Indian J. pure appl. Math., **34**(9) : 1339-1349, September 2003

NEW NUMERICAL TECHNIQUES FOR SOLVING NON-LINEAR EQUATIONS

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(*Received* 1 *January* 2002; *after final revision* 12 *March* 2003; *accepted* 22 *April* 2003)

We present here two cases of four different numerical techniques for solving the nonlinear algebraic and transcendental equations numerically. In these techniques, approximation process for finding the simple and real roots of any nonlinear equation is carried out with the help of circles. Let x_0 be an initial approximation to the required root and assume that $(x_1 = x_0 + h)$, where h is a small positive or negative quantity, be the first approximation to the required root of the given equation. In case I, a circle C_1 with centre at any point $((x_0 + h), f(x_0 + h))$ and radius equal to the ordinate of the same point is drawn on the curve of the given function $y = f(x)$. Then another circle C_2 with centre at $((x_0 - h), f(x_0 - h))$ and radius $f(x_0 - h)$ is drawn on the same curve of the given function such that it touches externally (or orthogonally) the circle C_1. The first approximation to the desired root is taken as the abscissa of the centre of circle C_1 (or circle C_2) according as x_0 lies in the right or left of true root. Here the increment h is obtained with the help of the conditions of external or orthogonal intersection of circles. The process is repeated by similar fashion till nearly a point circle is obtained on the x-axis at which the value of the function is sufficiently near zero and hence giving the required root. Similarly in case-II, two circles C_1 and C_2 with centres at $(x_0, f(x_0))$, $(x_0 + h, f(x_0 + h))$ and radii $f(x_0), f(x_0 + h)$ respectively are drawn on the curve of the given function such that C_1 touches C_2 externally (or orthogonally). These two cases give different general formulae for successive approximation of the required root. The performance of the Secant method is the best as compared to these circle based techniques.

Key Words : Algebraic and Transcendental Equations; Newton's Method; Order of Convergence

1. INTRODUCTION

Almost all the iterative techniques of solution to an equation require the prior knowledge of one or more initial guesses for the desired root. An initial guess can usually be found by using intermediate value property of continuous function. Once an interval is known to contain a root, several classical procedures are available to refine it further. Some of them are Bisection, Regula-falsi, Modified Regula-falsi[2], Secant, Newton, Muller, Laguerre, Lehmer, Graffe's root squaring and Bairstow's etc ([1]-[5]). These are the most important methods used in creating computer programs for solving the algebraic and transcendental equations. These methods proceed with varying degree of speed and sureness towards the answer. Unfortunately, the methods that are guaranteed to converge plod away most slowly, while those who rush to the solution can also fly off to infinity without warning if measures are not taken to avoid such behaviour. Many attempts have been made to speed up convergence. Out of all the above methods, Newton's method is the best known rootfinding technique for a wide variety of problems and hence used effectively in finding solutions especially, scalar problems. It is not always the best method for a given problem, but its formal simplicity and great speed often lead it to be the first choice. It works very fast for real or complex roots once the neighborhood of a root is found. But this method is quite sensitive to the starting value and is not suitable in those cases where the graph of function is nearly horizontal where it crosses the x-axis.

Due to the drawback of one kind or the other in the earlier available methods, the need to develop still more efficient and easy-to-use rootfinding techniques is being felt very badly. In the present paper, we present here two cases of four different new techniques by introducing circles, which touch externally or intersect orthogonally while moving along the curve for solving the non-linear equations numerically and throw open avenues for further research on this aspect of the problem. The most distinct feature of these techniques is that they compete with Newton's technique and work in the situation where Newton's technique fails. Further, we present here some comparative results of Newton's method and the present ones regarding root finding by means of various examples. The discussion here is carried out exclusively for real and simple roots.

2. APPROXIMATION TECHNIQUES BY CIRCLES

Case 1 — Consider the equation

$$f(x) = 0, \qquad \text{... (2.1)}$$

whose one or more roots are to be found.

Let

$$y = f(x), \qquad \text{... (2.2)}$$

represents the graph of the function $f(x)$ and assume that an initial estimate x_0 is known for the desired root of the above eq. (2.1). A circle C_1 of raidus $f(x_0+h)$ is drawn with centre at any point $(x_0+h, f(x_0+h))$ on the curve of the function (2.2), where h is a small positive or negative quantity. Another circle C_2 with radius $f(x_0-h)$ and centre at $(x_0-h, f(x_0-h))$ is drawn on the curve of function $f(x)$ such that it touches (or intersects) the circle C_1 externally (or orthogonally).

Let

$$x_1 = x_0 + h, \; |h| << 1 \qquad \text{... (2.3)}$$

be the first approximation to the required root of eq. (2.1).

Now considering the following two cases

(*a*) External touch technique of circles

(*b*) orthogonal intersection technique of circles

(a) External touch technique of circles

The circle C_2 will touch circle C_1 externally if and only if sum of their radii = distance between their centres

i.e. if

$$f(x_0+h) + f(x_0-h) = \sqrt{(x_0+h-x_0+h)^2 + (f(x_0+h) - f(x_0-h))^2}, \qquad \text{... (2.4)}$$

i.e. if

$$4h^2 - 4f(x_0+h)f(x_0-h) = 0, \qquad \text{... (2.5)}$$

i.e. if

$$h^2 - \left\{ \left\{ f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \ldots \right\} \right.$$

$$\times\left\{f(x_0) - hf'(x_0) + \frac{h^2}{2!}f''(x_0) - \ldots\right\} = 0, \quad \ldots (2.6)$$

(Expanding by Taylor's theorem)

i.e. if
$$h^2\left\{1 + f'^2(x_0) - f(x_0)f''(x_0)\right\} = f^2(x_0), \quad \ldots (2.7)$$

(Retaining the terms upto $o(h^2)$)

which on simplification gives

$$h = \pm\frac{f(x_0)}{\sqrt{1 + f'^2(x_0) - f(x_0)f''(x_0)}}, \quad \ldots (2.8)$$

where h can be taken positive or negative according as x_0 lies in the left or right of true root or slope of the curve at $(x_0, f(x_0))$ is positive or negative. If x_0 lies in the left of true root, then h is taken as positive otherwise, negative. Therefore, from eq. (2.3), we get the first approximation to the root as

$$x_1 = x_0 \pm h,$$

$$x_1 = x_0 \pm \frac{f(x_0)}{\sqrt{1 + f'^2(x_0) - f(x_0)f''(x_0)}}. \quad \ldots (2.9)$$

The general formula for successive approximation is, therefore, given by

$$x_{n+1} = x_n \pm \frac{f(x_n)}{\sqrt{1 + f'^2(x_n) - f(x_n)f''(x_n)}}, \ (n \geq 0). \quad \ldots (2.10)$$

The sufficient condition for convergence in the interval containing the root is given by

$$f(x_n)f''(x_n) < 1 + f'^2(x_n). \quad \ldots (2.11)$$

(b) Orthogonal intersection technique of circles

The circle C_2 will intersect the circle C_1 orthogonally if and only if

sum of the square of their radii = square of the distance between their centres

i.e. if

$$f^2(x_0 + h) + f^2(x_0 - h) = (x_0 + h - x_0 + h)^2 + (f(x_0 + h) - f(x_0 - h))^2, \quad \ldots (2.13)$$

i.e. if
$$4h^2 - 2f(x_0 + h)f(x_0 - h) = 0, \quad \ldots (2.14)$$

i.e. if
$$4h^2 - 2\left\{\left\{f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \ldots\right\}\right.$$
$$\left.\times\left\{f(x_0) - hf'(x_0) + \frac{h^2}{2!}f''(x_0) - \ldots\right\}\right\} = 0, \quad \ldots (2.15)$$

(Expanding by Taylor's theorem)

(Retaining the terms upto $O(h^2)$)

which on solving gives

$$h = \pm \frac{f(x_0)}{\sqrt{2 + f'^2(x_0) - f(x_0) f''(x_0)}} \qquad \text{... (2.16)}$$

Therefore, first approximation to the root is given by

$$x_1 = x_0 \pm \frac{f(x_0)}{\sqrt{2 + f'^2(x_0) - f(x_0) f''(x_0)}}. \qquad \text{... (2.17)}$$

And the general iteration process is

$$x_{n+1} = x_n \pm \frac{f(x_n)}{\sqrt{2 + f'^2(x_n) - f(x_n) f''(x_n)}}, \quad (n \geq 0). \qquad \text{... (2.18)}$$

The sufficient condition of convergence for the above formula in the interval containing the root is

$$f(x_n) f''(x_n) \leq 2 + f'^2(x_n). \qquad \text{... (2.19)}$$

(c) Special circle's formulae

if we retain the first two terms of the Taylor's series in eq. (2.6) and (2.15), then the formulae (2.10) and (2.18) reduce respectively to equivalent general formulae

$$x_{n+1} = x_n \pm \frac{f(x_n)}{\sqrt{1 + f'^2(x_n)}}. \qquad \text{... (2.20)}$$

and

$$x_{n+1} = x_n \pm \frac{f(x_n)}{\sqrt{2 + f'^2(x_n)}}. \qquad \text{... (2.21)}$$

These special formulae compete with Newton's formula for most of the cases.

Case II — As in the previous case, again assume that an initial estimate x_0 is known for the desired root of the above eq. (2.1). A circle C_1 of radius $f(x_0)$ is drawn with centre at any point $(x_0, f(x_0))$ on the curve of the function (2.2), where h is a small positive or negative quantity. Another circle C_2 with radius $f(x_0 + h)$ and centre at $(x_0 + h, f(x_0 + h))$ is drawn on the curve of function $f(x)$ such that it touches (or intersects) the circle C_1 externally (or orthogonally).

Again considering the following two cases

(*a*) External touch technique of circles

(*b*) Orthogonal intersection technique of circles

(a) Method of external touch of circles

The circle C_2 will touch circle C_1 externally if and only if sum of the radii of two circles = distance between their centres i.e. if $$f(x_0+h)+f(x_0)=\sqrt{(x_0+h-x_0)^2+(f(x_0+h)-f(x_0))^2}, \quad \text{... (2.21)}$$

i.e. if $$h^2-4f(x_0)f(x_0+h)=0, \quad \text{... (2.22)}$$

i.e. if $$h^2-4f(x_0)\left\{f(x_0)+hf'(x_0)+\ldots\ldots\right\} = 0, \quad \text{... (2.23)}$$

(Expanding by Taylor's theorem)

i.e. if $$h^2-4hf(x_0)f'(x_0)-4f^2(x_0)=0, \quad \text{... (2.24)}$$

from which we get the value of h as

$$h=2f(x_0)\left(f'(x_0)\pm\sqrt{1+f'^2(x_0)}\right), \quad \text{... (2.25)}$$

with the sign so chosen as to make the numerator smallest possible. Because of the loss of significant errors implicit in this formula, we rationalize the numerator to obtain the new formula

$$h=\frac{-2f(x_0)}{f'(x_0)\pm\sqrt{1+f'^2(x_0)}}, \quad \text{... (2.26)}$$

in which the sign should be so chosen as to make the denominator largest in magnitude.

Using (2.6) in (2.3), we get the first approximation to the required root as

$$x_1=x_0-\frac{2f(x_0)}{f'(x_0)\pm\sqrt{1+f'^2(x_0)}}. \quad \text{... (2.27)}$$

The general formula for successive approximation is, therefore, given by

$$x_{n+1}=x_n-\frac{2f(x_n)}{f'(x_n)\pm\sqrt{1+f'^2(x_n)}}, \quad (n\geq 0). \quad \text{... (2.28)}$$

(b) Orthogonal intersection of circles

The condition that circle C_2 will intersect the circle C_1 orthogonally is

sum of the square of radii = square of the distance between centres

i.e. if

$$f^2(x_0+h)+f^2(x_0)=(x_0+h-x_0)^2+(f(x_0+h)-f(x_0))^2=0, \quad \text{... (2.29)}$$

i.e. if $$h^2-2f(x_0)f(x_0+h)=0, \quad \text{... (2.30)}$$

o.e. if $$h^2-2f(x_0)\left\{f(x_0)+hf'(x_0)+\ldots\right\}=0, \quad \text{... (2.31)}$$

(Retaining the terms up to second)

which on solving gives

$$h=f(x_0)\left\{f'^2(x_0)\pm\sqrt{f'(x_0)+2}\right\} \quad \text{... (2.32)}$$

Therefore, first approximation to the root after rationalization is

$$x_1 = x_0 - \frac{2f(x_0)}{f'(x_0) \pm \sqrt{f'^2(x_0) + 2}}. \qquad \text{... (2.33)}$$

And the general iteration process is

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) \pm \sqrt{2 + f'^2(x_n)}}. \qquad \text{... (2.34)}$$

Now we are presenting here the geometrical interpretation of the circle methods of case-II. Geometrical interpretation of case-I can be obtained similarly.

3. Geometrical Interpretation (Case-II)

Let x_0 be an initial approximation to the desired root and $f(x_0)$ be the corresponding value of the function. Draw circle C_1 with centre $(x_0, f(x_0))$ on the curve of $f(x)$ having radius as $f(x_0)$. Draw another circle C_2 with centre $(x_0 + h, f(x_0 + h))$ and radius $f(x_0 + h)$, where h is a small positive or negative quantity, such that C_2 touches (or intersects) C_1 externally (or orthogonally). The first approximation to the root is taken as the abscissa of centre of C_2, where h is obtained by the external (or orthogonal) touch (or intersection) condition of circles. This process is repeated until we get a circle with radius sufficiently near to zero, which may be a point circle of the intersection of the curve with x-axis and hence giving the required approximate root. Geometrically, these methods consist in replacing the part of the curve between the point $(x_0, f(x_0))$ and x-axis by means of circles.

4. Error Estimation and Order of Convergence

The order of convergence of any numerical technique is one of the powerful tools of measuring the computational efficiency of the corresponding algorithms. The convergence (or divergence) conditions of any numerical technique strongly depend on properties like the continuity, monotonicity, etc. of the given function in a neighbourhood of the solution and of the quality of initial guess of the true root. Therefore, it is advantageous to know how rapidly the error in given iterative method decreases, assuming the convergence of the method. Now we are presenting here the mathematical proof for the order of convergence of the iterative technique (2.21) of case-I and the order of convergence for the remaining can be proved similarly.

Let x^* be a true solution of the given eq. (2.1) and assume that x_n is a close approximation of the true solution x^*. If $f(x)$ is continuous and continuously differentiable in the neighborhood of x^*, then the Taylor series expansion of $f(x)$ about x^* in this neighborhood is

$$f(x^*) = f(x_n) + (x^* - x_n) f'(x_n) + \frac{(x^* - x_n)^2}{2!} f''(x_n) + \ \dots . \qquad \text{... (4.1)}$$

Now we are going to discuss two cases depending upon the root lying in the left or right to x^*.

(*i*) If $x_n < x^*$

From the definition of error estimation at the nth iteration, we have

$$\varepsilon_n = x^* - x_n, \qquad \text{... (4.2)}$$

Similarly, the error estimation at the $(n+1)$th is given by

$$\varepsilon_{n+1} = x^* - x_{n+1}, \qquad \text{... (4.3)}$$

Substituting (4.2) and (4.3) in the iterative formula (2.21), we get

$$\varepsilon_{n+1} = \varepsilon_n + \frac{f(x_n)}{\sqrt{2 + f'^2(x_n)}}, \quad (n \geq 0). \qquad \text{... (4.5)}$$

Now if the numerical value of $f'(x)$ is small i.e. $|f'(x)| < 2$, then $\sqrt{2+f'^2(x_n)} \cong \sqrt{2}\left(1 + \frac{1}{4}f'^2(x_n)\right)$.

Using (4.2) and (4.5) in (4.4) and expanding by Taylor's theorem, we get after some simplifications and neglecting higher order terms of ε_n

$$\varepsilon_{n+1} = \varepsilon_n \left[1 - \frac{f'(x^*) - \frac{1}{2}\varepsilon_n f''(x^*) + \ldots\ldots\ldots}{\sqrt{2}\left(1 + \frac{1}{4}(f'^2(x^*) - 2\varepsilon_n f'(x^*) f''(x^*) + \ldots\right)} \right], \qquad \text{... (4.6)}$$

which shows that the method is linearly convergent.

Now if $|f'(x)| \geq 2$, then $|f'(x)|^2 > 2$

and therefore

$$\frac{2}{|f'(x)|^2} < 1. \qquad \text{... (4.7)}$$

Rewriting the formula (4.4), we get

$$\varepsilon_{n+1} = \varepsilon_n + f(x_n)\left\{\frac{1}{f'(x_n)} - \frac{1}{f'^3(x_n)}\right\}. \qquad \text{... (4.8)}$$

From inequality (4.7) it is very clear that

$$\left\{\frac{1}{f'(x_n)} - \frac{1}{f'^3(x_n)}\right\} \cong \frac{1}{f'(x_n)},$$

hence, (4.8) takes the form

$$\varepsilon_{n+1} = \varepsilon_n + \frac{f(x_n)}{f'(x_n)}. \qquad \text{... (4.9)}$$

From Taylor's formula (4.1) we have

$$f(x^*) = f(x_n) + \varepsilon_n f'(x_n) + \frac{\varepsilon_n^2}{\angle 2} f''(x_n) + \ldots.. = 0, \qquad \text{... (4.10)}$$

This implies

$$\frac{f(x_n)}{f'(x_n)} = -\varepsilon_n - \frac{\varepsilon_n^2}{2}\frac{f''(x_n)}{f'(x_n)}. \qquad \dots (4.11)$$

Therefore, (4.9) using (4.11) finally gives

$$\varepsilon_{n+1} \cong -\frac{\varepsilon_n^2}{2}\frac{f''(x_n)}{f'(x_n)}, \qquad \dots (4.12)$$

which shows that if $f''(x_n) \neq 0$, then the method converges quadratically.

(*ii*) If $x_n > x^*$

i.e. if the root lies on the right of the true root, then

$$\varepsilon_n = x_n - x^*$$

The iterative formula (2.21) takes the form

$$x_{n+1} = x_n - \frac{f(x_n)}{\sqrt{2 + f'^2(x_n)}}. \qquad \dots (4.13)$$

If $|f'(x_n)| < 2$, then repeating the process of case (*i*), we can easily show that it converges linearly. And if $|f'(x_n)| \geq 2$, it can be proved that the method has quadratic convergence. Similarly we can prove that the iterative formula (2.20) has linear convergence if $|f'(x_n)| < 1$, otherwise quadratic. The order of convergence for the remaining can be proved similarly.

6. CONCLUDING REMARKS

1. From the foregoing study, it is observed that if $|f'(x_n)| > 2$ (for circle formula (2.21)), then these special circle numerical techniques of case-I and case-II have approximately the same order of convergcne as that of Newton's technique for real and simple roots of any nonlinear equation. If $|f'(x_n)| \leq 2$, then these techniques have linear convergence. Orthogonal techniques are slower than external touch techniques of circles. However, the general formulae (2.10) and (2.18) of case-I have the fast rate of convergence than Newton's technique in most of the cases. But the major drawback of these techniques is that with each iteration, we have to evaluate three functional values viz. $f(x)$, $f'(x)$ and $f''(x)$, unlike Newton's technique.

2. It has been observed that Newton's technique will not necessary converge to the root that is nearest to the required root. For example, the eq. (1) of Table 1 (*a*) : $e^x - 1 - \cos(\pi x) = 0$, has an infinite number of roots. Of course the problem (2) is a cautionary problem, the Newton-Raphson method with an initial guess $x_0 = -1$ or -0.1 are found to be unsuitable due to numerical instability. It is therefore prudent to verify that the root found by the Newton-Raphson method is the desired one. This type of numerical instability has not been observed in these circle techniques.

3. Other major advantage of the above circle techniques over the Newton's technique is that, they do not fail if gradient of the function is zero or nearly zero during the iterative cycle.

4. If the root is required to four or five decimal degree of accuracy, then these circle techniques required the same number of iteration as Newton-Raphson and Secant technique. When the root is desired to several decimal places, circle techniques required more number of iteration than Newton-Raphson method.

5. These circle techniques are faster than that of bisection, Regula-falsi, etc. and are as simple as the earlier available numerical techniques. In routine problems, circle techniques are faster than secant technique. Finally, we conclude that if $|f'(x)| \geq 2$, then these proposed techniques are superior to Bisection, Regula-falsi, modified Regula-falsi and are as good as Newton-Raphson and Secant techniques.

6. The performance of the Secant method is the best as compared to these circle based techniques.

Case I — Comparison of the method(s) with Bisection, Regula-falsi, Newton-Raphson and Secant method for simple roots with termination criterion $|f(x)| < .5 \times 10^{-7}$

TABLE I:

(*a*) (1) Equation : $e^x - 1 - \cos(\pi x) = 0.$

Interval of Root	Initial Guess of Root	Bisection	Regula-falsi	N.R method	External	Orthogonal	Secant
(–1, 0)	–1	–0.699317	–0.699317	0.369256	–0.699317	–0.699317	–0.699317
	–0.10			–7.318241	0.369256	0.369256	
(0, 1)	0.00	0.369256	0.369256	0.369256	0.369256	0.369256	0.36256
	1.00			0.369256	0.369256	0.369256	

No. of iterations

Bisection	Regula-Falsi	Newton-Raphson	External	Orthogonal	Secant
26	7	10	9	10	6
		72	8	8	
25	7	7	7	8	6
		6	8	8	

(*b*) (2) Equation : $x^2 - (1 - x)^5 = 0.$

Interval of Root	Initial Guess of Root	Bisection	Regula-falsi	N.R. Method	External	Orthogonal	Secant
(0, 1)	0	0.345955	0.345955	0.345955	0.345955	0.345955	0.345955
	1			0.345955	0.345955	0.345955	

No. of iterations

Bisection	Regula-Falsi	Newton-Raphson	External	Orthogonal	Secant
25	20	6	11	13	7
		5	10	14	

(*c*) (3) Equation $\int_0^x \frac{\sin(xt)}{t}\,dt - \frac{1}{2} = 0.$

Interval of Root	Initial Guess of Root	Bisection	Regula-Falsi	N.R. Method	External	Orthogonal	Secant
(0, 1)	.1	0.712175	0.712175	–0.712175	0.712175	0.712175	0.712175
	1			0.712175	0.712175	0.712175	

No. of iterations

Bisection	Regula-Falsi	Newton-Raphson	External	Orthogonal	Secant
26	10	61	11	16	7
		4	11	15	

Case-II — Comparison of the method(s) with Bisection, Regula-falsi, Newton-Raphson and Secant method for simple roots with termination criterion

$$|f(x)| < .5 \times 10^{-7}.$$

TABLE II

(*d*) (1) Equation: $e^x - 1 - \cos(\pi x) = 0.$

Interval of Root	Initial Guess of Root	Bisection	Regula-falsi	N.R method	External	Orthogonal	Secant
(–1, 0)	–1	–0.699317	–0.699317	0.369256	–0.699317	–0.699317	–0.699317
	–0.10			–7.318241	0.369256	–0.699317	
(0, 1)	0.00	0.369256	0.369256	0.369256	0.369256	0.369256	0.36256
	1.00			0.369256	0.369256	0.369256	

No. of iterations

Bisection	Regula-falsi	Newton-Raphson	External	Orthogonal	Secant
26	7	10	8	9	6
		72	11	10	
25	7	7	7	7	6
		6	7	8	

(*e*) (3) Equation : $x^2 - (1-x)^5 = 0.$

Interval of Roots	Initial Guess of Root	Bisection	Regula-falsi	N.R method	External	Orthogonal	Secant
(0, 1)	0	0.345955	0.345955	0.345955	0.345955	0.345955	0.345955
	1			0.345955	0.345955	0.345955	

No. of iterations

Bisection	Regula-falsi	Newton-Raphson	External	Orthogonal	Secant
25	20	6	9	10	7
		5	8	10	

(*f*) (3) Equation : $\int_0^x \frac{\sin(xt)}{t}\,dt - \frac{1}{2} = 0.$

Interval of Root	Initial Guess of Root	Bisection	Regula-falsi	N.R Method	External	Orthogonal	Secant
(0, 1)	.1	0.712175	0.712175	–0.712175	0.712175	0.712175	0.712175
	1			.712175	0.712175	0.712175	

No. of iterations

Bisection	Regula-Falsi	Newton-Raphson	External	Orthogonal	Secant
26	10	61	9	9	7
		4	9	11	

ACKNOWLEDGEMENT

We would like to express our sincere thanks to Dr. R. C. Chauhan, Director, SLIET for supporting and providing necessary facilities. The authors wish to thank anonymous referee for making helpful suggestions.

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