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LIGHTLIKE HYPERSURFACES OF INDEFINITE SASAKIAN MANIFOLDS

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We study a new class of hypersurfaces called[6] lightlike hypersurfaces', of indefinite Sasakian manifolds. In particular, we prove that there exist no totally umbilical lightlike hypersurfaces of indefinite Sasakian space forms $\overline{M}(c)$ with $c \neq 1$ (Corollary 8).

Key Words : Lightlike Hypersurfaces; Indefinite Sasakian Manifolds; Screen Distribution; Indefinite Sasakian Space Form

0. INTRODUCTION

The theory of submanifolds of Riemannian (or semi-Riemannian) manifolds is one of the most important topics of differential geometry. In case the induced metric on the submanifold of semi-Riemannian manifolds is degenerate, the study becomes more difficult and is strikingly different from the study of nondegenerate submanifolds[5, 6, 7]. There exist few papers dealing with lightlike hypersurfaces[1, 2, 3, 5, 8, 10]. In this article we study a lightlike hypersurface when the ambient

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manifold is an indefinite Sasakian manifold. In particular, we prove that there exist no totally umbilical lightlike hypersurfaces of indefinite Sasakian space forms $\overline{M}(c)$ with $c \neq 1$. There are references of general lightlike submanifolds of semi-riemannian manifolds[6, 7, 9].

1. INDEFINITE SASAKIAN MANIFOLDS

Let $\overline{M}$ be a $(2m+1)$-dimensional manifold endowed with an almost contact structure $(\overline{\phi}, \xi, \eta)$, i.e. $\overline{\phi}$ is a tensor field of type (1.1), ξ is a vector field, and η is a 1-form satisfying

$$\overline{\phi}^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \ \eta \circ \overline{\phi} = 0. \qquad \dots (1.1)$$

Then $(\overline{\phi}, \xi, \eta, \overline{g}, \xi)$ is called a normal contact metric structure on $\overline{M}$, if $(\overline{\phi}, \xi, \eta)$ is an almost contact structure on $\overline{M}$, and $\overline{g}$ is a semi-riemannian metric on $\overline{M}$ such that for any vector field $\overline{X}, \overline{Y}$ on $\overline{M}$

$$\overline{g}(\xi, \xi) = \varepsilon, \ \varepsilon = +1 \text{ or } -1,$$

$$\eta(\overline{X}) = \varepsilon \overline{g}(\xi, \overline{X})$$

$$\overline{g}(\overline{\phi}\,\overline{X}, \overline{\phi}\,\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \varepsilon\, \eta(\overline{X})\, \eta(\overline{Y})$$

$$d\eta(\overline{X}, \overline{Y}) = \overline{g}(\overline{\phi}\,\overline{X}, \overline{Y})$$

$$(\overline{\nabla}_{\overline{X}} \overline{\phi})\, \overline{Y} = \varepsilon\, \eta(\overline{Y})\, \overline{X} - \overline{g}(\overline{X}, \overline{Y})\, \xi, \qquad \dots (1.2)$$

where $\overline{\nabla}$ indicates the Levi-Civita connection for a semi-Riemannian metric $\overline{g}$. In this case, we call $\overline{M}$ an *indefinite Sasakian manifold*. Note that we may assume that $\varepsilon = 1$ without loss of generality[11]. Hence, from now on we consider only the case $\varepsilon = 1$ in (1.2).

Remark : $(\overline{\phi}, \xi, \eta, \overline{g})$ on $\overline{M}$, is called an almost contact metric structure if an almost contact structure $(\overline{\phi}, \xi, \eta)$ satisfies the three conditions from above in (1.2). In this case,

$$(\overline{\nabla}_{\overline{X}} \overline{\phi})\, \overline{Y} = \eta(\overline{Y})\, \overline{X} - \overline{g}(\overline{X}, \overline{Y})\, \xi \qquad \dots (1.3)$$

implies

$$\overline{\nabla}_{\overline{X}}\, \xi = \overline{\phi}(\overline{X}), \qquad \dots (1.4)$$

$$\xi \text{ is Killing vector field}, \qquad \dots (1.5)$$

$$d\eta(\overline{X}, \overline{Y}) = \overline{g}(\overline{\phi}\,\overline{X}, \overline{Y}). \qquad \dots (1.6)$$

Let $\overline{M}$, be an indefinite Sasakian manifold. Let

$$D_p := \{ \overline{X} \in T_p(\overline{M});\ \eta(\overline{X}) = 0 \}.$$

For a non-null vector $\overline{X}$ in D_p, $\overline{X}$ and $\overline{\phi}\,\overline{X}$ span a non-degenerate 2-plane, and hence we can consider a sectional curvature

$$K(\overline{X}) := \frac{\overline{g}(\overline{R}(\overline{X}, \overline{\phi}\,\overline{X})\, \overline{\phi}\,\overline{X}, \overline{X})}{\overline{g}(\overline{X}, \overline{X})\, \overline{g}(\overline{\phi}\,\overline{X}, \overline{\phi}\,\overline{X}) - \overline{g}(\overline{X}\, \overline{\phi}\,\overline{X})^2},$$

where $\bar{R}$ denotes the curvature tensor of $\bar{M}$. If $K(\bar{X})$ is constant for all non-null vectors $\bar{X}$ in D_p, we call $\bar{M}$ to be of constant $\bar{\phi}$-sectional curvature at the point p. If $\bar{M}$ is of constant $\bar{\phi}$-sectional curvature at every point, $K(\bar{X})$ is a function of $p \in \bar{M}$, say $c(p)$. Moreover, if $c(p)$ is constant on $\bar{M}$, we call $\bar{M}$ to be constant $\bar{\phi}$-sectional curvature. In this case, we have for any tangent vectors $\bar{X}, \bar{Y}$ and $\bar{Z}$,

$$4\,\bar{R}(\bar{X},\bar{Y})\bar{Z} = (c+3)\{\bar{g}(\bar{Y},\bar{Z})\bar{X} - \bar{g}(\bar{X},\bar{Z})\bar{Y}\} + (c-1)\,\{\eta(\bar{X})\,\eta(\bar{Z})\bar{Y} - \eta(\bar{Y})\,\eta(\bar{Z})\bar{X} + \bar{g}(\bar{X},\bar{Z})\,\eta(\bar{Y})\,\xi - \bar{g}(\bar{Y},\bar{Z})\,\eta(\bar{X})\,\xi + \bar{g}(\bar{\phi}\bar{Y},\bar{Z})\,\bar{\phi}\bar{X} + \bar{g}(\bar{\phi}\bar{Z},\bar{X})\,\bar{\phi}\bar{Y} - 2\,\bar{g}(\bar{\phi}\bar{X},\bar{Y})\,\bar{\phi}\bar{Z}\}, \quad \ldots (1.7)$$

where we have assumed $\varepsilon = 1$. Thus, if $\bar{M}$ is of constant $\bar{\phi}$-sectional curvature $c = 1$, it is of constant curvature 1. An indefinite Sasakian manifold $\bar{M}$ is called an indefinite Sasakian space form if $\bar{M}$ has constant $\bar{\phi}$-sectional curvature c, and will be denoted by $\bar{M}(c)$[11].

Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote $\Gamma(E)$ the smooth sections of the vector bundle E.

2. Decompositions of Almost Contact Metric Manifolds

Let $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ be a $(2m+1)$-dimensional almost contact metric manifold, where $\bar{g}$ is a semi-Riemannian metric of index $\upsilon = s,\ 0 < s < 2m+1$. Let M be a hypersurface of $\bar{M}$. We consider for any $p \in M$

$$T_pM^{\perp} = \{E \in T_p\bar{M};\ \bar{g}_p(E, W),\ \forall\ W \in T_pM\},$$

and

$$\text{Rad } T_pM = T_pM \cap T_pM^{\perp},$$

where T_pM is a hyperplane of the semi-Euclidean space $(T_p\bar{M}, \bar{g}_p)$. M is said to be a lightlike (degenerate) hypersurface of $\bar{M}$ (or, the immersion of M into $\bar{M}$ is lightlike (degenerate)) if Rad $T_pM \neq 0$ $\{0\}$ at any point $p \in M$. In this case the induced metric g on M from the semi-Riemannian metric $\bar{g}$ on $\bar{M}$ is degenerate. Let (M, g) be a lightlike hypersurface of $(\bar{M}, \bar{g})$. Then, both the tangent space T_pM and the normal space $T_pM^{\perp}$ are degenerate subspaces of $T_p\bar{M}$ for each $p \in M$. If $E_p \in T_pM^{\perp}$, then we have $\bar{g}(E_p, E_p) = 0$, which implies $E_p \in T_pM$. Thus the normal bundle $TM^{\perp}$ becomes a one-dimensional distribution on M. Moreover $\bar{g}(\bar{\phi}E, E) = 0$, and so $\bar{\phi}E$ is tangent to M. Hence we get a distribution $\bar{\phi}(TM^{\perp})$ on M of rank 1. Now we choose a complementary distribution (called a screen distribution) SM to $TM^{\perp}$ in TM such that it contains $\bar{\phi}E$ and ξ. Since the screen distribution SM is nondegenerate, there exists a complementary orthogonal vector subbundle $SM^{\perp}$ to SM in $T\bar{M}$ over M. Hence we have the orthogonal decomposition

$$T\bar{M} = SM \perp SM^{\perp}. \quad \ldots (2.1)$$

Note that $TM^{\perp}$ is a lightlike vector subbundle of the nondegenerate vector bundle $SM^{\perp}$ with two-dimensional fibres. Then, for any local section $E \in \Gamma(TM^{\perp})$ there exists a unique lightlike local section $N \in \Gamma(SM^{\perp})$ (page 79, [7]) such that

$$\bar{g}(N, E) = 1. \quad \ldots (2.2)$$

Hence, N is not tangent to M and $\{E, N\}$ is a local field of frames of $SM^{\perp}$. Moreover we have a one-dimensional vector subbundle NM of $T\bar{M}$ over M, which is locally spanned by N. Then we set

$$SM^{\perp} = \mathrm{TM}^{\perp} \oplus NM, \quad \ldots (2.3)$$

where the decomposition is not orthogonal. Thus we have the following decompositions of $T\bar{M}$

$$T\bar{M} = SM \perp (TM^{\perp} \oplus NM) = TM \oplus NM. \quad \ldots (2.4)$$

Then N is orthogonal to $\bar{\phi}E$ and we have

$$\bar{g}(\bar{\phi}N, E) = -\bar{g}(N, \bar{\phi}, E) = 0, \bar{g}(\bar{\phi}N, N) = 0, \quad \ldots (2.5)$$

which means that $\bar{\phi}N$ is also tangent to M and belongs to SM. From (1.2) with $\varepsilon = 1$ we have

$$\bar{g}(\bar{\phi}E, \bar{\phi}N) = 1 \quad \ldots (2.6)$$

Therefore, $\bar{\phi}(TM^{\perp}) \oplus \bar{\phi}(NM)$ (direct sum but not orthogonal) is a non-degenerate vector subbundle of SM of rank 2. Then there exists a non-degnerrate distribution D_0 on M such that

$$SM = \{\bar{\phi}(TM^{\perp}) \oplus \bar{\phi}(NM)\} \perp D_0, \quad \ldots (2.7)$$

where $\xi \in \Gamma(D_0)$ and D_0 is an invariant distribution with respect to $\bar{\phi}$, i.e., $\bar{\phi}(D_0) = D_0$. Hence, from (2.1), (2.3), (2.4) and (2.7) we obtain the decompositions

$$TM = \{\bar{\phi}(TM^{\perp}) \oplus \bar{\phi}(NM)\} \perp D_0 \perp TM^{\perp} \quad \ldots (2.8)$$

and

$$T\bar{M} = \{\bar{\phi}(TM^{\perp}) \oplus \bar{\phi}(NM)\} \perp D_0 \perp \{TM^{\perp} \oplus NM\}, \quad \ldots (2.9)$$

respectively.

3. A Brief Review of Lightlike Immersions

In the present section, we first recall some results from the general theory of lightlike hypersurfaces[1,5,7].

Let $\bar{\nabla}$ be the Levi-Civita connection on an indefinite Sasakian manifold $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ and (M, g) be a lightlike hypersurface of $\bar{M}$.

Then, according to the decomposition (2.4), we write for any $X, Y \in \Gamma(TM)$

$$\bar{\nabla}_X Y = \nabla_X Y = B(X, Y)N \quad \ldots (3.1)$$

and

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \quad \ldots (3.2)$$

where $\nabla_X Y$ and $A_N X$ belong to $\Gamma(TM)$, while $B(X, Y)$ and $\tau(X)$ are smooth functions on M. We call B, A_N and τ the second fundamental form, the shape operator, and the transversal 1-form, respectively, for the lightlike immersion of M in $\overline{M}$. As in the non-degenerate case we call (3.1) and (3.2) the formulas of Gauss and Weingarten of the lightlike hypersurface M. It is easy to check that B is a symmetric tensor field of type (0, 2), τ is a differential 1-form, A_N is a tensor field of type (1, 1) and ∇ is a torsion-free linear connection on M. Moreover the second fundamental form B is independent of the choice of screen distributions, in fact, from (3.1) and (2.2) we obtain $B(X, Y) = \overline{g}(\overline{\nabla}_X Y, E)$ for any $X, Y \in \Gamma(TM)$.

The tensor fields B and A_N are not related by means of g, and therefore, in general, A_N is not symmetric with respect to g. The 1-form τ, in general, does not vanish on M as it is in the nondegenerate case. The induced linear connection ∇ is not a metric connection. More precisely, we obtain from (3.1) and the fact that $\overline{\nabla}$ is a metric connection.

$$(\nabla_X g)(Y,) = B(X, Y)\,\theta(Z) + B(X, Z)\,\theta(Y) \qquad \text{... (3.3)}$$

for any $X, Y \in \Gamma(TM)$, where θ is a differential 1-form locally defined on M by

$$\theta(X) := \overline{g}(X, N), \quad \forall X \in \Gamma(TM). \qquad \text{... (3.4)}$$

Next, we denote by P the projection morphism of TM on SM with respect to the orthogonal decomposition $TM = SM \perp TM^{\perp}$.

Taking into account of this decomposition, we can put

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\,E, \qquad \text{... (3.5)}$$

and

$$\nabla_X E = -A^*_E X + n(X)\,E \qquad \text{... (3.6)}$$

for any $X, Y \in \Gamma(TM), E \in \Gamma(TM^{\perp})$, where $\nabla^*_X PY$ and A^*_E belong to $\Gamma(SM)$, while $C(X, PY)$ and $C(X, PY)$ and $n(X)$ are smooth functions on M. By using (2.2), (3.1), (3.2) and (3.6) we obtain

$$n(X) = \overline{g}(\nabla_X E, N) = \overline{g}(\overline{\nabla}_X E, N) = -\overline{g}(E, \overline{\nabla}_X N) = -\tau(X).$$

Hence (3.6) becomes

$$\nabla_X E = -A^*_E X - \tau(X)\,E. \qquad \text{... (3.7)}$$

It follows that C is $C^{\infty}(M)$-bilinear on $\Gamma(M) \times \Gamma(SM)$, but in general, it is not symmetric on $\Gamma(SM) \times \Gamma(SM)$, A^*_E is a tensor field of type (1, 1) on M and ∇^* is a linear connection on the screen distribution SM. We call C and A^*_E the second fundamental form and the shape operator of the screen distribution SM, respectively.

The following identities are valid.

$$g(A_N X, PY) = C(X, PY), \; g(A_N X, N) = 0 \qquad \text{... (3.8)}$$

and $$g(A^*_E X, PY) = B(X, PY),\ g(A^*_E X, N) = 0 \qquad \text{... (3.9)}$$

for any $X, Y \in \Gamma(TM)$. From $\bar{g}(\bar{\nabla}_X E, E) = 0$, we get

$$B(X, E) = 0, \ \forall X \in \Gamma(TM) \qquad \text{... (3.10)}$$

Finally we are concerned with the structure equations of the immersion of a lightlike hypersurface in a semi-riemannian manifold. By direct calculation, using (3.1) and (3.2) we obtain

$$[\bar{R}](X, Y)Z = R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X$$
$$+ \{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z)$$
$$- \tau(Y)B(X, Z)\}\ N \qquad \text{... (3.11)}$$

for any $X\, Y, Z \in \Gamma(TM)$, where $\bar{R}$ and R are the curvature tensor fields of $\bar{M}$ and M, respectively, and we put as usually

$$(\nabla_X B)(Y, Z) = XB(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z). \qquad \text{... (3.12)}$$

It follows from (3.11) that for any $X, Y, Z \in \Gamma(TM)$

$$\bar{g}(\bar{R}(X, Y)Z, PW) = g(R(X, Y)Z, PW) + B(X, Z)C(Y, PW)$$
$$- B(Y, Z)C(X, PW), \qquad \text{... (3.13)}$$

$$\bar{g}(\bar{R}(X, Y)Z, E) = (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z), \qquad \text{... (3.14)}$$

and $$\bar{g}(\bar{R}(X, Y)Z, N) = g(R(X, Y)Z, N). \qquad \text{... (3.15)}$$

4. LIGHTLIKE HYPERSURFACES

Let $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ be an indefinite Sasakian manifold and (M, g) be its lightlike hypersurface. We consider the distributions on M

$$D := TM^{\perp} \perp \bar{\phi}(TM^{\perp}) \perp D_0,\ D' := \bar{\phi}(NM).$$

Then D is invariant under $\bar{\phi}$ and

$$TM = D \oplus D'. \qquad \text{... (4.1)}$$

Now we consider the local lightlike vector fields

$$U := -\bar{\phi}N,\ V := -\bar{\phi}E. \qquad \text{... (4.2)}$$

Then, from (4.1), any $X \in \Gamma(TM)$, is written as

$$X = SX + QX,\ QX = u(X)U, \qquad \text{... (4.3)}$$

where S and Q are the projection morphisms of TM into D and D', respectively, and u are 1-forms locally defined on M by

$$u(X) := g(X, V). \qquad \text{... (4.4)}$$

Applying $\bar{\phi}$ to (4.3) and using (1.1) (note that $\bar{\phi}^2 N = -N$), we obtain

$$\bar{\phi} X = \phi X + u(X) N, \qquad \text{... (4.5)}$$

where ϕ is a tensor field of type (1.1) defined on M by

$$\phi X := \bar{\phi} SX, X \in \Gamma(TM). \qquad \text{... (4.6)}$$

Again, applying $\bar{\phi}$ to (4.5) and using (1.1), we also have

$$\phi^2 X = -X + \eta(X)\xi + u(X) U, X \in \Gamma(TM). \qquad \text{... (4.7)}$$

Differentiating (4.5) and comparing both sides with respect to the decomposition (2.4), we obtain for any $X, Y \in \Gamma(TM)$

$$(\nabla_X \phi) Y = \eta(Y) X - g(X, Y)\xi - B(X, Y) U + u(Y) A_N X, \qquad \text{... (4.8)}$$

$$(\nabla_X u) Y = -B(X, \phi Y) - u(Y)\tau(X), \qquad \text{... (4.9)}$$

where we have used (1.3), (3.1), (3.2) and (4.4).

***Theorem* 1** — *A lightlike hypersurface M of an indefinite Sasakian manifold $\bar{M}$ is totally geodesic, i.e., $B = 0$ if and only if*

$$(\nabla_X \phi) Y = \eta(Y) X - g(X, Y)\xi, \quad \forall X \in \Gamma(TM), Y \in \Gamma(D),$$

$$A_N X = -\phi(\nabla_X U) + g(X, U)\xi, \quad \forall X \in \Gamma(TM). \qquad \text{... (4.10)}$$

PROOF : Note that $u(Y) = 0$, $\forall Y \in \Gamma(D)$. Then (4.8) is reduced to the equation

$$(\nabla_X \phi) Y = \eta(Y) X - g(X, Y)\xi - B(X, Y) U, \qquad \text{... (4.11)}$$

where $Y \in \Gamma(D)$. On the other hand, replacing Y by U in (4.8), we also obtain

$$A_N X = -\phi(\nabla_X U) + g(X, U)\xi - B(X, U) U, \qquad \text{... (4.12)}$$

with the aid of $u(U) = 1$ and $g(U, \xi) = 0$. Therefore if we assume that M is totally geodesic, then (4.10) follows from (4.11) and (4.12). The converse is clear. Thus we complete the proof. □

Proposition 2 — Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$. Then we have for any $X \in \Gamma(TM)$

(i) If the vector field U is parallel, then

$$A_N X = \eta(A_N X)\xi + u(A_N X) U, \ \tau(X) = 0, \qquad \text{... (4.13)}$$

(ii) If the vector V is parallel, then

$$A^*_E X = \eta(A^*_E X)\xi + u(A^*_E X) U, \ \tau(X) = 0. \qquad \text{... (4.14)}$$

PROOF : (i) Applying ϕ to (4.12) and using (4.7), we have

$$\phi(A_N X) = \nabla_X U - \eta(\nabla_X U)\xi - u(\nabla_X U) U, X \in \Gamma(TM).$$

If U is parallel, i.e., $\nabla_X U = 0$, then this equation reduces to

$$\phi(A_N X) = 0.$$

From which and (4.5), we have

$$\bar{\phi}(A_N X) = u(A_N X)\, N.$$

Applying $\bar{\phi}$ to this equation and using (1.1), we get the first equation in (4.13).

Replacing Y in (4.9) by U and noting that $u(U) = 1$, we have $(\nabla_X u)(U) = -\tau(X)$, and so $\tau(X) = 0$, since $(\nabla_X u)(U) = -u(\nabla_X U) = 0$.

(*ii*) Suppose that the vector field V is parallel. Replacing Y by E in (4.8) and remembring $E \in \Gamma(TM^{\perp})$ and (3.10), we have $(\nabla_X \phi) E = 0$. Hence

$$0 = (\nabla_X \phi) E = X(\phi(E)) - \phi(\nabla_X E) = X(\bar{\phi}(E)) - \phi(\nabla_X E)$$

$$= -\nabla_X V - \phi(-A^*_E X - \tau(X) E) \quad \because (3.7)$$

$$= \phi(A^*_E X + \tau(X) E) \quad \because \nabla_X V = 0.$$

And so $\phi(A^*_E X) = \tau(X) V$. Applying ϕ to this equation and using (4.7), we obtain

$$-A^*_E X + \eta(A^*_E X)\,\xi + u(A^*_E X)\, U = \tau(X)\,\phi(V) = \tau(X) E, \qquad \text{... (4.15)}$$

where the last equality follows from the fact that $\phi(V) = \bar{\phi}(V) = -\bar{\phi}^2(E) = E$. The left hand side of (4.15) belongs to $\Gamma(SM)$, while the right hand side belongs to $\Gamma(TM^{\perp})$. This proves (4.14). □

Corollary 3 — Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$ such that U and V are parallel with respect to the induced connection ∇ on M. Then, for any point p in M the type numbers of M and $S(M)$ satisfy $t(p) \le 2$ and $t^*(p) \le 2$, respectively.

Now, according to the decomposition (2.8), we consider a local field of frames on M, i.e.,

$$\left\{ \bar{\phi}E, \bar{\phi}N, \xi, E_a, \bar{\phi}E_a, E;\ a = 1, 2, \ldots, n-2 \right\},$$

where $\left\{ \xi, E_a, \bar{\phi}E_a, E;\ a = 1, 2, \ldots, n-2 \right\}$ is an orthonormal field of frames of D_0. Put

$$D_0 = D_0' \perp [\xi].$$

Then, using (1.1)-(1.5), (3.1), (3.2) and (3.7)-(3.10), we obtain

Proposition 4 — Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$.

(*i*) $TM^{\perp} \perp \bar{\phi}(TM^{\perp})$ is integrable if and only if

$$B(X, Y) = 0,\ \forall X \in \Gamma(\bar{\phi}\ (TM)^{\perp})),\ Y \in \Gamma(\bar{\phi}(TM^{\perp}) \perp D_0'). \qquad \text{... (4.16)}$$

(*ii*) $TM^{\perp} \perp \bar{\phi}(TM^{\perp}) \perp [\xi]$ is integrable if and only if (4.16) holds.

(*iii*) $\bar{\phi}(TM^{\perp}) \oplus \bar{\phi}(NM)$ is integrable if and only if

$$C(U, V) = C(V, U), \qquad \text{... (4.17)}$$

and $$B(U, \bar{\phi}X) = C(V, \bar{\phi}X), \ \forall X \in \Gamma(D_0). \quad \text{... (4.18)}$$

(*iv*) D_0 is integrable if and only if

$$C(X, Y) = C(Y, X), \quad \text{... (4.19)}$$

$$C(X, \bar{\phi}Y) = C(Y, \bar{\phi}X), \ \forall X, Y \in \Gamma(D_0) \quad \text{... (4.20)}$$

and $$B(X, \bar{\phi}Y) = B(Y, \bar{\phi}X), \ \forall X, Y \in \Gamma(D_0'). \quad \text{... (4.21)}$$

(*v*) $TM^{\perp} \perp D_0$ is integrable if and only if (4.20) and (4.21) hold

$$B(X, V) = 0. \quad \text{... (4.22)}$$

and $$C(E, \bar{\phi}X) + B(X, U) = 0, \ \forall X \in \Gamma(D_0'). \quad \text{... (4.23)}$$

(*vi*) $\bar{\phi}(TM^{\perp}) \perp D_0$ is integrable if and only if (4.19) and (4.20) hold and

$$C(U, \bar{\phi}X) = 0, \quad \text{... (4.24)}$$

and $$C(X, U) = C(U, X), \ \forall X \in \Gamma(D_0'). \quad \text{... (4.25)}$$

(*vii*) $\bar{\phi}(NM) \perp D_0$ is integrable if and only if (4.19) and (4.20) hold and

$$C(X, U) = C(U, X), \quad \text{... (4.26)}$$

and $$C(U, \bar{\phi}X) = 0 \ \forall X \in \Gamma(D_0'). \quad \text{... (4.27)}$$

(*viii*) D is integrable if and only if (4.21) and (4.22) hold and

$$B(V, V) = 0. \quad \text{... (4.28)}$$

(*iv*) $\bar{\phi}(NM) \perp TM^{\perp} \perp D_0$ is integrable if and only if (4.20), (4.23) and (4.24) hold and

$$B(U, U) = 0. \quad \text{... (4.29)}$$

Remark : Proposition 4 indicates that the integrability of distributions involved is expressed as both second fundamental forms of M and SM. In the proof of Proposition 4 the following identities have been used.

$$B(X, \xi) = 0, \ \forall X \in \Gamma(TM^{\perp}) \perp \Gamma(D_0) \perp \Gamma(\bar{\phi}(TM^{\perp})), \ B(U, \xi) = 1, \quad \text{... (4.30)}$$

$$C(X, \xi) = 0, \ \forall X \in \Gamma(TM^{\perp}) \perp \Gamma(D_0) \perp \Gamma(\bar{\phi}(TM^{\perp})), \ C(V, \xi) = 1, \quad \text{... (4.31)}$$

where (4.30) follows from (1.4) and (3.1), and (4.31) follows from (1.4), (3.1) and (3.8).

Corollary 5 — Let M be a totally geodesic, lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$. Then the distributions $TM^{\perp} \perp \bar{\phi}(TM^{\perp})$, $TM^{\perp} \perp \bar{\phi}(TM^{\perp}) \perp [\xi]$ and D on M are integrable.

Corollary 6 — Let M be a totally geodesic, lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$. Then (*i*) On each leaf L of the distribution D on M the followings are valid :

$$\varphi^2 = -I + \eta \otimes \xi,$$

$$(\nabla_X \phi)(Y) = \eta(Y) X - g(X Y)\xi, \quad \forall \; X, Y \in \Gamma(D).$$

(*ii*) The distribution D is parallel with respect to the induced connection ∇.

PROOF : (*i*) follows from (4.7) and (4.11). For the assertion (*ii*), using (1.3), (3.1) and (3.3), we obtain

$$g(\nabla_X E, \overline{\phi} E) = -\overline{g}(\overline{\nabla}_X \overline{\phi} E, E) = -B(X, \overline{\phi} E) = 0,$$

$$g(\nabla_X \overline{\phi} E, \overline{\phi} E) = 0,$$

$$g(\nabla_X Y, \overline{\phi} E) = -\overline{g}(\overline{\nabla}_X \overline{\phi} Y, E) = -B(X, \overline{\phi} Y) = 0$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D_0)$. Hence D is a parallel with respect to ∇. □

5. TOTALLY UMBILICAL LIGHTLIKE HYPERSURFACES

A submanifold M is called a totally umbilical lightlike hypersurface of a semi-riemannian manifold $\overline{M}$ if the local second fundamental form B of M satisfies

$$B(PX, PY) = \rho\, g(PX, PY), \quad \forall X, Y \in \Gamma(TM) \qquad \text{... (5.1)}$$

where ρ is a smooth function on M. In this section, we suppose that M is a lightlike hypersurface of indefinite Sasakian space forms $\overline{M}(c)$.

***Theorem* 7** — *Let M be a totally umbilical lightlike hypersurface of a semi-riemannian manifold $\overline{M}(c)$. Then $c = 1$ ($\overline{M}(c)$ is of constant curvature 1) and ρ satisfies the partial differential equations*

$$E\rho + \rho\,\tau(E) - \rho^2 = 0, \qquad \text{... (5.2)}$$

and

$$PX_\rho + \rho\,\tau(PX) = 0, \quad \forall X \in \Gamma(TM). \qquad \text{... (5.3)}$$

PROOF : From (1.7) we get

$$4\,\overline{g}(\overline{R}(X, Y)Z, E) = (c-1)\;\{\overline{g}(\overline{\phi} Y, Z)\,\overline{g}(\overline{\phi} X, E) + \overline{g}(\overline{\phi} Z, X)\,\overline{g}(\overline{\phi} Y, E) - 2\,\overline{g}(\overline{\phi} X, Y)\,\overline{g}(\overline{\phi} Z, E)\}$$

for any $X, Y, Z \in \Gamma(TM)$. Substituting (3.14) into the left hand side of (5.4) and using (4.4) yields

$$\frac{c-1}{4}\{\overline{g}(\overline{\phi} Y, Z)\,u(X) + \overline{g}(\overline{\phi} Z, X)\,u(Y) - 2\overline{g}(\overline{\phi} X, Y)\,u(Z)\}$$

$$= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)\,B(Y, Z) - \tau(Y)\,B(X, Z).$$

Replacing X, Y, Z in (5.5) by PX, E, PZ respectively, we deduce

$$-\frac{3}{4}(c-1)\,u(PX)\,u(PZ) = \{-E\rho - \rho\,\tau(E) + \rho^2\}\;g(PX, PZ),$$

where we have used (3.3),(3.9), (3.10), (3.12) and (5.1). Taking $PX = PZ = U$ in this equation yields $c = 1$. Next, substituting $X = E$, $Y = PY$ and $Z = PY$ into (5.5) with $c = 1$. Next, substituting $X = E$, $Y = PY$ and $Z = PY$ into (5.5) with $c = 1$ gives

$$\{E\rho - \rho^2 + \rho\tau(E)\}\ g(PY, PY) = 0,$$

which means (5.2), since we can take Y such that $g(PY, PY) \neq 0$ (locally). Finally, substituting $X = PX$, $Y = PY$ and $Z = PZ$ into (5.5) with $c = 1$ and taking into account that SM is nondegenerate, we get

$$\{PX(\rho) + \rho\,\tau(PX)\}\, PY = \{PY(\rho) + \rho\,\tau(PY)\}\, PX. \qquad \dots (5.6)$$

Now suppose that there exists a vector field X_0 on some neighbourhood of M such that $PX_0(\rho) + \rho\,\tau(PX_0) \neq 0$ at some point p in the neighbourhood. Then from (5.6) it follows that all vectors of the fibre SM_p are collinear with $(PX_0)_p$.. This contradicts dim $(SM_p) > 1$. This implies (5.3). □

From Theorem 7 we obtain

Corollary 8 — There exist no totally umbilical lightlike hypersurfaces of indefinite Sasakian space forms $\overline{M}(c)$ with $c \neq 1$.

Next, we say that the screen distribution SM is totally umbilical if we have

$$C(X, PY) = \lambda\, g(X, PY), \quad \forall X, Y \in \Gamma(TM), \qquad \dots (5.7)$$

where λ is a smooth function on M.

Proposition 9 — Let (M, g, SM) be a lightlike hypersurface of $\overline{M}(c)$ such that SM is totally umbilical. Then SM is totally geodesic, i.e. $C(X, PY) = 0$ for any $X, Y \in \Gamma(TM)$.

PROOF : First, the direct calculation of the right hand side in (3.15) shows that

$$\overline{g}(\overline{R}(X, Y)PZ, N) = (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ)$$
$$+\ \tau(Y)\, C(X, PZ) - \tau(X)\, C(Y, PZ)$$

with the aid of (3.5) and (3.7), where we have put

$$(\nabla_X C)(Y, PZ) := X(C(Y, PZ)) - C(\nabla_X Y, PZ) - C(Y, \nabla^*_X PZ).$$

Taking $X = E$, $Y = PZ = U$ in (5.8) and using (5.7) we have

$$\overline{g}(\overline{R}(E, U)U, N) = (\nabla_E \lambda g)(U, U) - (\nabla_U \lambda g)(E, U)$$
$$= \lambda(\nabla_E g)(U, U) - \lambda(\nabla_U g)(E, U)$$
$$= -\lambda(\nabla_U g)(E, U) \quad \because \dots (3.3)$$
$$= \lambda\, g(\nabla_U E, U)$$
$$= -\lambda\, \overline{g}(\overline{\nabla}_U E, \overline{\phi} N) \quad \because \dots (3.10)$$

$$= \lambda \bar{g}(\bar{\phi} \bar{\nabla}_U E, N)$$

$$= \lambda \bar{g}(\bar{\nabla}_U \bar{\phi} E, N) \because \ldots (1.3)$$

$$= -\lambda \bar{g}(\bar{\phi} E, \bar{\nabla}_U N) = \lambda \bar{g}(\bar{\phi} E, A_N U)$$

$$= \lambda C(U, \bar{\phi} E) \because \ldots (3.8)$$

$$= -\lambda^2 \because \ldots (1.2), (4.2), (5.7)$$

On the other hand, it is clear from (1.7) that $\bar{g}(\bar{R}(\xi, U) U, N) = 0$, which means that $S(M)$ is totally geodesic. □

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