)$ as follows :

$$X_T(F) = \{x \in X : \text{there exists an analytic } X\text{-valued function}$$

$$f : C \setminus F \to X \text{ such that } (T-\lambda)f(\lambda) = x\}.$$

If T has the SVEP, then we have $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$. Laursen and Mbekhta[10] defined an interesting class, denoted by $Z_G(X)$, of operators by

$$Z_G(X) := \{T \in B(X) : X_T(\mathbf{C} \setminus G) = (0)\} \text{ for an arbitrary set } G \subseteq \mathbf{C}.$$

For $T \in Z_G(X)$, it is well known ([10, Proposition 1]) that every clopen subset of $\sigma(T)$ meets G. So every $T \in Z_G(X)$ is named as operators with tacked (to G) spectra. If $G = \mathbf{C}$, then $Z_C = \{T \in B(X) : T$ has the SVEP$\}$. For a Hilbert space H, this is a generalization of the class $\mathcal{KA}_0(H) = \{T \in B(H) : K(T) = \{0\}\}$ defined in [12] because $\mathcal{KA}_0(H)$ is actually identical with $Z_{\{0\}}(H)$. As an important subset of $Z_G(X)$ to impose our concern we consider

$$z_G(X) := \{T \in B(X) : T \in Z_G(X), T^* \in Z_G(X^*)\}.$$

In the context of a Hilbert space H, this notion is defined in a different way as follows:

$$Z_G(H) := \{T \in B(H) : T \in Z_G(H), T^* \in Z_{G^*}(H)\},$$

where G^* denotes the set of all complex conjugate numbers in G. It is well known that the class $z_G(H)$ contains the class $\mathcal{BT}_G$ of bitriangular operators with diagonal contained in G and the class $\mathcal{QN}$ of quasinilpotent operators, and is contained in the class $\mathcal{BQT}$ of biquasitriangular operators (cf. [10, p. 25]). In this paper we study Weyl's theorem for the class of operators with tacked spectra. In section 2, we give necessary and sufficient conditions that Weyl's theorem holds for $T \in Z_G(X)$, in section 3 it is shown that under mild conditions on the isolated points asymptotic quasisimilarity preserves Weyl's theorem, and in section 4 specializing to a Hilbert space we study spectral continuity for an operator $T \in z_G(H)$ with int $(G \cap \sigma(T)) = \phi$.

2. WEYL'S THEOREM

For an operator $T \in B(X)$, let $\mathcal{H}(\sigma(T))$ denote the set of all complex-valued functions which are defined and holomorphic on some neighbourhood of $\sigma(T)$ and let $\sigma_0(T)$ be the isolated points of $\sigma(T)$ for which the corresponding Riesz projection has finite dimensional range.

***Theorem* 2.1** — *Let $T \in Z_G(X)$. Then the following conditions are equivalent :*

1. Weyl's theorem holds for T.

2. $R(T-\lambda)$ is closed for all $\lambda \in \pi_{00}(T)$.

3. $X_T(\{\lambda\})$ is finite dimensional for all $\lambda \in \pi_{00}(T)$.

PROOF : It is clear from the definition of local spectra that if $G_1 \subseteq G_2$ then $Z_{G_1} \subseteq Z_{G_2}$. Thus we shall prove theorem when $G = \mathbf{C}$.

$1 \Rightarrow 2$: Let $\lambda \in \pi_{00}(T)$. Since T obeys Weyl's theorem, $\lambda \in \sigma(T) \setminus \sigma_\omega(T)$. Therefore $R(T-\lambda)$ is closed.

$2 \Rightarrow 3$: Let $\lambda \in \pi_{00}(T)$. Since $R(T-\lambda)$ is closed, by the punctured neighborhood theorem we have that $\lambda \in \sigma(T) \setminus \sigma_b(T)$, and so $\lambda \in \sigma_0(T)$. Thus if we consider the spectral Riesz projection $P \in B(X)$ corresponding to λ such that

$$P = \frac{1}{2\pi i} \int_{\partial D_\lambda} (zI - T)^{-1}\, dz,$$

where D_λ is an open disk of center λ which contains no other points of $\sigma(T)$, then we have

$$R(P) = \left\{ x \in X : \| (T-\lambda)^n x \|^{\frac{1}{n}} \to 0 \right\} \text{ is finite dimensional.}$$

Since $T \in Z_G(T)$ has the SVEP, we have

$$\left\{ x \in X : \| (T-\lambda)^n x \|^{\frac{1}{n}} \to 0 \right\} = X_T(\{\lambda\}).$$

Hence $X_T(\{\lambda\})$ is finite dimensional.

$3 \Rightarrow 1$: Suppose $\lambda \in \sigma(T) \setminus \sigma_\omega(T)$. $T-\lambda$ is Weyl and not invertible. We first show that $\lambda \in \partial\sigma(T)$. Assume to the contrary that $\lambda \in \mathrm{int}\,\sigma(T)$. Then there exists a neighborhood N_λ of λ such that $\dim N(T-\mu) > 0$ for all $\mu \in N_\lambda$. It follows from [5, Theorem 10] that T doesn't have the SVEP. This is contrary to the assumption that $T \in Z_G(X)$. Therefore $\lambda \in \partial\sigma(T)$. It follows from the punctured neighborhood theorem that $\lambda \in iso\,\sigma(T)$. Hence $\lambda \in \pi_{00}(T)$. Conversely, suppose $\lambda \in \pi_{00}(T)$. Then $X_T(\{\lambda\})$ is finite dimensional, and so $T-\lambda$ is in $\Phi(X)$ by [11, Lemma 1]. Hence by the index continuity we have that $\lambda \in \sigma(T) \setminus \sigma_\omega(T)$. □

Now, if we take G as a 'thin' subset of $\mathbf{C}$ we get the following result :

Corollary 2.1 — Let $T \in z_G(X)$ with int $(G \cap \sigma(T)) = \phi$. Then Weyl's theorem holds for T and the spectral mapping theorem holds for $\sigma_w(T)$.

PROOF : Let an operator $T \in B(X)$ obey conditions of theorem. Then by [10, Theorem 1] we have that $\sigma(T) = \sigma_e(T) \cup \sigma_0(T)$ disjointly. Now by the simple consideration we have

$$\sigma_e(T) = \sigma_w(T) = \sigma_b(T).$$

Assume that $\lambda \in \pi_{00}(T)$. Then $\lambda \in iso\, \sigma(T)$ and by [10, Proposition 1. (c)] follows that $\lambda \in G$. Now by the proof of [10, Theorem 1] we have that $T - \lambda$ is a semi-Fredholm operator, and so $R(T-\lambda)$ is closed for all $\lambda \in \pi_{00}(T)$. Hence by Theorem 2.1 Weyl's theorem holds for T. Next, to show that the spectral mapping theorem holds for $\sigma_w(T)$, let $f \in \mathcal{H}(\sigma(T))$. Since by [1, Theorem 3.2] follows that $\sigma_w(f(T)) \subset f(\sigma_w(T))$, we have to show only opposite inclusion. Since for operator T holds that $\sigma_e(T) = \sigma_w(T)$ we have

$$f(\sigma_w(T)) = f(\sigma_e(T)) = \sigma_e(f(T)) \subset \sigma_w(f(T)),$$

i.e. the spectral mapping theorem holds for T. □

3. Asymptotic Quasisimilarity

For given operators $T \in B(X), S \in B(Y)$, and $A \in B(X, Y)$, we consider the corresponding commutator $C(S, T) : B(X, Y) \to B(X, Y)$ defined by

$$C(S, T)(A) = SA - AT.$$

Then for every $n \in N$ and all $A \in B(X, Y)$ we have (see [9])

$$C(S, T)^n(A) = C(S, T)^{n-1}(SA - AT) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k S^{n-k} A T^k.$$

We say that an operator $A \in B(X, Y)$ intertwines S and T asymptotically if

$$\| C(S, T)^n (A) \|^{1/n} \to 0 \text{ as } n \to \infty.$$

An operator A is a *quasiaffinity* if A is injective and has dense range. Also, we shall call S and T *asymptotically quasisimilar*, denoted by $T \sim^a S$, if there are quasiaffinites operators $A \in B(X, Y)$ and $B \in B(Y, X)$ such that

$$\| C(S, T)^n (A) \|^{1/n} \to 0 \text{ and } \| C(T, S)^n (B) \|^{1/n} \to 0 \text{ as } n \to \infty.$$

For details, the reader is refereed to[9]. Now, we are ready to write the following results:

***Theorem* 3.1** — *Let $T \in Z_G(X)$ and $S \in Z_G(Y)$, and $iso\sigma(T) = iso\sigma(S)$. If $T \sim^a S$, then Weyl's theorem holds for T if and only if Weyl's theorem holds for S.*

PROOF : Assume Weyl's theorem holds for T and let $\lambda \in \pi_{00}(S)$. Since $T \sim^a S$, it is easy to see that $\pi_0(T) = \pi_0(S)$. Since $iso\sigma_0(T) = iso\sigma_0(S)$ by the hypothesis, we have $\pi_{00}(T) = \pi_{00}(S)$. Since $\lambda \in \pi_{00}(S) = \pi_{00}(T) = \sigma(T) \setminus \omega(T)$, by Theorem 2.1, we have $X_T(\{\lambda\})$ is finite dimensional. Since $T \sim^a S$ there exists a quasiaffinity $B \in B(Y, X)$ such that

$$\| C(T, S)^n (B) \|^{1/n} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

and then [9, Proposition 2.2] implies that

$$BY_s(\{\lambda\}) \subseteq X_T(\{\lambda\}).$$

So $X_S(\{\lambda\})$ is also finite dimensional for each $\lambda \in \pi_0(S)$. Thus by Theorem 2.1 we have that Weyl's theorem holds for $S \in Z_G(Y)$. Hence the proof completes by symmetry. □

For the next results we need some basic notations from spectral theory. Recall ([9], [10]) that an operator $t \in B(X)$ has weak 2-spectral decomposition property (weak 2-SDP) if for every open covering $\{U_1, U_2\}$ of the complex plane **C** there are T-invariant closed linear subspaces Y_1 and Y_2 of X such that $Y_1 + Y_2$ be dense in X and $\sigma(T \mid Y_i) \subset U_i$, $i = 1, 2$. Also, an operator $T \in B(X)$ has *decomposition property* (δ) if for arbitrary open covering $\{U_1, U_2\}$ of **C** every $x \in X$ has a decomposition $x = u_1 + u_2$ where is satisfy $u_i = (T - \lambda) f_k(\lambda)$ for all $\lambda \in \mathbf{C} \setminus \overline{U}_i$ and some analytic function $f_i : \mathbf{C} \setminus \overline{U}_i$ for $i = 1, 2$. It is known that operators with weak 2-SDP need not have property (δ) and also there are operators with property (δ) which do not have the weak 2-SDP. Also, we say that an operator $T \in B(X)$ has *Dunford's property (C)* if the spaces $X_T(F)$ are closed whenever $F \subseteq \mathbf{C}$ is closed.

Corollary 3.1 — Let $T \in B(X)$ and $S \in B(Y)$ have weak 2-SDP or decomposition property (δ) or Dunford's property (C). If $T \sim^a S$, then Weyl's theorem holds for T if and only if Weyl's theorem holds for S.

PROOF : Let $T \in B(X)$ and $S \in B(Y)$ obey the hypothesis. Then if $T \sim^a S$, we can see that $\sigma(T) = \sigma(S)$ from [9, Corollary 4.5]. Thus the proof immediately follows by Theorem 3.1. □

Also, taking G as a thin subset of **C** we have the following result :

Corollary 3.2 — Let $T \in z_G(X)$ with int $(G \cap \sigma(T)) = \phi$ and $S \in B(Y)$. If $T \sim^a S$, then Weyl's theorem holds for S.

PROOF : First, we show that $S \in Z_G(X)$ with int $(G \cap \sigma(S)) = \phi$. By [10, Proposition 3] we have that $S \in Z_G(Y)$ and by [9, Lemma 3.4] we have that $S^* \in Z_G(Y^*)$. Hence $S \in z_G(Y)$. Also, suppose now that int $(\sigma(S) \cap G) \neq \phi$. Then there exist $\lambda \in \sigma(S) \cap G$ and $\varepsilon > 0$ such that

$B(\lambda, \varepsilon) \subset \sigma(S) \cap G$. By [10, p.21] we can assume that $G \subset \sigma(T)$ and we have a contradiction. So $\text{int}(\sigma(S) \cap G) \neq \phi$ and hence the proof immediately follows from Corollary 2.1. □

4. THE SPECTRAL CONTINUITY

In this secion the underlying space is assumed to be a Hilbert space H and then we study the continuity of $\sigma(T)$ at an oprator $T \in z_G(H)$ with int $(G \cap \sigma(T)) = \phi$. Conway and Morrel[3] gave necessary and sufficient conditions that a biquasitriangular operator be a point of spectral continuity as follows :

Lemma 4.1 — ([3, Corollary 3.3]) — Let $T \in B(H)$ be biquasitriangular. Then σ is continuous at T if and only if for each $\lambda \in \sigma(T)$ and $\varepsilon > 0$ the ε-neighborhood of λ contains a component of $\sigma_e(T) \cup \pi_{00}(T)$.

***Theorem* 4.1** — *Let $T \in z_G(H)$ with int $(G \cap \sigma(T)) = \phi$. Then σ is continuous at T if and only if for each $\lambda \in \sigma(T)$ and $\varepsilon > 0$ the ε-neighborhood of λ contains a component of $\sigma(T)$. Furthermore, the following conditions are equivalent :*

1. *σ is continuous at T;*
2. *σ_b is continuous at T;*
3. *σ_w is continuous at T;*
4. *σ_e is continuous at T.*

PROOF : From [10, Theorem 1] and a notice in proof of Corollary 2.1 we can see that

$$\sigma(T) \setminus \sigma_e(T) = \sigma_0(T) = \pi_{00}(T).$$

Also, since $z_G(H) \subseteq \mathcal{BQT}$ by [8, Theorem 2] the first statement of this theorem follows by Lemma 4.1. Next, let $T \in z_G(H)$ with int $(G \cap \sigma(T)) = \phi$. Since by Corollary 2.1 Weyl's theorem holds for T, by [4, Theorem 2.2] follows that σ is continuous at T if and only if σ_w is continuous. Also, by [4, Theorem 2.3] continuity of σ_w and σ_b are equivalent at operator T. Since by [10, Theorem 1] T is a biquasitriangular operator, equivalence of continuity of σ_w and σ_e follows by [3, p. 195]. □

***Theorem* 4.2** — *Let a quasiaffinity $A \in B(H)$ intertwine $S \in B(H)$ and $T \in B(H)$ asymptotically and let $T \in Z_G(H)$ with int $(G \cap \sigma(T)) = \phi$. If T has either property (δ) or weak 2-SDP and some of $\sigma, \sigma_b, \sigma_w$ or σ_e is continuous at S, then everyone of just mentioned spectra is continuous at T.*

PROOF : Let T and S obey conditions of the theorem. Then by the proof of Corollary 2.1 we have that int $(G \cap \sigma(S)) = \phi$ and by [10, Theorem 1] follows that S is biquasitriangular operator and $\sigma(S) = \sigma_e(S) \cup \sigma_0(S)$. Now the assumption that one of spectrum σ, σ_b, σ_w or σ_e is continuous at S together with Theorem 4.1 implies that all of those spectra is continuous at S. In our case is important that spectrum σ is continuous at S because by [8, Theorem 3] we have that $\sigma(S)$ is the closure of its isolated points. Now since quasiaffinity $A \in B(H)$ intertwines $S \in B(H)$ and $T \in B(H)$ we have that $\sigma(S) \subset \sigma(T)$ and by [9, Theorem 4.1] $\sigma(T) \subset \sigma(S)$. Hence $\sigma(S) = \sigma(T)$ and $\sigma(T)$ is the closure of its isolated points. Since T is biquasitriangular by Lemma 4.1 follows that σ is continuous at T and by Theorem 3.1 we have that σ_b, σ_w and σ_e are continuous at T. □

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