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SOME RESULTS ON *H*-LINE GRAPHS

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In this paper, we discuss the behaviour of the sequence $\{HL^k(G)\}$ for a unicyclic graph G when $H \cong P_n$ and for a wheel $G \cong W_n$ when $H \cong G$. In addition, we derive a formula to obtain the number of edges in $HL(G)$ and find H-line graphs of n-pan graphs which are interesting.

Key Words : Unicyclic Graphs; *H*-Line Graphs; Iterated *H*-Line Graphs; Convergence

1. INTRODUCTION

Given a graph G, the line graph $L(G)$ is defined to be the graph whose vertices are the edges of G: two vertices are adjacent if they share an endvertex as edges in G, that is, a P_3 in G gives rise to an edge in $L(G)$.

Chartrand *et al.*[1] generalized this construction by defining the H-line graph $HL(G)$ for a connected graph H on at least three vertices. The vertices of the graph $HL(G)$ correspond to the edges of G. Two vertices are adjacent if the corresponding edges in G share an endvertex and there is a copy of H containing both the edges, that is, a P_3 in a copy of H gives rise to an edge in $HL(G)$. Clearly, when $H = P_3$ the H-line graph $HL(G)$ is the standard line graph $L(G)$ and for any connected graph G on at least three vertices $GL(G) = L(G)$.

The kth iterated H-line graph of G is defined by $HL^k(G) = HL\{HL^{k-1}(G)\}$, where $HL^0(G) = G$ and $HL^{k-1}(G)$ is assumed to exist. If there is some integer N such that $HL^{k+1}(G) = HL^k(G)$ whenever $k \geq N$, then the sequence $\{HL^k(G)\}$ is said to converge, and $HL^N(G)$ is called the limit H-line graph of G. If the sequence $\{HL^k(G)\}$ is finite, then it is said to terminate. Chartrand *et al.*[1] discussed the behaviour of the sequence $\{HL^k(G)\}$ when $H = P_4$ and $H = P_5$.

2. EARLIER RESULTS ON H-LINE GRAPHS

The following are some of the results we have used in our work.

***Theorem* 1** [Chartrand et al.] — *Let H be a connected graph of order at least 3. If* $\{HL^k(G)\}$ *converges to a connected limit graph, then H is a cycle or a path.*

***Theorem* 2** [Chartrand et al.] — *Let* $H \cong P_4$. *Then the sequence* $\{HL^k(G)\}$ *converges if and only if each component of G is isomorphic to* C_n *for some* $n \geq 4$ *or is the graph* G_0 *shown in*

Fig. 1. In particular, if G is connected and $G \cong C_n$ *for some* $n \geq 4$ *or* $G \cong G_0$, *then* $\{HL^k(G)\}$ *converges to* C_n *or* C_4, *respectively.*

Theorem 3 [*Chartrand et al.*] — *Let* $H \cong P_5$. *Then the sequence* $\{HL^k(G)\}$ *converges if and only if each component of G is isomorphic to* C_n *for some* $n \geq 5$ *or, each component of G is one of the graphs of Fig. 2, namely* G_1, G_2 *or* A_j $(j \geq 0)$. *In particular, if G is connected and* $G \cong C_n$ *the sequence* $\{HL^k(G)\}$ *converges to* C_n *and if* $G \cong G_i$ *for i = 1, 2 or* $G \cong A_j$, *the sequence* $\{HL^k(G)\}$ *converges to* C_5.

Note — F_m is a graph of order m and size $m + 1$ consisting of an m-cycle together with an edge joining a pair of vertices on the cycle whose distance is 2.

Theorem 4 [*Chartrand et al.*] — *Let H and G be graphs. If there exists a positive integer n such that* $HL^n(G)$ *contains atleast two edge-disjoint copies of G, then* $\{HL^k(G)\}$ *diverges.*

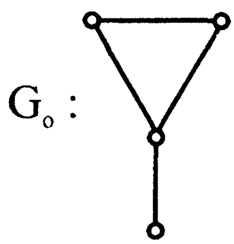

FIG. 1

In this paper, we derive a formula that gives the number of edges in a H-line graph. We discuss the behaviour of the sequence $\{HL^k(G)\}$for a unicyclic graph G when $H \cong P_n$ and prove that the sequence $\{HL^k(G)\}$ is divergent for $G \cong W_n, n \geq 7$ when $H \cong G$. Finally, we deal with H-line graphs of n-pan graphs.

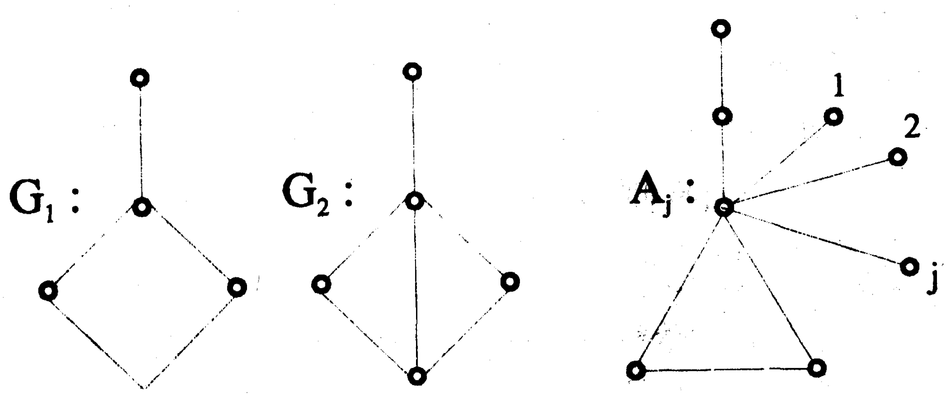

FIG. 2

3. RESULTS

We prove four main results in this section.

Theorem A — *For a unicyclic graph G which consists of a cycle C_t and a path P_m originating from a vertex v_i on the cycle such that C_t and P_m have only one vertex v_i in common, the sequence $\{HL^k(G)\}$ for $H \cong P_n$*

(i) converges to C_n, whenever $n = t + m - 1$

(ii) diverges, whenever $n < t + m - 1$.

PROOF : Suppose G is a unicyclic graph as described above. Let v_i be the vertex on the cycle C_t from which the path P_m originates and v_1, v_j be vertices on C_t that are adjacent to v_i and u_{m-1} be the vertex on P_m adjacent to v_i. The graph G has only two copies of H, viz., $u_1 u_2 \dots u_{m-1} v_i v_1 \dots v_j$ and $u_1 u_2 \dots u_{m-1} v_i v_j \dots v_1$. In $HL(G)$, the vertex corresponding to the edge $u_{m-1} v_i$ in G is adjacent to both the vertices that correspond to the edges $v_i v_1$ and $v_i v_j$ in G. As a result $HL(G)$ consists of a cycle C_{t+1} and a path P_{m-1} originating from a vertex on C_{t+1}. Likewise $HL^2(G)$ contains C_{t+2} and a path P_{m-2} originating from a vertex on C_{t+2}. Thus, $HL^{m-1}(G)$ contains a cycle $C_{t+m-1} = C_n$ and a path P_0 originating from a vertex on C_n..

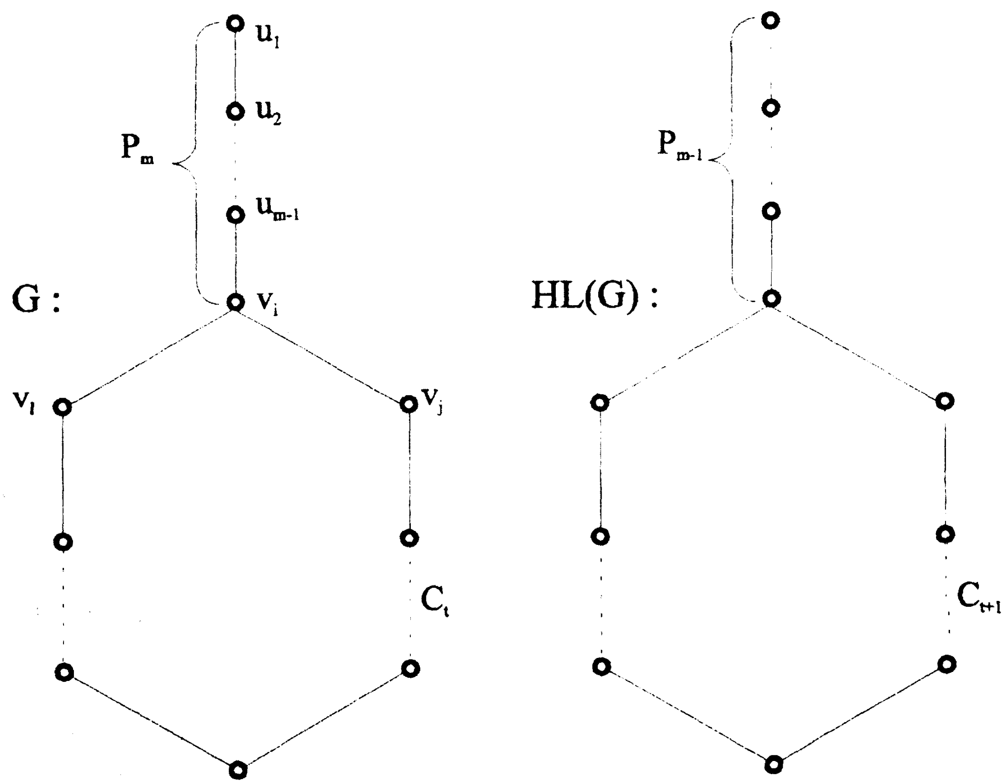

FIG. 3. Unicyclic graph and its *H*-line graph

Next, it is easy to see that $HL^m(G) = HL^{m+1}(G) = \dots = C_n$, that is, the sequence $\{HL^k(G)\}$ converges to C_n.

(*ii*) Suppose $n < t + m - 1$. Let $t < n$

As before there are two copies of H in G and $HL(G)$ contains a cycle C_{t+1} and a path P_{m-1} originating from a vertex on C_{t+1} After $n - t$ such iterations of H-line graph we obtain $HL^{n-t}(G)$ which contains C_n and a path P_{m-n+t} $(m - n + t > 0)$ originating from a vertex on C_n. Now in $HL^{n-t}(G)$ we have more than two copies of H. The new copies of H are found on C_n. However, $HL^{n-t+1}(G)$ is determined by three copies of H, *viz.*, $P_{m-n+t}\, v_i\, v_1 \dots, P_{m-n+t}\, v_i\, v_j \dots,$ and $v_1\, v_i\, v_j \dots$ and it contains F_{n+1}. But the sequence $\{HL^k(G)\}$ is divergent [see Lemma[1]]. Hence $\{HL^k(G)\}$ is divergent.

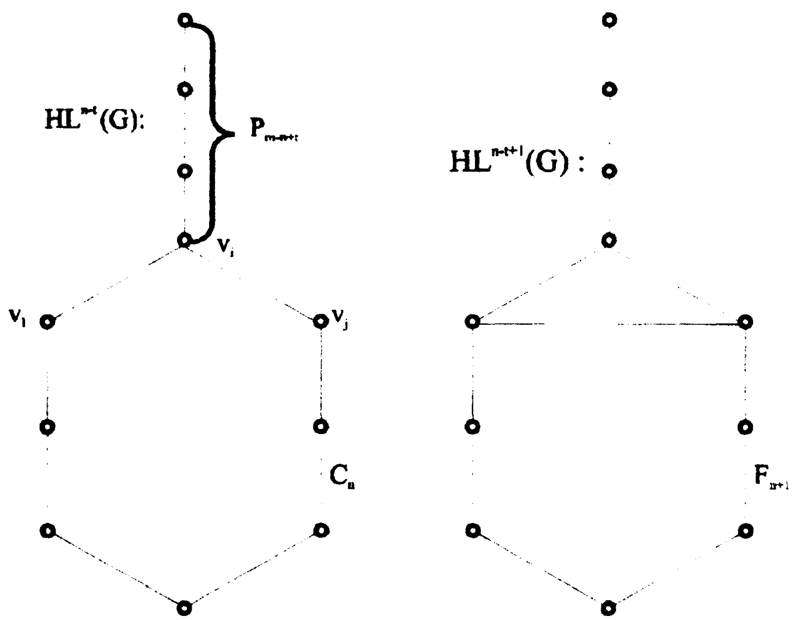

FIG. 4. Iterated H-line graphs.

Remark : Let $t \geq n$. Then there are several copies of H on C_t itself and these copies produce C_t back in $HL(G)$. But copies of H that contain edges of P_m will produce a C_{t+1} in $HL(G)$. As a result $HL(G)$ contains a F_{t+1}. Therefore the sequence $\{HL^k(G)\}$ diverges.

Theorem B — *Let* $G \cong W_n$ $(n \geq 7)$ *and* $H \cong G$. *Then* HL (G) *contains two edge-disjoint copies of* W_n.

PROOF : The wheel $W_n : C_{n-1} + K_1$ contains 2 $(n - 1)$ edges. Of these 2 $(n - 1)$ edges $(n - 1)$ are associated with C_{n-1} and the remaining $(n - 1)$ edges join the centre of W_n with each of the $n - 1$ vertices on C_{n-1}. (We refer to them as spokes).

Consider $G \cong W_n$ and $H \cong G$. In $HL\ (G)$, there are $2\ (n - 1)$ vertices. Let the $(n - 1)$ vertices that correspond to $(n - 1)$ edges on C_{n-1} in W_n be denoted by $V = \{v_1, v_2 \ldots v_{n-1}\}$ and the $(n - 1)$ vertices that correspond to $(n - 1)$ spokes in W_n be denoted by $U = \{u_1, u_2, \ldots, u_{n-1}\}$.

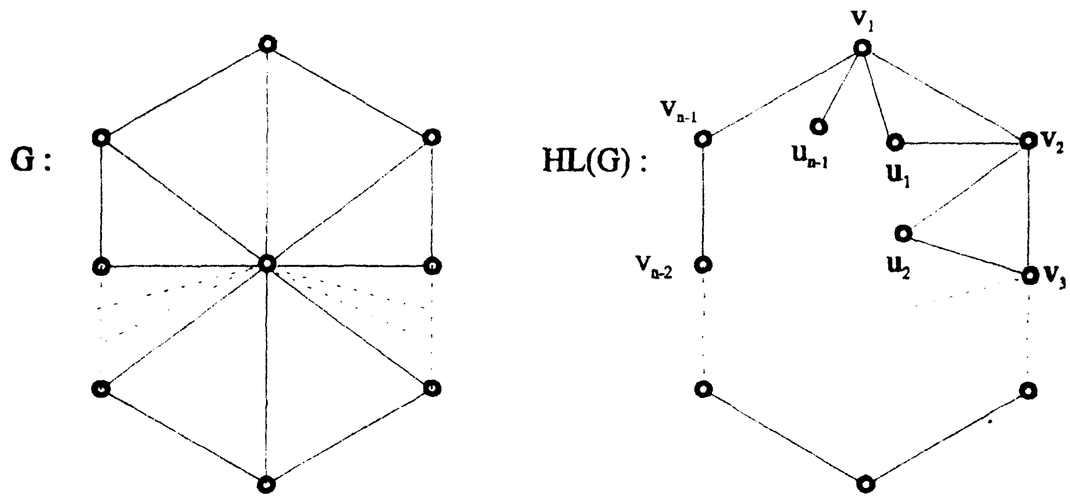

FIG. 5. A wheel and its *H*-line graph.

Clearly, for each $v_i \,\varepsilon\, V$ deg $v_i = 4$ $(i = 1$ to $n - 1)$ in $HL\ (G)$. Out of these 4 edges 2 are incident with vertices v_{i-1} and v_{i+1} and the other 2 with u_{i-1} and u_i (the subscripts are all addition modulo n). Also, each $u_j \,\varepsilon\, U\ (j = 1$ to $n - 1)$ is adjacent to 2 vertices $v_j, v_{j+1} \,\varepsilon\, V$ and to all the remaining $(n - 2)$ u_k's $\varepsilon\, U, (k \neq j)$. Therefore, deg $u_j = 2 + n - 2 = n$.

We now show that $HL(G)$ contains two edge-disjoint copies of W_n. A copy of W_n in $HL(G)$ is lifted as follows.

We can choose any $u_i\ (i = 1$ to $n - 1)$ for the centre of a copy of W_n since degree of each u_i is n. No v_i can serve as the centre since deg $v_i = 4$. So, suppose we take u_1 for the centre of a copy of W_n. Then immediately four more vertices get automatically selected to form a copy of W_n. [Note that u_i's or v_i's alone cannot form a copy of W_n]. Those four vertices are v_1, v_2, u_2, u_{n-1}. We need $(n - 5)$ more vertices to form C_{n-1} and we are left with $(n - 4)$ u_j's $\varepsilon\, U - \{u_1, u_2, u_{n-1}\}$ and $(n - 3)\ v_i$'s $\varepsilon\, V - \{v_1, v_2\}$. These $(n - 5)$ vertices cannot be any of $v_i \,\varepsilon\, V - \{v_1, v_2\}$, since u_1 the centre is not adjacent to them. So these $(n - 5)$ vertices are chosen from $U - \{u_1, u_2, u_{n-1}\}$. Then there is one vertex left, say, u_k after choosing $(n - 5)$ vertices. Then $v_1\, v_2\, u_2 \ldots u_{k-1} \ldots u_{n-1}\, v_1$ forms C_{n-1} and with u_1 as the centre we have a copy of W_n. Here u_1 is adjacent to all the vertices on C_{n-1} forming spokes of the wheel.

In forming a copy of W_n we have used: edges $v_1 v_2, v_1 u_{n-1}, v_2 u_2, v_1 u_1, v_2 u_1$ and the remaining $2n - 2 - 5 = 2n - 7$ edges are all amongst the vertices $U - \{ u_k \}$, of which $(n - 3)$ are incident with the centre u_1.

To form a second copy of W_n, we take u_k to be the centre. As before four more vertices, $v_k, v_{k+1}, u_{k-1}, u_{k+1}$ immediately get selected and all different from the one selected for first copy in a similar way. Therefore, corresponding edges are $v_k v_{k+1}, v_k u_{k-1}$, and $v_k u_{k+1}$, are all distinct. Aso, we choose $(n - 5)$ vertices to complete the cycle C_{n-1} from $U - \{ u_k, u_{k-1}, u_{k+1}, u_1 \}$. Though there are $n - 7$ vertices common in the two copies of C_{n-1}, the edges that form copies of C_{n-1} can be made distinct since all u_j's are mutually adjacent. Moreover, the spokes in the two copies of W_n are distinct since the centres are distinct.

Corollary — Let $G \cong W_n$ $(n \geq 7)$ and $H \cong G$. Then the sequence $\{HL^k(G)\}$ diverges.

PROOF : This is obtained by applying theorem B and theorem 4.

Also, it is easy to see that if $n = 4$ then $HL\,(G)$ does not contain a copy of W_4 whereas if $n = 5$ or 6, $HL^2(G)$ contains two edge-disjoint copies of W_5 or W_6.

Theorem C — *The number of edges in HL (G), is,*

$$m\,[HL\,(G)] = k_0\,Q_L + k_1\,(q_L - 1) + k_2\,(q_L - 2) + \dots k_{qL-1}.$$

Where $Q_L = m\,[HL\,(H)] = m\,[L\,(H)]$

k_0 = No. of P_3 -disjoint copies of H.

k_1 = No. of copies of H with $1\,P_3$ common

k_2 = No. of copies of H with $2P_3$ common

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k_{qL-1} = No. of copies of H with $(q_L - 1)\,P_3$ common.

PROOF : We know that every P_3 in G gives rise to an edge in $L(G)$. Likewise, every P_3 in a copy of H in G gives rise to an edge in $HL(G)$. Also, for any graph G, $GL(G)$ is same as $L(G)$. Therefore, the number of edges in $HL(H)$ is equal to the number of edges in $L(H)$, that is, $m\,[HL\,(H)] = m\,[L\,(H)]$. Thus,

$$m\,[HL\,(H)] = q_L = 1/2\,\Sigma\,d_i^2 - q.$$

where d_i is the degree of each vertex v_i in H and q is the number of edges in H. Here q_L also dentoes the number of P_3's in a copy of H.

Two copies of H are said to be P_3-disjoint if there are no P_3's in common in them. Suppose there are $K_0\,P_3$ disjoint copies of H. Then corresponding to these k_0 copies we have $k_0\,q_L$ edges in $HL(G)$. Next we find copies of H which have $1P_3$ common with any of the $k_0\,P_3$ - disjoint copies of H considered above. Denote these copies of H by H_1. Let there be k_1 copies of H_1. Now. in each of these k_1 copies one P_3 has already been considered, the edges corresponding to these P_3 are counted once. So, each of these k_1 copies yield only $(q_L - 1)$ distinct edges. Thus, these k_1 copies together contribute $k_1\,(q_L - 1)$ edges. Likewise, we next find copies of H which have exactly $2P_3$ common with any of the $K_0\,P_3$ - disjoint copies of H [or $1P_3$ common with any of H_1]. Denote a copy of H which has $2P_3$ common by H_2. Let k_2 denote the number of copies of H_2. Then corresponding to these k_2 copies of H_2 we have $k_2\,(q_L - 2)$ edges. In general, let H_t denote a copy of H which has tP_3 common and k_t denote the number of copies of H_t (t = 1 to $q_L - 1$). Then these k_t copies of H_t contribute $k_t\,(q_L - t)$ distinct edges.

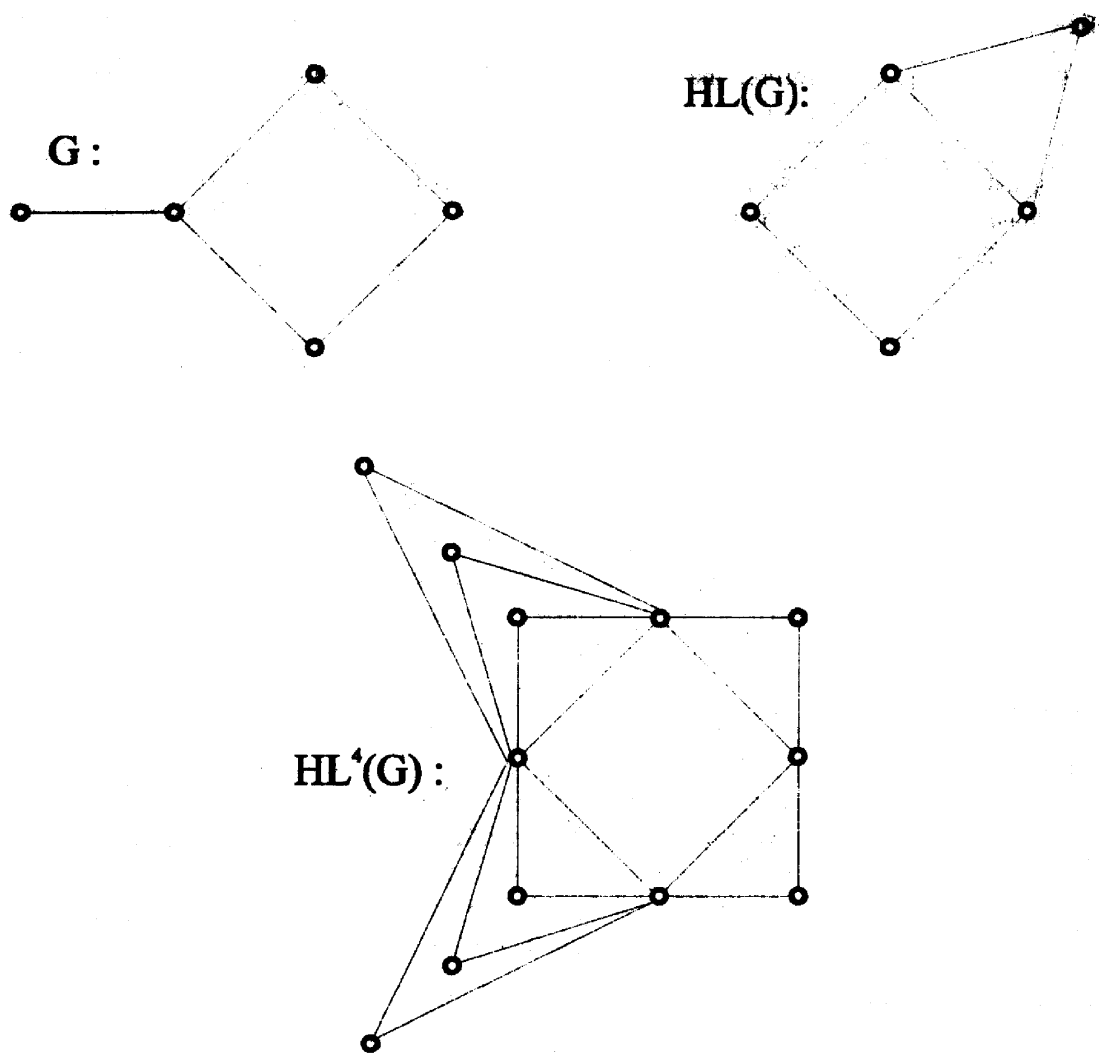

FIG. 6. A n-pan graph and its H-line graphs.

Thus, $$m\,HL(G) = \sum_{t=0}^{q_L - 1} k_t\,(q_L - t)$$

Theorem D — *If G is a n-pan graph and $H \cong G, HL^{m+1}(G)$ contains an n-pan graph with 2^m triangles.*

PROOF : An n-pan is made of a cycle C_n and a pendant edge incident on one of the vertices in C_n. So, we see that a n-pan contains a $K_{1,3}$ as an induced subgraph.

We prove the theorem by induction on m. In $HL(G)$, the cycle C_n (of G) reappears as C_n while $K_{1,3}$ produces a triangle K_3. The triangle K_3 and the cycle C_n have a common edge in $HL(G)$. At each of the end vertices of this common edge there is one copy of $K_{1,3}$. Again in $HL^2(G)$ the cycle C_n reappears and each $K_{1,3}$ gives rise to a triangle. Thus, $HL(G)$ contains a n-pan with 2^0-triangle, and $HL^2(G)$ contains a n-pan with 2^1-triangles.

We now assume that the result holds for $m = k$, a positive integer, that is, $HL^k(G)$ contains an n-pan with 2^{k-1} triangles. Each of these triangles have one edge common with C_n. At every endvertex of these common edges there is a copy of $K_{1,3}$, that is, $HL^k(G)$ contains a n-pan with 2.2^{k-1} copies of $K_{1,3}$. Therefore, the next iteration contains an n-pan with 2^k triangles.

REFERENCES

1. G. Chartrand, H. Galvas, M. Schultz, *Discrete Mathematics*, **147** (1995), 73-86.
2. D. Dorrough, *Discrete Math.*, **161** (1996), 79-86.
3. R. A. Britto - Pacumio, *Discrete Math.*, **199** (1999), 1-10.