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PERTURBATION THEORY FOR DAMPED NONLINEAR SYSTEMS WITH VARYING COEFFICIENTS

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Krylov-Bogoliubov-Mitropolskii method is modified and applied to certain damped nonlinear systems with slowly varying coefficients. The method is illustrated by an example.

Key Words : Damped Oscillation - Strong Damping - Varying Coefficients

1. INTRODUCTION

Most of the perturbation methods were developed to find periodic solutions of nonlinear systems, but Krylov and Bogoliubov[1] introduced a perturbation method to discuss transients in equation

$$\ddot{x} + \omega_0^2 x = -\varepsilon f(x, \dot{x}). \qquad \ldots (1)$$

Then the method was amplified and justified by Bogoliubov and Mitropolskii[2] and extended by Mitropolskii[3] to similar systems involving some slowly varying coefficients described by

$$\ddot{x} + \omega_0^2(\tau)\, x = -\varepsilon f(x, \dot{x}, \tau), \quad \tau = \varepsilon t. \qquad \ldots (2)$$

On the other hand, Popov[4] extended the method to nonlinear damped oscillatory systems with constant coefficients

$$\ddot{x} + 2k\dot{x} + \omega^2 x = -\varepsilon f(x, \dot{x}), \quad k > 0. \qquad \ldots (3)$$

Sometimes Popov's[4] solution is useless when damping coefficient, k is much greater than 0 or the damping force approaches toward the critical damping. Recently, Shamsul[5] has found an approximate solution of (3) as a complement of Popov's[4] solution; *i.e.*, Shamsul's solution gives desired results for the cases where Popov's[4] solution was unable. As a limit $k \to \omega$, Shamsul's[5] solution can be brought to a critically damped solution. Murty, Deekshatulu and Krisna[6] used Popov's[4] solution as an over-damped solution replacing only the trigonometric functions by similar type of hyperbolic functions. Murty[7] used the solution obtained in[6] as a unified solution for both under-damped and over-damped systems and later Bojadziev and Edwards[8] studied some under-damped and over-damped systems with slowly varying coefficients.

In accordance to Shamsul's[5] observation, Bojadziev and Edward's[8] solution is not useful for strong damping forces (especially when the damping force is slightly smaller than the critical damping force). In the present paper, an approximate solution is found for strong damping effects and with slowly varying coefficients in where Bojadziev and Edwards[8] solution is unable to give desired results. Actually the new solution (concerned of this paper) is a complement of Bojadziev and Edwards'[8] solution.

2. The Method

Let us consider the nonlinear differential system with slowly varying coefficients

$$\ddot{x} + 2k(\tau)\dot{x} + \omega^2(\tau)\, x = -\varepsilon f(x, \dot{x}, \tau). \qquad \dots (4)$$

For $\varepsilon = 0$, eq. (4) has two eigen-values, $-k(\tau_0) \pm i\,\omega_0(\tau_0)$, $\omega_0^2(\tau_0) = \omega^2(\tau_0) - k^2(\tau_0)$ and the unperturbed solution of (4) becomes

$$x(t, 0) = \alpha_0\, e^{-k(\tau_0)\,t} \cos[\omega_0(\tau_0)\, t + \varphi_0], \qquad \dots (5)$$

where α_0 and φ_0 are arbitrary constants and τ_0 represents the value of τ when $\varepsilon = 0$.

When $\varepsilon \neq 0$, we propose an approximate solution of (4) in the form

$$x(t, \varepsilon) = \alpha \cos\psi + \varepsilon\, u_1(\alpha, \psi, \tau) + \varepsilon^2 \dots, \qquad \dots (6)$$

where α and ψ satisfy first order differential equations

$$\dot{\alpha} = -k(\tau) + \varepsilon A_1(\alpha, \psi, \tau) + \varepsilon^2 \dots,$$

$$\dot{\psi} = \omega_0(\tau) + \varepsilon B_1(\alpha, \psi, \tau) + \varepsilon^2 \dots . \qquad \dots (7)$$

It was restricted by Krylov, Bogoliubov and Mitropolskii[1, 2, 3] that the unknown functions $u_1, u_2, \dots$ exclude first harmonic terms and the functions $A_1, A_2, \dots$ and $B_1, B_2, \dots$ are independent of phase variable ψ. It has already been mentioned that Krylov, Bogoliubov and Mitropolskii studied nonlinear systems with small damping effects in which $k = 0$ or $k(\tau) = O(\varepsilon)$. Popov[4], Murty, Deekshatulu and Krisna[6], Murty[7] and Bojadziev and Edwards[8] strictly followed Krylov, Bogoliubov and Mitropolskii's assumptions that $A_1, A_2, \dots$ and $B_1, B_2, \dots$ are independent of ψ even the system is strongly damped, *i.e.*, $k = O(1)$. It has also been mentioned that Shamsul[5] (developed by himself[9]) investigated damped nonlinear systems with constant coefficients based on a critically damped solution and observed that Popov's solution is useful for some significant damping forces. In the case of strong damping effects, Shamsul[5] observed that $u_1, u_2, \dots$ are not independent of first harmonic terms and for this reason $A_1, A_2, \dots$ and $B_1, B_2, \dots$ depend on both amplitude, α and phase, ψ (see Appendix *A*).

Now differentiating (6) twice with respect to t, substituting for the derivatives $\dot{x}, \ddot{x}$ and x in (4), utilizing relation (7) and comparing the coefficients of ε, we obtain

$$\omega_0' \alpha \sin \psi - k' \alpha \cos \psi - \left(k \alpha \frac{dA_1}{d\alpha} - kA_1 - 2\,\omega_0\, \alpha B_1 \right)$$

$$\cos\ \psi - \left(2\,\omega_0 A_1 - k\,\alpha^2 \frac{dB_1}{d\alpha} \right) \sin \psi \qquad \ldots (8)$$

$$+ \left(-k\,\alpha \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \psi} \right)^2 u_1 + 2k \left(-k\,\alpha \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \psi} \right) u_1 + \omega^2 u_1 = -f^{(0)}(\alpha, \psi),$$

where $f^{(0)}(\alpha, \psi) = f(x_0, \dot{x}_0)$ and $x_0 = \alpha \cos \psi$.

In accordance to formal Krylov, Bogoliubov and Mitropolskii method[1, 2, 3] $f^{(0)}(\alpha, \psi)$ be expanded in a Fourier series

$$f^{(0)}(\alpha, \psi) = \sum_{n=0}^{\infty} F_n(\alpha) \cos n\,\psi + G_n(\alpha) \sin n\,\psi, \qquad \ldots (9)$$

and that u_1 has the terms in $\cos \psi$ and $\sin \psi$ missing, so that

$$u_1(\alpha, \psi) = U_0(\alpha) + \sum_{n=2}^{\infty} U_n(\alpha) \cos n\,\psi + V_n(\alpha) \sin n\,\psi. \qquad \ldots (10)$$

Substituting the expressions for $f^{(0)}$ and u_1 in (8), we obtain the following equations for A_1, B_1 and u_1 as

$$k' \alpha + k\,\alpha \frac{dA_1}{d\alpha} - kA_1 - 2\,\omega_0\, \alpha B_1 = F_1, \qquad \ldots (11)$$

$$-\omega_0' \alpha + 2\,\omega_0 A_1 - k\,\alpha^2 \frac{dB_1}{d\alpha} = G_1, \qquad \ldots (12)$$

and

$$\left(\left(-k\,\alpha \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \psi} + k \right)^2 + \omega_0^2 \right) u_1 = -F_0 - \sum_{n=2}^{\infty} F_n \cos n\,\psi + G_n \sin n\psi. \qquad \ldots (13)$$

The particular solution of (11)-(13) gives three unknown functions A_1, B_1 and u_1, which complete the determination of the first order Bojadziev and Edwards'[8] solution of (4). It is clear that both the functions A_1 and B_1 are independent of phase variable ψ in Bojadziev and Edwards' solution and u_1 excludes all first harmonic terms.

However, in accordance to Shamsul's[5] assumption A_1, B_1 and u_1 satisfy the following equations [instead of (11)-(13)].

$$-k' \alpha - k\alpha \frac{dA_1}{d\alpha} + kA_1 + 2\omega_0\, \alpha B_1 = -F_1 \cos^2(\psi - \omega t), \qquad \ldots (14)$$

$$\omega_0' \alpha - 2\omega_0 A_1 + k\alpha^2 \frac{dB_1}{d\alpha} = -G_1 \cos^2(\psi - \omega t), \quad \text{... (15)}$$

and

$$\left(\left(-k\alpha \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \psi} + k \right)^2 + \omega_0^2 \right) u_1 \quad \text{... (16)}$$

$$= -F_0 - (F_1 \cos \psi + G_1 \sin \psi) \sin^2(\psi - \omega t) - \sum_{n=2}^{\infty} F_n \cos n\psi + G_n \sin n\psi.$$

The particular solution of (14)-(16) gives three unknown functions A_1, B_1 and u_1. Thus the determination of the first order solution is clear. In this case A_1 and B_1 depend on both α and ψ and u_1 is not independent of first harmonic terms (for details derivations of (14)-(16), see Appendix A). In order to solve (14)-(16), we restrrict (according to Shamsul's[5] assumption) that $\cos^2(\psi - \omega t)$ and $\sin^2(\psi - \omega t)$ are constant in the right hand sides, since $\psi - \omega t$ is small. For sake of definiteness, we substitute $\psi - \omega t \approx \psi_0$ in right hand terms (14)-(16) [since the solution of the second equation of (7) takes the form $\psi = \psi_0 + \omega t + O(\varepsilon)$].

3. Example

As an example of the above procedure we may consider the Duffing equation with a linear damping and varying coefficients as

$$\ddot{x} + 2k(\tau)\dot{x} + \omega(\tau) x = -\varepsilon x^3. \quad \text{... (17)}$$

Here $f = x^3$, $f^{(0)} = \alpha^3 \cos^3 \psi = \frac{1}{4} \alpha^3 (3 \cos \psi + \cos 3\psi)$; so that non-zero coefficients are $F_1 = \frac{3}{4}$, $F_3 = \frac{1}{4}$. Substituting the values of F_1 and F_3 into (14)-(16) and solving them, we obtain

$$A_1 = -\frac{k' \alpha}{2\omega_0} + \frac{3k\alpha^3 \cos^2 \psi_0}{8\omega^2}, \quad B_1 = -\frac{\omega_0'}{2\omega_0} + \frac{3\omega_0 \alpha^2 \cos^2 \psi_0}{8\omega^2}, \quad \text{... (18)}$$

and

$$u_1 = \frac{3\alpha^3(-\cos \psi + \omega_0 \sin \psi / k) \sin^2 \psi_0}{16\omega^2}$$

$$+ \frac{\alpha^3(-(k^2 - 2\omega_0^2)\cos 3\psi + 3k\omega_0 \sin 3\psi)}{16\omega^2(k^2 + 4\omega_0^2)}. \quad \text{... (19)}$$

$$A_1 = -\frac{-k' \alpha}{2\,\omega_0} + \frac{3k\,\alpha^3}{8\,\omega^2},\; B_1 = \frac{-\,\omega_0'}{2\,\omega_0} + \frac{3\,\omega_0^2\,\alpha^2}{8\,\omega^2}, \qquad \text{... (20)}$$

and

$$u_1 = \frac{\alpha^3\,(-k^2 - 2\omega_0^2)\cos 3\psi + 3k\omega_0 \sin 3\psi}{16\,\omega^2\,(k^2 + 4\,\omega_0^2)}. \qquad \text{... (21)}$$

Now substituting the functional values of A_1, B_1 from (18) into (7), we obtain

$$\dot{\alpha} = -\,k\,(\tau)\,\alpha - \frac{\varepsilon k'\,\alpha}{2\,\omega_0} + \frac{3\,\varepsilon\,k\,\alpha^3 \cos^2 \psi_0}{8\,\omega_2},$$

$$\dot{\psi} = \omega\,(\tau) - \frac{\varepsilon\,\omega_0'}{2\,\omega_0} + \frac{3\,\varepsilon\,\omega_0\,\alpha^2 \cos^2 \psi_0}{8\,\omega^2}. \qquad \text{... (22)}$$

In general (22) is solved by a numerical method (see [10] for details). However, for some particular and important cases it has approximate solution, We may divide the first equation of (22) by a α and then integrate both equations of (22) with respect to t, by replacing α's unperturbed value, i.e., $\alpha_0\, e^{-k(\tau_0)t}$. Thus an approximate solution of (22) become

$$\ln\,(\alpha/\alpha_0) = -\,\varepsilon^{-1} \int_{\tau_0}^{\tau} k\,(\tau)\,d\,\tau - \int_{\tau_0}^{\tau} \frac{k'\,d\,\tau}{2\,\omega_0} + \varepsilon \int_0^t \frac{3k\,\alpha_0^2 \cos^2 \psi_0\, e^{-\,2k\,(\tau_0)\,t}\,dt}{8\,\omega^2\,(\varepsilon\,t)},$$

$$\psi = \psi_0 + \varepsilon^{-1} \int_{\tau_0}^{\tau} \omega\,(\tau)\,d\,\tau - \int_{\tau_0}^{\tau} \frac{\omega_0'\,d\,\tau}{2\,\omega_0} + \varepsilon \int_0^t \frac{3\,\omega_0\,\alpha^2 \cos^2 \psi_0\, e^{-\,2k\,(\tau_0)\,t}\,dt}{8\,\omega^2\,(\varepsilon\,t)}. \qquad \text{... (23)}$$

Therefore, the first order solution of (17) is

$$x\,(t,\,\varepsilon) = \alpha \cos\,\psi + \varepsilon\,u_1\,(a,\,\psi,\,\tau), \qquad \text{... (24)}$$

where α and ψ are given by (23) and u_1 is given by (19). Substituting the values of A_1, B_1 from (20) into (7) and solving them, Bojadziev and Edwards'[8] found the solution of (7) similar to (23).

4. Results and Discussions

In order to test the accuracy of an approximate solution, some authors compare perturbation solution to numerical solution (considered to be exact). With regard such a comparison concerning the presented KBM method of this paper, we refer to works of Murty, Deeshatulu and Krisna[6], Mendelson[11] and Shamsul[5]. In this paper, for different damping forces, $-\,2k\,(\tau_0)\,\dot{x}$, the new perturbation solution (24) and Bojadziev and Edwards'[8] solution have been compared to the corresponding numerical solution.

For a significant damping force $-e^{-0.5\tau}\dot{x}$, i.e., for $k(\tau) = 0.5\,e^{-0.5\tau}$ and for $\omega^2(\tau) = e^{-\tau}$ or $\omega_0(\tau) = \frac{\sqrt{3}}{2} e^{-\tau}$, $\varepsilon = 0.1$, $x(\varepsilon, t)$ has been computed by solution (24) with initial conditions $[x(0) = 1, \dot{x}(0) = 0]$. Then $x(\varepsilon, t)$ has been computed by Bojadziev and Edwards'[8] solution. Finally, the numerical solution (by Runge-Kutta fourth-order procedure) has been obtained and percentage errors have been calculated. All the results (with percentage errors) are shown in Table 1. From Table 1 it is seen that most of the time, t, errors of the results obtained by solution (24) and Bojadziev and Edwards' solution are less than 1% and on an average percentage errors of Bojadziev and Edwards' solution are less than those computed by solution (24).

If the damping force is increased, solution (24) shows a good coincidence with the numerical solution. Contrary, errors of Bojadziev and Edwards'[8] solution increase. For verifying this, $x(\varepsilon, t)$ has again been computed by solution (24), Bojadziev and Edwards' solution and *Runge-Kutta method* for a strong damping force $-\sqrt{2}\,e^{-0.5\tau}\dot{x}$, i.e., for $k(\tau) = \frac{1}{\sqrt{2}} e^{-0.5\tau}$ and for the same values of $\omega(\tau)$ and ε and with same initial conditions. The results (with percentage errors) are given in Table 2. Table 2 shows that percentage errors of solution (24) occur in an order of 1% (except at t = 3 and t = 8), while errors of Bojadziev and Edwards[8] solution are many times greater than 1% and almost twice of those obtained by solution (24). It is to be noted that in the case of a nonlinear system with constant coefficients, the error of a first order approximate solution occurs in an order of ε^2 due to a nonlinear term. However in the case of a nonlinear system with variable coefficients the error occurs due to both nonlinear terms and the varying properties of the coefficients. Thus error sometimes occurs much more than 1% for both solutions (i.e. solution (24) and Bojadziev and Edwards'[8] solution), yet the error of the present solution (24) is smaller than the previous solution obtained by Bojadziev and Edwards[8] when the damping force is strong.

TABLE I

t	x	x_{nu}	E (%)	x_{BE}	E_{BE} (%)
0	1.000000	1.000000	0.0000	1.000000	0.0000
1	0.642193	0.641101	0.1703	0.641957	0.1335
2	0.136156	0.134311	1.3744	0.134611	0.2241
3	–0.140639	–0.141801	–0.8195	–0.142327	0.3709
4	–0.192790	–0.192868	–0.0404	–0.193747	0.4558
5	–0.131988	–0.131299	0.5248	–0.132070	0.5872
6	–0.050690	–0.049836	1.7136	–0.050253	0.8367
8	0.033573	0.033807	–0.6922	0.033981	0.5147
10	0.027473	0.027256	0.7962	0.027513	0.9429

TABLE 2

t	x	x_{nu}	$E(\%)$	x_{BE}	E_{BE} (%)
0	1.000000	1.000000	0.0000	1.000000	0.0000
1	0.678614	0.677989	0.0922	0.684010	0.8881
2	0.265293	0.263317	0.7504	0.268888	2.1157
3	0.033068	0.031519	4.9145	0.034209	8.5345
4	-0.049626	-0.050305	-1.3498	-0.049713	-1.1768
5	-0.057106	-0.057181	-0.1311	-0.057548	0.6418
6	-0.039592	-0.039411	0.4593	-0.039994	1.4793
8	-0.007227	-0.007075	2.1484	-0.007350	3.8869
10	0.002924	0.002941	-0.5780	0.002932	-0.3060

5. CONCLUSION

An approximate solution of a second order nonlinear differential system with slowly varying coefficients has been found based on the works of Krylov, Bogoliubov and Mitropolskii[1, 2, 3]. In the case of *Duffing equation* (i.e. equation (17) in Sec. 3), the solution concerned of this paper (i.e. solution (24)) is allowable for the damping coefficients $k(\tau) > \frac{1}{2}\omega(\tau)$ while Bojadziev and Edwards'[8] solution is useful for $k(\tau) \le \frac{1}{2}\omega(\tau)$.

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Appendix A

In the case of similar type nonlinear systems with constant coefficients [i.e. for eq. (3) in Sec. 1], Shamsul[5] found an asymptotic solution in the form

$$x(t, \varepsilon) = e^{-kt}(a \cos \omega_0 t + b \sin \omega_0 t) + \varepsilon u_1(a, b, t) + \varepsilon^2 \ldots, \qquad \ldots (A.1)$$

where a and b are functions of t, defined by the first order differential equations

$$\dot{a} = \varepsilon A_1(a, b, t) + \varepsilon^2 \ldots,$$

$$\dot{b} = \varepsilon B_1(a, b, t) + \varepsilon^2 \ldots, \qquad \ldots (A.2)$$

Differentiatin (A.1) twice with respective to t, substituting for the derivatives $\dot{x}, \ddot{x}$ and x into (2), utilizing relations (A.2) and comparing the coefficients of various powers of ε, the following equation for the coefficient of ε was found in[5]

$$e^{-kt}\left(\left(\frac{\partial A_1}{\partial t} + 2\omega_0 B_1\right)\cos \omega_0 t + \left(-2\omega_0 A_1 + \frac{\partial B_1}{\partial t}\right)\sin \omega_0 t\right)$$

$$+ \frac{\partial^2 u_1}{\partial t^2} + 2k\frac{\partial u_1}{\partial t} + \omega^2 u_1 = -f^{(0)}(a, b, t), \qquad \ldots (A.3)$$

where, $\quad f^{(0)} = f(x_0, \dot{x}_0)$ and $x_0 = e^{-kt}(a \cos \omega_0 t + b \sin \omega_0 t)$.

In order to solve eq. (A.3) for the unknown functions A_1, B_1 and u_1, Shamsul[5] assumed that u_1 does not contain first harmonic terms, which are contained in $(a_0 \cos \omega_0 t)^r$ and $(a_0 \cos \omega_0 t)^r b_0 \sin \omega_0 t, r > 1$ of $f^{(0)}$. Under these restrictions the following equations of A_1, B_1 and u_1 for the *Duffing equation* (17) with constant coefficients were found

$$\frac{\partial A_1}{\partial t} + 2\omega_0 B_1 = -\frac{3a^3 e^{-2kt}}{4}, \qquad \ldots (A.4)$$

$$-2\omega_0 A_1 + \frac{\partial B_1}{\partial t} = -\frac{3a^2 b e^{-2kt}}{4}, \qquad \ldots (A.5)$$

and

$$\frac{\partial^2 u_1}{\partial t^2} + 2k\frac{\partial u_1}{\partial t} + \omega^2 u_1 = -e^{-3kt}\left(\frac{1}{4}a^{33} \cos 3\omega_0 t + \frac{3}{4}a^2 b \sin 3\omega_0 t^2\right.$$

$$\left. + 3ab^2 \cos \omega_0 t \sin^2 \omega_0 t + b^3 \sin^3 \omega_0 t\right), \qquad \ldots (A.6)$$

since

$$f^{(0)} = e^{-kt}(a^3 \cos^3 \omega_0 t + 3a^2 b \cos^2 \omega_0 t \sin \omega_0 t$$

$$+ 3ab^2 \cos \omega_0 t \sin^2 \omega_0 t + b^3 \sin^3 \omega_0 t)$$

$$= e^{-3kt}\left(\frac{3}{4}a^3\cos\omega_0 t + \frac{3}{4}a^2 b\sin\omega_0 t + \frac{1}{4}a^3\cos\omega_0 t + \frac{3}{4}a^2 b\sin 3\omega_0 t\right.$$

$$\left. + 3ab^2\cos\omega_0 t\sin^2\omega_0 t + b^3\sin^3\omega_0 t\right). \quad \text{... (A.7)}$$

The particular solution fo (A.4)-(A.6) is

$$A_1 = \frac{3a^2(ka+\omega_0 b)e^{-2kt}}{8\omega^2}, B_1 = \frac{3a^2(-\omega_0 a + kb)e^{-2kt}}{8\omega^2}. \quad \text{... (A.8)}$$

and

$$u_1 = -\frac{e^{-ekt}}{16\omega^2}\left(\frac{3ab^2(k\cos\omega_0 t - \omega_0\sin\omega_0 t)}{k} + \frac{3b^3(\omega_0\cos\omega_0 t + k\sin\omega_0 t)}{k}\right.$$

$$+ (a^3 - 3ab^2)\left(\frac{(k^2 - 2\omega_0^2)\cos 3\omega_0 t - 3k\omega_0\sin 3\omega_0 t}{k^2 + 4\omega_0^2}\right) \quad \text{... (A.9)}$$

$$\left. + (3a^2 b - b^3)\left(\frac{3k\omega_0\cos 3\omega_0 t + (k^2 - 2\omega_0^2)\sin 3\omega_0 t}{k^2 + 4\omega_0^2}\right)\right).$$

Substituting the values of A_1 and b_1 form (A.8) into (A.2), we get

$$\dot{a} = \frac{3\varepsilon a^2(ka+\omega_0 b)e^{-2kt}}{8\omega^2}, \dot{b} = \frac{3\varepsilon a^2(-\omega_0 a + kb)e^{-2kt}}{8\omega^2}. \quad \text{... (A.10)}$$

Shamsul[5] integrated (A.10) with respect to t, by assuming that a and b are constants in the right hand sides as :

$$a = a_0 + \frac{3\varepsilon a_0^2(ka_0 + \omega_0 b_0)(1 - e^{-2kt})}{16k\omega^2}, b = b_0 + \frac{3\varepsilon a_0^2(-\omega_0 a_0 = kb_0)(1 - e^{-2kt})}{16k\omega^2}.$$

... (A.10)

Therefore, in the case of constant coefficients, the first approximate solution of (17) is

$$x(t, \varepsilon) = e^{-kt}(a\cos\omega_0 t + b\sin\omega_0 t) + \varepsilon u_1, \quad \text{... (A.11)}$$

where a, b and u_1 are given by respectively (A.10) and (A.9).

The eq. (A.10) has an another approximated solution. One can replace a and b by $\beta\cos\varphi$ and $-\beta\sin\varphi$, and then one obtains

$$\dot{\beta} = \frac{3\varepsilon k\beta^3 e^{-2kt}\cos^2\varphi}{8\omega^2}, \dot{\varphi} = \frac{3\varepsilon\omega_0\beta^2 e^{-2kt}\cos^2\varphi}{8\omega^2}. \quad \text{... (A.12)}$$

Now under the transformations $\alpha = \beta e^{-kt}$, $\psi = \varphi + \omega t$, eq. (A.12) takes the form

$$\dot{\alpha} = -k\,\alpha + \frac{3\,\varepsilon k\,\alpha^3 \cos^2(\psi - \omega t)}{8\,\omega^2},\ \dot{\psi} = \omega_0 + \frac{3\,\varepsilon\,\omega_0\,\alpha^2 \cos^2(\psi - \omega t)}{8\,\omega^2}. \qquad \text{... (A.13)}$$

It is obvious that eq. (A.13) is similar to (22) (is Sec. 3). Moreover, in the case of constant coefficients, eq. (22) reduces to the exact form of (A.13). It is obvious that (A.13) has an approximated solution of the form (23) (discussed in Section 3) in where both $k' = \omega_0' = 0$.