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STRESS DISTRIBUTION DUE TO A PAIR OF COPLANAR GRIFFITH CRACKS AT THE INTERFACE OF TWO BONDED DISSIMILAR ELASTIC HALF-PLANES

S. K. DE

Department of Mathematics, Kharagpur College, Kharagpur 721 305, *India*

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The stress distribution in the neighbourhood of a pair of coplanar Griffith cracks opened at the interface of two bonded dissimilar media has been considered in this paper and dislocation layers have been utilized to solve the problem. Here the cracks are opened by normal pressure on the crack surfaces. This problem has been considered earlier by other methods. The superposition principle is exploited here to adapt the solution of this problem into the frame work of our formulation.

Key Words : Dislocation Theory; Crack Problems; Hilbert Problem

1. INTRODUCTION

The cracks opened by the internal pressure in a two dimensional infinite isotropic medium have been considered by Willmore[1], Tranter[2] and Lowengrub and Srivastava[3]. Willmore considered the complex variable method developed by Muskhelishvili[4]. But Tranter, Lowengrub and Srivastava have utilized the theory of double and triple integral equations based on Fourier transforms techniques developed by Sneddon[5].

Lardner[6] has utilized dislocation layers more effectively in solving boundary value problems of plane elasticity by considering a continuous distribution of edge dislocations along the boundary and exploited dislocation method to solve two-dimensional problems of general regions and half-spaces, crack and contact problems by formulating them in terms of singular integral equations. In this paper an alternative solution to the title problem by exploiting continuum theory of dislocations has been derived. This approach is based on modified Somigliana integral established by Maiti *et al.*[7,8,9]). The corresponding problem for a homogeneous medium has been considered by Maiti *et al.*[10]. The stress distribution in the neighbourhood of a pair of coplanar Griffith cracks opened at the interface of two bonded dissimilar material media has been derived here. The solution found in this case is physically inadmissible, because violent oscillations are found to occur both in displacements and stresses near the crack tips. Salganik[11] has differed from this view and claimed that the integral transform techniques smooth out this characteristic oscillatory phenomena. But he has been proved to be wrong on this account by Lowengrub[12]. England[13] pointed out that the gap between crack faces is negative on the parts of the crack and has shown that these effects are confined only to a narrow region near the crack tips and was able to solve this problem based on complex variable formulation describing a reasonable estimate of the physical state at the points far away from the crack tips. Comninou[14] and Gautesen and Dundurs[15] observed that the solutions of such type of problems are oscillating in nature near the crack tips.

In this problem the cracks are opened by normal pressure on the crack surfaces at the interface of two dissimilar elastic half-planes and the solution of this problem reduces to that of a Hilbert problem for a constant pressure. It may be noted that for varying pressure the problem still reduces to Hilbert problem. This will be considered later on in a separate publication. It may be mentioned here that the plane strain conditions prevailled in the present problem. This problem has been considered by Lowengrub[16] in another approach.

2. Displacements and Stresses in a Half-Plane

Starting from the Somigliana integral it has been shown by Maiti *et al.*[8] that the displacements u_i in the upper half-plane $y > 0$ may be given by

$$u_x(x,y) = -\int_{-\infty}^{\infty} f_x(x')\left[\frac{1}{2\pi}\tan^{-1}\left(\frac{y}{x-x'}\right)+\frac{1}{4\pi(1-\nu)}\frac{y(x-x')}{(x-x')^2+y^2}\right]dx'$$

$$-\int_{-\infty}^{\infty} f_y(x')\left[\frac{1-2\nu}{8\pi(1-\nu)}\log\left\{(x-x')^2+y^2\right\}+\frac{1}{4\pi(1-\nu)}\frac{y^2}{(x-x')^2+y^2}\right]dx' \quad \ldots (2.1)$$

$$u_y(x,y) = \int_{-\infty}^{\infty} f_x(x')\left[\frac{1-2\nu}{8\pi(1-\nu)}\log\left\{(x-x')^2+y^2\right\}\right.$$

$$\left.+\frac{1}{4\pi(1-\nu)}\frac{(x-x')^2}{(x-x')^2+y^2}\right]dx'$$

$$+\int_{-\infty}^{\infty} f_y(x')\left[\frac{1}{2\pi}\tan^{-1}\left(\frac{x-x'}{y}\right)+\frac{1}{4\pi(1-\nu)}\frac{y(x-x')}{(x-x')^2+y^2}\right]dx', \quad \ldots (2.2)$$

where ν is the Poisson's ratio. The displacement field, given by (2.1) and (2.2), is the superposition of two displacement fields considered as due to the layers of edge dislocation of densities $-f_x(x)$ and $f_y(x)$ distributed along the line $y = 0$; (see, Hirth and Lothe[17]), the corresponding, stresses σ_{ij} in the upper half-plane are given by

$$\sigma_{xx}(x,y) = \frac{\mu}{2\pi(1-\nu)}\left[\int_{-\infty}^{\infty} f_x(x')\frac{y\left\{3(x-x')+y^2\right\}}{\left\{(x-x')^2+y^2\right\}^2}dx'\right.$$

$$\left.+\int_{-\infty}^{\infty} f_y(x')\frac{(x-x')\left\{y^2-(x-x')^2\right\}}{\left\{(x-x')^2+y^2\right\}^2}dx'\right], \quad \ldots (2.3)$$

$$\sigma_{yy}(x, y) = \frac{\mu}{2\pi(1-\nu)}\left[\int_{-\infty}^{\infty} f_x(x') \frac{y\{y^2-(x-x')^2\}}{\{(x-x')^2+y^2\}^2}\, dx'\right.$$

$$\left. - \int_{-\infty}^{\infty} f_y(x') \frac{(x-x')\{(x-x')^2+3y^2\}}{\{(x-x')^2+y^2\}^2}\, dx'\right], \quad \text{... (2.4)}$$

$$\sigma_{xy}(x, y) = \frac{\mu}{2\pi(1-\nu)}\left[\int_{-\infty}^{\infty} f_x(x') \frac{(x-x')\{y^2-(x-x')^2\}}{\{(x-x')^2+y^2\}^2}\, dx'\right.$$

$$\left. + \int_{-\infty}^{\infty} f_y(x') \frac{y\{y^2-(x-x')^2\}}{\{(x-x')^2+y^2\}^2}\, dx'\right], \quad \text{.... (2.5)}$$

where μ is the shear modulus.

The boundary displacements and stresses can now be derived, in the limit as $y \to 0$ as

$$u_x(x, 0) = -\frac{1}{2}\int_{x}^{\infty} f_x(x')\, dx' - \frac{1-2\nu}{4\pi(1-\nu)}\int_{-\infty}^{\infty} f_y(x') \log|x-x'|\, dx', \quad \text{... (2.6)}$$

$$u_y(x, 0) = \frac{1-2\nu}{4\pi(1-\nu)}\int_{-\infty}^{\infty} f_x(x') \log|x-x'|\, dx'$$

$$+ \frac{1}{4\pi(1-\nu)}\int_{-\infty}^{\infty} f_x(x')\, dx' - \frac{1}{2}\int_{x}^{\infty} f_y(x')\, dx', \quad \text{... (2.7)}$$

$$\sigma_{yy}(x, 0) = \frac{\mu}{2\pi(1-\nu)}\int_{-\infty}^{\infty} \frac{f_y(x')\, dx'}{x'-x}, \quad \text{... (2.8)}$$

$$\sigma_{xy}(x, 0) = \frac{\mu}{2\pi(1-\nu)}\int_{-\infty}^{\infty} \frac{f_x(x')\, dx'}{x'-x}. \quad \text{... (2.9)}$$

From the above relations one can easily derive

$$\frac{du_x(x, 0)}{dx} = \frac{1}{2} f_x(x) + \frac{1-2\nu}{2\mu}\sigma_{yy}(x, 0), \quad \text{... (2.10)}$$

$$\frac{du_y(x, 0)}{dx} = \frac{1}{2} f_y(x) - \frac{1-2\nu}{2\mu}\sigma_{xy}(x, 0), \quad \text{... (2.11)}$$

which give dislocation densities in terms of boundary displacements and stresses.

The application of infinite Hilbert transforms to (2.8) and (2.9) yields

$$f_x(x) = -\frac{2(1-\nu)}{\pi\mu}\int_{-\infty}^{\infty}\frac{\sigma_{xy}(x',0)\,dx'}{x'-x}, \quad \text{... (2.12)}$$

$$f_y(x) = -\frac{2(1-\nu)}{\pi\mu}\int_{-\infty}^{\infty}\frac{\sigma_{yy}(x',0)\,dx'}{x'-x}, \quad \text{... (2.13)}$$

which, when substituted into (2.10) and (2.11), give

$$U'(x,0) = \frac{1-2\nu}{2\mu}\sigma_{yy}(x,0) - \frac{1-\nu}{\pi\mu}\int_{-\infty}^{\infty}\frac{\sigma_{xy}(x',0)\,dx'}{x'-x}, \quad \text{... (2.14)}$$

$$V'(x,0) = -\frac{1-2\nu}{2\mu}\sigma_{xy}(x,0) - \frac{1-\nu}{\pi\mu}\int_{-\infty}^{\infty}\frac{\sigma_{yy}(x',0)\,dx'}{x'-x}, \quad \text{... (2.15)}$$

where $u_x(x,0) = U(x,0)$, $U_y(x,0) = V(x,0)$. Using subscripts 1 and 2 in the material constants for the upper half-plane and lower half-plane respectively, one can derive

$$U'(x,0+) = \frac{1-2\nu_1}{2\mu_1}\sigma_{yy}(x,0+) - \frac{1-\nu_1}{\pi\mu_1}\int_{-\infty}^{\infty}\frac{\sigma_{xy}(x',0+)\,dx'}{x'-x}, \quad \text{... (2.16)}$$

$$V'(x,0+) = -\frac{1-2\nu_1}{2\mu_1}\sigma_{xy}(x,0+) - \frac{1-\nu_1}{\pi\mu_1}\int_{-\infty}^{\infty}\frac{\sigma_{yy}(x',0+)\,dx'}{x'-x}, \quad \text{... (2.17)}$$

$$U'(x,0-) = \frac{1-2\nu_2}{2\mu_2}\sigma_{yy}(x,0-) + \frac{1-\nu_2}{\pi\mu_2}\int_{-\infty}^{\infty}\frac{\sigma_{xy}(x',0-)\,dx'}{x'-x}, \quad \text{... (2.18)}$$

$$V'(x,0-) = -\frac{1-2\nu_2}{2\mu_2}\sigma_{xy}(x,0-) + \frac{1-\nu_2}{\pi\mu_2}\int_{-\infty}^{\infty}\frac{\sigma_{yy}(x',0-)\,dx'}{x'-x}. \quad \text{... (2.19)}$$

The solution to the crack problem essentially hinges on the boundary relations (2.16)-(2.19).

3. The Crack Problem and its Solution

Consider a two-dimensional infinite heterogeneous medium containing a pair of coplanar Griffith cracks, which occupy the regions $-b \le x \le -a$, $a \le x \le b$, $y = 0$ as shown in Fig. 1 and are opened at the interface of two bonded dissimilar elastic half-planes by the normal pressure $p(x)$, where $p(x)$ is an even function of x. Then the relevant boundary conditions on the line $y = 0$ are given by

$$\sigma_{yy}(x,0+) = -p(x),\ \sigma_{xy}(x,0+) = 0,\ a \le |x| \le b, \quad \text{... (3.1)}$$

$$\sigma_{yy}(x,0-) = -p(x),\ \sigma_{xy}(x,0-) = 0,\ a \le |x| \le b, \quad \text{... (3.2)}$$

and outside the crack regions

$$\sigma_{yy}(x, 0+) = \sigma_{yy}(x, 0-), |x| < a, |x| > b, \qquad \text{... (3.3)}$$

$$\sigma_{xy}(x, 0+) = \sigma_{xy}(x, 0-), |x| < a, |x| > b, \qquad \text{... (3.4)}$$

$$u_x'(x, 0+) = u_x(x, 0-), |x| \le a, |x| \ge b, \qquad \text{... (3.5)}$$

$$u_y(x, 0+) = u_y(x, 0-), |x| \le a, |x| \ge b. \qquad \text{... (3.6)}$$

The boundary conditions (3.1)-(3.4), when applied to (2.16) and (2.18), yield for $a \le |x| \le b$

$$\frac{1-\nu_2}{\mu_2} U'(x, 0+) + \frac{1-\nu_1}{\mu_1} U'(x, 0-) = -Mp(x) \qquad \text{... (3.7)}$$

where

$$M = \frac{(1-2\nu_1)(1-\nu_2) + (1-2\nu_2)(1-\nu_1)}{2\mu\mu_2} \qquad \text{... (3.8)}$$

Further, from the boundary conditions (2.17) and (2.19) one can derive for $a \le |x| \le b$

$$\frac{1-\nu_2}{\mu_2} V'(x, 0+) + \frac{1-\nu_1}{\mu_1} V'(x, 0-) = 0 \qquad \text{... (3.9)}$$

From (3.7) and (3.9) it is easy to derive

$$\phi^-(x) + g\,\phi^-(x) = -Tp(x) \qquad \text{... (3.10)}$$

for $a \le |x| \le b$, where

$$\phi^-(x) = \phi(x + i0) = U'(x, 0+) + iV'(x, 0-), \qquad \text{... (3.11)}$$

$$\phi^-(x) = \phi(x - i0) = U'(x, 0-) + iV'(x, 0-), \qquad \text{... (3.12)}$$

$$g = \frac{(1-\nu_1)\mu_2}{(1-\nu_2)\mu_1}, \qquad \text{... (3.13)}$$

$$T = \frac{M\mu_2}{1-\nu_2}. \qquad \text{... (3.14)}$$

To solve the Hilbert problem (3.10) a sectionally holomorphic function $\phi(z)$ is to be found in the entire plane cut along the segments $[-b, -a]$ and $[a, b]$ for a constant pressure $p(x) = p$, which is given by

$$\phi(z) = -\frac{TPX(z)}{2\pi i} \int_N \frac{dx'}{X^+(x')(x'-z)} + (c_1 z + c_2) X(z), \qquad \text{... (3.15)}$$

where $N = [-b, -a]\ U\ [a, b]$, c_1 and c_2 are complex constants and

$$X(z) = (z+b)^{-\gamma}(z+a)^{\gamma-1}(z-a)^{-\gamma}(z-b)^{\gamma-1}, \quad \text{... (3.16)}$$

is a Plemelj function satisfying the relation

$$X^{+}(x) = -gX^{-}(x), x \in N, \quad \text{... (3.17)}$$

with

$$e^{2\pi i\gamma} = -g. \quad \text{... (3.18)}$$

Following England[18] one can easily derive from (3.15)

$$\phi(z) = -\frac{TPX(z)}{2\pi i(1+g)}\left[\frac{2\pi i}{X(z)} - L(z)\right] + (c_1 z + c_2) X(z), \quad \text{... (3.19)}$$

where $L(z)$ is an integral over $C_\alpha(c)$ is the union of the lacets around the segments $[-b, -a]$ and $[a, b]$; see Fig. 1).

Now

$$L(z) = \lim_{R\to\infty} \int_0^{2\pi} \frac{i\,\mathrm{Re}^{i\theta}\, d\theta}{X(Re^{i\theta})(Re^{i\theta}-z)}$$

$$= \lim_{R\to\infty} \int_0^{2\pi} \left(1 - \frac{z}{\mathrm{Re}^{i\theta}}\right)^{-1} \{R^2 e^{2i\theta} + (b-a)(2\gamma-1)\ \mathrm{Re}^{i\theta}$$

$$+ 2\gamma(\gamma-1)(a^2+b^2) - ab(4\gamma^2 - 4\gamma + 1) + O(R^{-1})\}\ d\theta$$

$$= \lim_{R\to\infty} \int_0^{2\pi} \{R^2 e^{2i\theta} + [z + (b-a)(2\gamma-1)]\,\mathrm{Re}^{i\theta} + [z^2 + (b-a)(2\gamma-1)z$$

$$+ 2\gamma(\gamma-1)(a^2+b^2) - ab(4\gamma^2 - 4\gamma+1)] + O(R^{-1})\}\ d\theta$$

with

$$\frac{1}{X(z)} = z^2 + (b-a)(2\gamma-1)z + 2\gamma(\gamma-1)(a^2+b^2)$$

$$- ab(4\gamma^2 - 4\gamma + 1) + O\left(\frac{1}{z}\right) \quad \text{... (3.20)}$$

$$\therefore L(z) = 2\pi i\left\{z^2 + (b-a)(2\gamma-1)z + 2\gamma(\gamma-1)(a^2+b^2) - ab(4\gamma^2 - 4\gamma+1)\right\}. \quad \text{... (3.21)}$$

Hence from (3.19) and (3.21) it is easy to get

$$\phi(z) = -\frac{TP}{1+g}\left\{1 - (z^2 + Az + B)X(z)\right\}, \quad \text{... (3.22)}$$

where A and B are constants given by

$$A = (b-a)(2\gamma-1) - \frac{1+g}{TP}c_1 = A_1 + iA_2, \text{ say}, \quad \text{... (3.23)}$$

$$B = 2\,\gamma(\gamma - 1)\,(a^2 + b^2) - ab\,(4\,\gamma^2 - 4\,\gamma + 1) - \frac{1+g}{TP}\,c_2,$$

$$= B_1 + iB_2, \text{ say}, \qquad \dots (3.24)$$

whence one can easily derive

$$\phi\,(x + i0) - \phi\,(x - i0) = S\,(x) - iR\,(x), \qquad \dots (3.25)$$

where

$$S\,(x) = \frac{TP}{g^{3/2}} \left[\frac{(x^2 + A_1\,x + B_1)\sin\beta\,\theta + (A_2\,x + B_2)\cos\beta\,\theta}{\left\{(x^2 - a^2)\,(b^2 - x^2)\right\}^{1/2}} \right], \qquad \dots (3.26)$$

$$R\,(x) = \frac{TP}{g^{3/2}} \left[\frac{(x^2 + A_1\,x + B_1)\cos\beta\,\theta - (A_2\,x + B_2)\sin\beta\,\theta}{\left\{(x^2 - a^2)\,(b^2 - x^2)\right\}^{1/2}} \right], \qquad \dots (3.27)$$

with

$$\beta = \frac{\log g}{2\,\pi}, \qquad \dots (3.28)$$

$$\theta = \log\left(\frac{x-a}{x+a}\,\frac{x+b}{b-x} \right), \qquad \dots (3.29)$$

for $a < x < b$.

Similarly, one can obtain for $-b < x < -a$

$$\phi\,(x + i0) - \phi\,(x - i0) = S_1\,(x) - iR_1\,(x), \qquad \dots (3.30)$$

where

$$S_1\,(x) = -\frac{TP}{g^{3/2}} \left[\frac{(x^2 + A_1\,x + B_1)\sin\beta\,\theta + (A_2\,x + B_2)\cos\beta\,\theta}{\left\{(x^2 - a^2)\,(b^2 - x^2)\right\}^{1/2}} \right], \qquad \dots (3.31)$$

$$R_1\,(x) = -\frac{TP}{g^{3/2}} \left[\frac{(x^2 + A_1\,x + B_1)\cos\beta\,\theta - (A_2\,x + B_2)\sin\beta\,\theta}{\left\{(x^2 - a^2)\,(b^2 - x^2)\right\}^{1/2}} \right], \qquad \dots (3.32)$$

Now in order that the relation $S\,(-x) = S_1\,(x)$ and $-R\,(-x) = R_1\,(x)$ on $-b < x < -a$ holds one must choose $A_1 = B_2 = 0$.

Hence from (3.25) and (3.9), it is found that

$$V'\,(x, 0+) = -\frac{TP}{(1+g)\,g^{1/2}} \left[\frac{(x^2 + B_1)\cos\beta\,\theta - A_2\,x\sin\beta\,\theta}{\left\{(x^2 - a^2)\,(b^2 - x^2)\right\}^{1/2}} \right], \qquad \dots (3.33)$$

$$V'(x, 0-) = -\frac{TP}{(1+g)\,g^{3/2}}\left[\frac{(x^2+B_1)\cos\beta\,\theta - A_2\,x\sin\beta\,\theta}{\left\{(x^2-a^2)(b^2-x^2)\right\}^{1/2}}\right], \quad \ldots (3.34)$$

for $a < x < b$. Now for $x > b$ the displacements and stresses may be derived, where

$$X^+(x) = X^-(x) = X(x), \text{ say}, \quad \ldots (3.35)$$

then in this case

$$\phi(x+i0) + \phi(x-i0) = -\frac{2TP}{1+g}\left[1 - \frac{(x^2+B_1)\cos\beta\,\theta_2 - A_2\,x\sin\beta\,\theta_2}{\left\{(x^2-a^2)(x^2-b^2)\right\}^{1/2}}\right.$$

$$\left. - i\,\frac{(x^2+B_1)\sin\beta\,\theta_2 - A_2\,x\cos\beta\,\theta_2}{\left\{(x^2-a^2)(x^2-b^2)\right\}^{1/2}}\right], \quad \ldots (3.36)$$

which, separating real and imaginary parts and then using the boundary conditions (3.5) and (3.6), gives

$$U'(x, 0+) = U'(x, 0-) = -\frac{TP}{1+g}\left[1 - \frac{(x^2+B_1)\cos\beta\,\theta_2 - A_2\,x\sin\beta\,\theta_2}{\left\{(x^2-a^2)(x^2-b^2)\right\}^{1/2}}\right], \quad \ldots (3.37)$$

$$V'(x, 0+) = V'(x, 0-) = \frac{TP}{1+g}\left[1 - \frac{(x^2+B_1)\sin\beta\,\theta_2 + A_2\,x\cos\beta\,\theta_2}{\left\{(x^2-a^2)(x^2-b^2)\right\}^{1/2}}\right], \quad \ldots (3.38)$$

where $$\theta_2 = \log\left(\frac{x-a}{x+a}\cdot\frac{x+b}{x-b}\right). \quad \ldots (3.39)$$

Further for $x > b$

$$U'(x, 0+) = U'(x, 0-) = \frac{T}{1+g}\,\sigma_{yy}(x, 0+) = \frac{T}{1+g}\,\sigma_{yy}(x, 0-), \quad \ldots (3.40)$$

whence one can derive

$$\sigma_{yy}(x, 0, +) = \sigma_{yy}(x, 0-) = -P\left[1 - \frac{(x^2+B_1)\cos\beta\,\theta_2 - A\,x\sin\beta\,\theta_2}{\left\{(x^2-a^2)(x^2-b^2)\right\}^{1/2}}\right] \quad \ldots (3.41)$$

Also $$\sigma_{xy}(x, 0+) = \sigma_{xy}(x, 0-) = -\frac{1+g}{T}\,V'(x, 0+) = -\frac{1+g}{T}\,V'(x, 0-)$$

$$= -P\left[\frac{(x^2+B_1)\sin\beta\,\theta_2 + A_2\,x\cos\beta\,\theta_2}{\left\{(x^2-a^2)(x^2-b^2)\right\}^{1/2}}\right], \quad \ldots (3.42)$$

The stresses and displacements will be computed for $0 \le x < a$. In this case also the relation (3.35) holds. Processing exactly in the same way one can easily obtain

$$\sigma_{yy}(x, 0+) = \sigma_{yy}(x, 0-) = -P\left[1 + \frac{(x^2+B_1)\cos\beta\theta_1 - A_2 x \sin\beta\theta_1}{\{(a^2-x^2)(b^2-x^2)\}^{1/2}}\right], \quad \ldots (3.43)$$

$$\sigma_{xy}(x, 0+) = \sigma_{xy}(x, 0-) = P\left[1 + \frac{(x^2+B_1)\sin\beta\theta_1 + A_2 x \cos\beta\theta_1}{\{(a^2-x^2)(b^2-x^2)\}^{1/2}}\right], \quad \ldots (3.44)$$

where
$$\theta_1 = \log\left(\frac{a-x}{a+x}\cdot\frac{b+x}{b-x}\right). \quad \ldots (3.45)$$

But in our formulations $\sigma_{ij} \sim O(r^{-2})$ at a large distance and it is seen that

$$\sigma_{xy} \sim -P\left\{\frac{2\beta(b-a)}{x} + \frac{A_2}{x} + O(x^{-2})\right\}, \quad \ldots (3.46)$$

as $x \to \infty$. This shows that $2\beta(b-a) + A_2 = 0$, implying

$$A_2 = -2\beta(b-a). \quad \ldots (3.47)$$

With this value of A_2 the eqs. (3.33) and (3.34) take the forms :

$$V'(x, 0+) = -\frac{TP}{(1+g)g^{1/2}}\left[\frac{(x^2+B_1)\cos\beta\theta + 2\beta(b-a)x\sin\beta\theta}{\{(x^2-a^2)(b^2-x^2)\}^{1/2}}\right], \quad \ldots (3.48)$$

$$V'(x, 0-) = \frac{TP}{(1+g)g^{3/2}}\left[\frac{(x^2+B_1)\cos\beta\theta + 2\beta(b-a)x\sin\beta\theta}{\{(x^2-a^2)(b^2-x^2)\}^{1/2}}\right]. \quad \ldots (3.49)$$

Whence for $a < x < by$

$$V(x, 0+) = -\frac{TP}{(1+g)g^{1/2}}\int_a^x \frac{(u^2+B_1)\cos\beta\theta + 2\beta(b-a)u\sin\beta\theta}{\{(u^2-a^2)(b^2-u^2)\}^{1/2}}\,du, \quad \ldots (3.50)$$

$$V(x, 0-) = \frac{TP}{(1+g)g^{3/2}}\int_a^x \frac{(u^2+B_1)\cos\beta\theta + 2\beta(b-a)u\sin\beta\theta}{\{(u^2-a^2)(b^2-u^2)\}^{1/2}}\,du. \quad \ldots (3.51)$$

To determine the constant B_1 the condition that at the crack tips the limiting values of the displacements must be zero will be considered and hence it will be assumed that $V(b-, 0+) = 0$. This implies that

$$\int_a^b \frac{(u^2 + B_1)\cos\beta\theta + 2\beta(b-a)\,u\sin\beta\theta}{\{(u^2-a^2)(b^2-u^2)\}^{1/2}}\,du = 0, \qquad \text{... (3.52)}$$

implying

$$B_1 = -a^2 - \frac{2}{C(\beta; a)}\int_a^b x\,C(\beta; x)\,dx$$

$$-\frac{2\beta(b-a)}{C(\beta; a)}\left[a\,S(\beta; a) + \int_a^b S(\beta; x)\,dx\right], \qquad \text{... (3.53)}$$

where $$C(\beta; x) = \int_x^b \frac{\cos\beta\theta\,du}{\{(u^2-a^2)(b^2-u^2)\}^{1/2}}, \qquad \text{... (3.54)}$$

$$S(\beta; x) = \int_x^b \frac{\sin\beta\theta\,du}{\{(u^2-a^2)(b^2-u^2)\}^{1/2}}, \qquad \text{... (3.55)}$$

with $$\theta = \log\left(\frac{u-a}{u+a}\cdot\frac{u+b}{b-u}\right). \qquad \text{... (3.56)}$$

Thus all constants have been determined.

For the homogeneous material medium $\beta = 0$ and from (3.53) one can get

$$B_1 = -a^2 - \frac{2}{K(k)}\int_a^b xK(k, \phi)\,dx, \qquad \text{... (3.57)}$$

where $K(k)$ and $K(k, \phi)$ are complete and incomplete elliptic integrals of the first kind with

$$\phi = \sin^{-1}\left(\frac{b^2-x^2}{b^2-a^2}\right)^{1/2}, \qquad \text{... (3.58)}$$

and $$k^2 = \frac{b^2-a^2}{b^2}. \qquad \text{... (3.59)}$$

Using the properties of the elliptic integrals one can obtain from (3.57)

$$B_1 = -b^2\frac{E(k)}{K(k)}, \qquad \text{... (3.60)}$$

where $E(k)$ is the complee elliptic integral of the second kind. In the limit as $a \to 0$, $B_1 = 0$ and

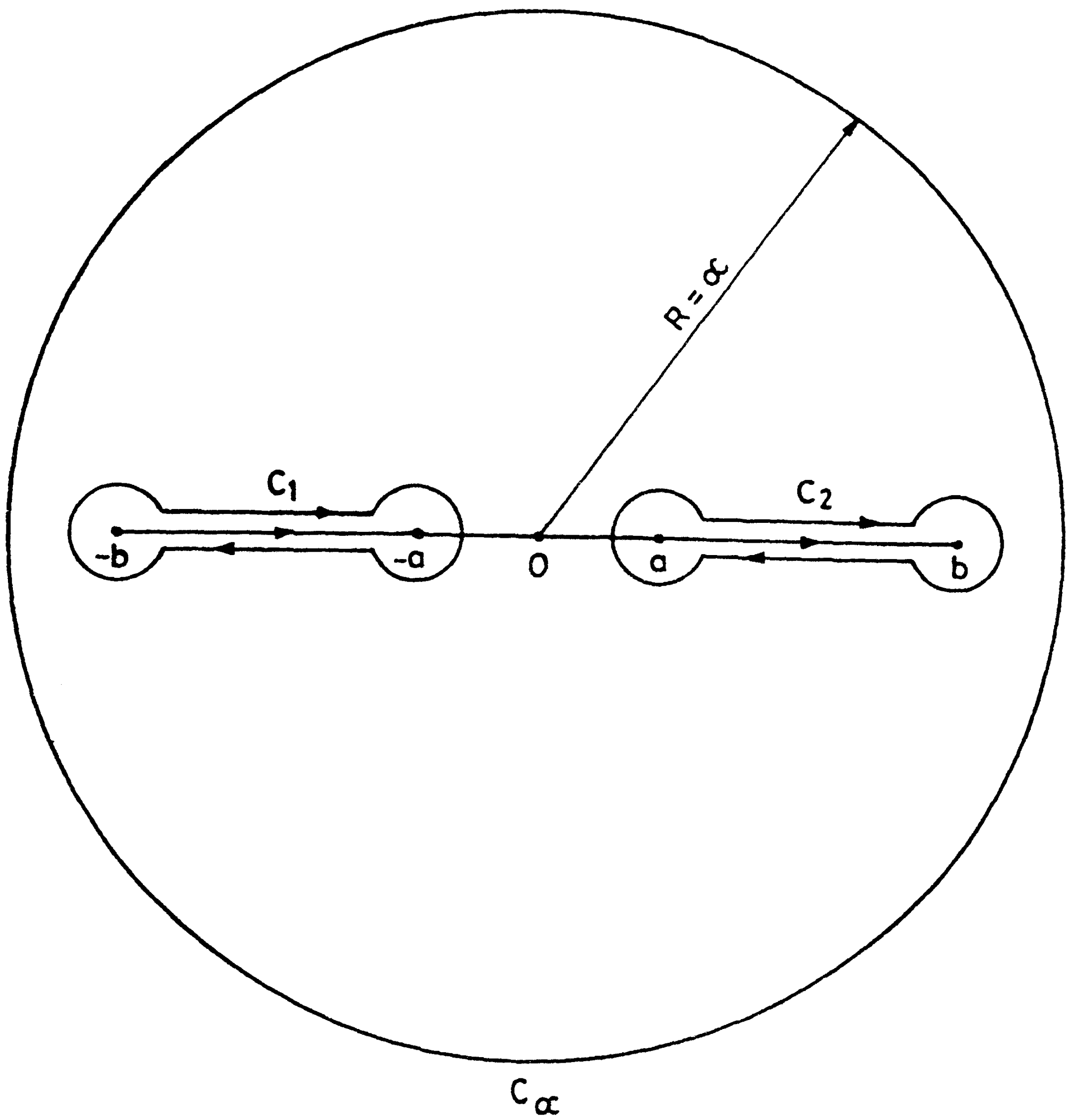

FIG. 1. The circle C_∞ around lacets C_1 and C_2.

the eqs. (3.43) and (3.41) are in complete agreement with the results observed by Tranter[2] in the case of a single crack of width 2b in the homogeneous medium. In other cases B_1 must be computed numerically and results obtained in (3.41), (3.42), (3.43) and (3.44) agree with those derived by Lowengrub[16].

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