Indian J. pure appl. Math., **34**(9) : 1351-1357, September 2003

ON SOME NEW SEQUENCE SPACES OF FUZZY NUMBERS

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(*Received* 13 *December* 2002; *accepted* 11 *May* 2003)

In this paper, we introduce and study some new sequence spaces of fuzzy numbers generated by non-negative regular matrix $A = (a_{nk})$ $(n, k = 1, 2)$.

Key Words : Fuzzy Numbers; Sequence Spaces; Paranorm; Infinite Matrix

1. INTRODUCTION AND PRELIMINARIES

Let D be the set of all bounded intervals $A = [\underline{A}, \overline{A}]$ on the real line $\mathbb{R}$. For $A, B \in D$, define

$$A \le B \text{ if and only if } \underline{A} \le \underline{B} \text{ and } \overline{A} \le \overline{B},$$

$$d(A, B) = \max\{\underline{A} - \underline{B}, \overline{A} - \overline{B}\}.$$

Then it can be easily see than d defines a metric on D (cf [1]) and (D, d) is a complete metric space.

A fuzzy number is a fuzzy subset of the real line $\mathbb{R}$ which is bounded, convex and normal. Let $L(\mathbb{R})$ denote the set of all fuzzy numbers which are upper semicontinuous and have compact support, i.e. if $X \in L(\mathbb{R})$ then for any $\alpha \in [0, 1], X^\alpha$ is compact where

$$X^\alpha = \begin{cases} t : X(t) \ge \alpha & \text{if } 0 < \alpha \le 1, \\ t : X(t) > 0 & \text{if } \alpha = 0, \end{cases}$$

For each $0 < \alpha \le 1$, the α-level set X^α is a nonempty compact subset of $\mathbb{R}$. The linear structure of $L(\mathbb{R})$ includes addition $X + Y$ and scalar multiplication λX, (λ a scalar) in terms of α-level sets, by

$$[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha \text{ and } [\lambda X]^\alpha = \lambda [X]^\alpha,$$

for each $0 \le \alpha \le 1$.

Define a map $\bar{d} : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ by

$$\bar{d}(X, Y) = \sup_{0 \le \alpha \le 1} d(X^\alpha, Y^\alpha).$$

For $X, Y \in L(\mathbb{R})$ define $X \leq Y$ if and only if $X^\alpha \leq Y^\alpha$ for any $\alpha \in [0, 1]$. It is known that $L(\mathbb{R}), \bar{d})$ is a complete metric space (cf [2]).

We will need the following definitions (cf [2]).

Definition 1.1 — A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set $\mathbb{N}$ of natural numbers into $L(\mathbb{R})$. The fuzzy number X_k denotes the value of the function at $n \in \mathbb{N}$ and is called the nth term of the sequence. We denote by $w(F)$ the set of all sequences $X = (X_k)$ of fuzzy numbers.

Definition 1.2 — A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to a fuzzy number X_0, written as $\lim_k X_k = X_0$, if for every $\varepsilon > 0$ there exists a positive integer N_0 such that

$$\bar{d}(X_k, X_0) < \varepsilon \text{ for } k > N_0.$$

Let $c(F)$ denote the set of all convergent sequences of fuzzy numbers.

Definition 1.3 — A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded. We denote by $l_\infty(F)$ the set of all bounded sequences of fuzzy numbers.

It is straightforward to see that

$$c(F) \subset l_\infty(F) \subset w(F).$$

In [5], it was shown that $c(F)$ and $l_\infty(F)$ are complete metric spaces. In [4], we have shown that $L(\mathbb{R})$ and $w(F)$ are Frechet spaces and $c(F)$ and $l_\infty(F)$ are Banach spaces.

For further studies we refer [7], [8] and [9].

In this paper we define some new sequence spaces of fuzzy numbers by using regular matrices $A = (a_{nk})$, $(n, k = 1, 2, \ldots)$. By the regularity of A we mean that the matrix which transform convergent sequence into a convergent sequence leaving the limit invariant (cf. Maddox[2]). We prove that these spaces are complete paranormed spaces.

By a paranorm we mean a function $g : E \to \mathbb{R}$ (where E is a linear space) which satisfies the following conditions :

(p.1) $g(0) = 0$,

(p.2) $g(x) \geq 0$ for all $x \in E$,

(p.3) $g(-x) = g(x)$ for all $x \in E$,

(p.4) $g(x + y) \leq g(x) + g(y)$ for all $x, y \in E$,

(p.5) If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda\ (n \to \infty)$ and (x_n) is a sequence of the elements of E with $g(x_n - x) \to 0\ (n \to \infty)$, then $g(\lambda_n x_n - \lambda x) \to 0\ (n \to \infty)$.

The space E is called the paranormed space with the paranorm g.

2. Some new Sequence Spaces

Recently Nuray and Savas[6] have defined the following space of sequences of fuzzy numbers.

$$l(p) = \left\{ X = (X_k) : \sum_k [\bar{d}(X_k, 0)]^{p_k} < \infty \right\},$$

where (p_k) is a bounded sequence of strictly positive real numbers. If $p_k = p$ for all k, then $l(p) = l_p$, the space due to Nanda[5].

In this paper we define the following :

$$F_0(p) = \left\{ X = (X_k) : n^{-1} \sum_{k=1}^{n} (\bar{d}(X_k, 0))^{p_k} \to 0 \ (n \to \infty) \right\},$$

$$F(p) = \left\{ X = (X_k) : n^{-1} \sum_{k=1}^{n} [\bar{d}(X_k, X_0)]^{p_k} \to 0 \ (n \to \infty) \right\},$$

$$F_\infty(p) = \left\{ X = (X_k) : \sup_n n^{-1} \sum_{k=1}^{n} [\bar{d}(X_k, 0)]^{p_k} < \infty \right\},$$

and call them respectively the spaces of sequences of fuzzy numbers which are strongly convergent to zero, strongly convergent to X_0 and strongly bounded.

We further generalize these spaces as follows. Let $A = (a_{nk})$ $(n, k = 1, 2, ...)$ be a non-negative regular matrix. We define

$$F_0[A, p] = \left\{ X = (X_k) : \sum_k a_{nk} [\bar{d}(X_k, 0)]^{p_k} \to 0 \ (n \to \infty) \right\},$$

$$F(A, p] = \left\{ X = (X_k) : \sum_k a_{nk} [\bar{d}(X_k, X_0)]^{p_k} \to 0 \ (n \to \infty) \right\},$$

$$F_\infty[A, p] = \left\{ X = (X_k) : \sup_n \left(\sum_k a_{nk} [\bar{d}(X_k, 0)]^{p_k} \right) < \infty \right\},$$

and call them respectively the spaces of strongly A-convergent to zero, strongly A-convergent to X_0 and strongly A-bounded sequences of fuzzy numbers $X = (X_k)$. We can specialize these spaces as follows.

(*i*) If $a_{nk} = \begin{cases} 1, & 1 \le k \le n \\ 0, & k > n \end{cases}$ then $F_\infty[A, p] = l(p)$, the space due to Nuray and Savas[6]

(*ii*) If $A = I$, the unit matrix, then we get another set of new sequence spaces for fuzzy numbers, i.e.

$$F_0[A, p] = c_0(F, p) = \left\{ X = (X_k) : [\bar{d}(X_k, 0)]^{p_k} \to 0 \ (k \to \infty) \right\},$$

$F[A, p] = c(F, p)$ and $F_\infty [A, p] = l_\infty (F, p)$; which on further taking $p_k = p$ for all k, are reduced to $c_0(F), c(F)$ and $l_\infty(F)$ respectively (cf. [4])

(*iii*) If $A = (a_{nk})$ is a Cesàro matrix of order 1, i.e.

$$a_{nk} = \begin{cases} 1/n, & k \le n \\ 0, & k > n \end{cases}$$

then $F_0[A, p] = F_0(p), F[A, p] = F(p), F_\infty[A, p] = F_\infty(p)$ and further on taking $p_k = p$ for all k, these are reduced to the following new sequence spaces :

$$F_0^p = \left\{ X = (X_k) : n^{-1} \sum_{k=1}^{n} [\bar{d}(X_k, 0)]^p \to 0 \ (n \to \infty) \right\},$$

$$F^p = \left\{ X = (X_k) : n^{-1} \sum_{k=1}^{n} [\bar{d}(X_k, X_0)]^p \to 0 \ (n \to \infty) \right\},$$

$$F_\infty^p = \left\{ X = (X_k) : \sup_n \left(n^{-1} \sum_{k=1}^{n} [\bar{d}(X_k, 0)]^p \right) < \infty \right\},$$

A metric $\bar{d}$ on $L(\mathbb{R})$ is said to be a translation invariant if $\bar{d}(X+Z, Y+Z) = \bar{d}(X, Y)$ for $X, Y, Z \in L(\mathbb{R})$.

Proposition 2.1 — If $\bar{d}$ is a translation invariant metric on $L(\mathbb{R})$ then

(*i*) $\bar{d}(X+Y, 0) \le \bar{d}(X, 0) + \bar{d}(Y, 0)$,

(*ii*) $\bar{d}(\lambda X, 0) \le |\lambda| \bar{d}(X, 0), |\lambda| > 1$.

PROOF : (*i*) By the triangle inequality

$$\bar{d}(X = Y, 0) \le \bar{d}(X+Y, Y) = \bar{d}(Y, 0) = \bar{d}(X+Y, Y+0) + \bar{d}(Y, 0) = \bar{d}(X, 0) + \bar{d}(Y, 0)$$

since $\bar{d}$ is a translation invariant.

(*ii*) It follows easily by using (i) and induction.

If $\bar{d}$ is a translation invariant, we have the following straightforward results.

Proposition 2.2 — Let (p_k) be a bounded sequence of strictly positive real numbers. Then $F_0[A, p], F[A, p]$ and $F_\infty[A, p]$ are linear spaces over the complex field $\mathbb{C}$.

Proposition 2.3 —$F_0[A, p], F[A, p]$ and $F_\infty[A, p]$ are absolutely convex subsets of the space $w(F)$ of all sequences of fuzzy numbers, where $0 < p_k \le 1$.

3. MAIN RESULTS

***Theorem* 3.1** — *$F_0[A, p]$ and $F[A, p]$ are complete paranormed spaces with the paranorm g defined by*

$$g(X) = \sup_n \left(\sum_k a_{nk} [\bar{d}(X_k, 0)]^{p_k} \right)^{1/M}$$

where $M = \max \{1, \sup_k p_k\}$, where $\bar{d}$ is a translation invariant.

PROOF : Clearly $g(\theta) = 0$, $g(-X) = g(X)$. it can also be seen easily that $g(X + Y) \leq g(X) + g(Y)$ for $X = (X_k), Y = (Y_k)$ in $F_0[A, p]$ since $\bar{d}$ is a translation invariant.

Now for any scalar λ, we have $|\lambda|^{p_k} < \max \{1, |\lambda|^H\}$, where $H = \sup_k p_k < \infty$, so

$$g(\lambda X) < (\sup_k |\lambda|^{p_k})^{1/M} \cdot g(X) \text{ on } F_0[A, p].$$

Hence $\lambda \to 0, X \to \theta$ implies $\lambda X \to \theta$ and also $X \to \theta, \lambda$ fixed implies $\lambda X \to \theta$. Now let $\lambda \to 0, X$ fixed. For $|\lambda| < 1$ we have

$$\sum_k a_{nk} [\bar{d}(\lambda X_k, 0)]^{p_k} < \varepsilon \text{ for } n > N(\varepsilon).$$

Also, for $1 \leq n \leq N$, since $\sum_k a_{nk} [\bar{d}(X_k, 0)]^{p_k} < \infty$, there exists m such that

$$\sum_{k=m}^{\infty} a_{nk} [\bar{d}(\lambda X_k, 0)]^{p_k} < \varepsilon$$

Taking λ small enough we then have

$$\sum_k a_{nk} [\bar{d}(\lambda X_k, 0)]^{p_k} < 2\varepsilon \text{ for all } n.$$

Hence $g(\lambda X) \to 0$ as $\lambda \to 0$. Therefore g is a paranorm on $F_0[A, p]$. Completeness can be proved on the same lines as in [6] for $l(p)$.

The case $F[A, p]$ has exactly the same proof.

Similarly we can prove the following

***Theorem* 3.3** — *If* $0 < \inf_k p_k \leq \sup_k p_k < \infty$, *then* $F_\infty[A, p]$ *is a paranormed space with the above paranorm.*

***Theorem* 3.3** — *Let* $0 < p_k \leq q_k$ *and* (q_k/p_k) *be bounded. Then* $F[A, q] \subseteq F[A, p]$.

PROOF : Let $X = (X_k) \in F[A, q]$. Put $t_k = [\bar{d}(X_k, X_0)]^{q_k}$ and $\lambda_k = q_k/p_k$. Of course $0 < \lambda_k \leq 1$. Take $0 < \lambda < \lambda_k$. Define $u_k = \begin{cases} t_k, & t_k \geq 1 \\ 0, & t_k < 1 \end{cases}$ and $v_k = \begin{cases} 0, & t_k \geq 1 \\ t_k, & t_k < 1 \end{cases}$. Then we have $t_k = u_k + v_k$ and $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$ and it follows that $u_k^{\lambda k} \leq u_k \leq t_k$ and $v_k^{\lambda k} \leq v_k^{\lambda}$. Therefore

$$\sum_k a_{nk} [\bar{d}(X_k, X_0)]^{p_k} = \sum_k a_{nk} t_k^{\lambda k} = \sum_k a_{nk} (u_k^{\lambda k} + vk^{\lambda k})$$

$$\sum_k a_{nk} t_k + \sum_k a_{nk} v_k^{\lambda} \to 0 \ (n \to \infty).$$

Since $X \in F[A, q]$, $\sum_k a_{nk} t_k$ is convergent, and since $v_k < 1$ and A is regular, $\sum_k a_{nk} v_k^{\lambda}$ is also convergent. Hence $X \in F[A, p]$, i.e. $F[A, q] \subseteq F[A, p]$.

***Theorem* 3.4** — *Let* m_1 *and* m_2 *be constants such that* $0 < m_1 \le p_k \le m_2$. *Then* $X \in c(F)$ *implies* $X \in F[A, p]$ *with*

$\lim_k X_k = F[A, p] - \lim_k X_k = X_0$ *if and only if* $A = (a_{nk})$ *transforms null sequence into null sequence, i.e.* $A \in (c_0(F), c_0(F))$.

PROOF : *Sufficiency.* Since $p_k \ge m_1 > 0$, we have

$$[\bar{d}(X_k, X_0)] \to 0 \Rightarrow [\bar{d}(X_k, X_0)]^{p_k} \to 0$$

Hence $A \in (c_0(F), c_0(F))$ implies that $\sum_k a_{nk} [\bar{d}(X_k, X_0)]^{p_k} \to 0 \ (n \to \infty)$, i.e. $X \in c(F) \Rightarrow X \in F[A, p]$ with the same limit X_0.

Necessity — Suppose $[\bar{d}(X_k, X_0)] \to 0 \Rightarrow \sum_k a_{nk} [\bar{d}(X_k, X_0)]^{p_k} \to 0 \ (n \to \infty)$. Then (3.4.1)

$[\bar{d}(X_k, X_0)]^{q_k} \to 0 \ (k \to \infty) \Rightarrow \sum_k a_{nk} [\bar{d}(X_k, X_0)] \to 0$, where $q_k = 1/p_K$. Since $q_k \ge 1/m_2 > 0$, (3.4.2)

$[\bar{d}(X_k, X_0)] \to 0 \ (k \to \infty) \Rightarrow [\bar{d}(X_k, X_0)] \to 0 \ (n \to \infty)$.

Therefore by (3.4.1) and (3.4.2), we have

$$[\bar{d}(X_k, X_0)] \to 0 \ (k \to \infty) \Rightarrow \sum_k a_{nk} [\bar{d}(X_k, X_0)] \to 0 \ (n \to \infty).$$

Hence $\quad A \in (c_0(F), c_0(F))$.

ACKNOWLEDGEMENT

This paper was completed when the first author visited Sakarya University for 8 weeks (Sep. and Oct. 2001) under the U.N. Programme TOKTEN-UNISTAR. The author is very much thankful to TUBITAK and Sakarya University for financial support and hospitalities provided to him during the visit.

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