Indian J. pure appl. Math., 34(9) : 1393-1396, September 2003

GENERALIZED DERIVATIONS AND COMMUTATIVITY OF PRIME RINGS

M. A. QUADRI, M. SHADAB KHAN AND N. REHMAN

Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, *India*
e-mail: km_shadab@postmark.net

(*Received* 16 *April* 2002; *after final revision* 12 *March* 2002; *accepted* 29 *April* 2003)

The concept of derivations as well as of generalized inner derivations (i.e. $I_{a,b}(x) = ax + xb$, for fixed $a, b \in R$) have been generalized as : an additive function $F : R \to R$ satisfying $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, where d is a derivation on R. Such a function F is said to be a generalized derivation. In the present paper we have discussed the commutativity of prime rings admitting a generalized derivation F satisfying $F([x, y]) = [x, y]$ for each pair x, y of elements in a specified subset of R.

Key Words : Prime Ring; Generalized Derivator; Derivation; Ideal and Commutativity

1. INTRODUCTION

Let R denote an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x \circ y$ denotes the anticommutatr $xy + yx$. Recall that a ring R is called prime if for any $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation.

An additive function $F_{a,b} : R \to R$ is called a generalized inner derivation if $F_{a,b}(x) = ax + xb$ for some fixed $a, b \in R$. It is straightforward to note that if $F_{a,b}$ is a generalized inner derivation, then for any $x, y \in R$

$$F_{a,b}(xy) = F_{a,b}(x)y + x[y, b]$$

$$= F_{a,b}(x)y + xI_b(y)$$

where I_b is an inner derivation. In view of the above observation, the concept of generalized derivation is introduced as follows : an additive mapping $F : R \to R$ is called a generalized derivation associated with a derivation d if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Generally, we do not mention the derivation d associated with a generalized derivation F; rather we prefer to call F simply a generalized derivation. One may observe that the concept of generalized derivation includes the concept of derivations and generalized inner derivations, also of the left multipliers when $d = 0$. Hence it should be interesting to extend some results concerning these notions to generalized derivations. Recently, some authors have also studied generalized derivations in the theory of operator algebras and C^*-algebras (see for Example[5, 8]). In the present paper we shall attempt to generalize some known results for derivations to generalized derivations.

Throughout the present paper we shall make extensive use of the following basic commutator identities without any specific mention :

$$[xy, z] = x[y, z] + [x, z]y;\ [x, yz] = y[x, z] + [x, y]z$$

$$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$$

$$(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z].$$

The proofs of the following results can be seen in [2, Lemma 3] and [11, Lemma 3] respectively.

Lemma 1.1 — Let R be a prime ring and I be a non-zero right ideal of R. If d is a nonzero derivation on R, then d is nonzero on I.

Lemma 1.2 — If a prime ring R contains a nonzero commutative right ideal, then R is commutative.

2. Commutativity of Rings Admitting Generalized Derivations

During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations of R. In the year 1992, Daif and Bell[6] established that if in a semiprime ring R there exists a nonzero ideal I of R and a derivation d such that $d([x, y]) = [x, y]$ for all $x, y \in I$, then $I \subseteq Z(R)$. It is natural to ask what we can say about the commutativity of R if the derivation d is replaced by a generalized derivation F. In this direction, we succeeded in establishing the following result for prime rings.

***Theorem* 2.1** — *Let R be a prime ring and I a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F([x, y]) = [x, y]$ for all $x, y \in I$, then R is commutative.*

Proof : We may assume that F is nonzero, otherwise I is commutative and so is R. For any $x, y \in I$, we have $F([x, y]) = [x, y]$, which gives

$$F(x)y + xd(y) - F(y)x - yd(x) - [x, y] = 0, \text{ for all } x, y \in I. \quad \dots (1)$$

replacing y by yz in (1), we get

$$F(x)yz + xd(y)z + xyd(z) - F(y)zx - yd(z)x - yzd(x) - y[x, z] - [x, y]z = 0.$$

Using (1) to substitute for $F(x)y$ in the above equation, we obtain

$$F(y)xz + yd(x)z + xyd(z) - F(y)zx - yd(z)x - yzd(x) - y[x, z] = 0.$$

This can be written as

$$F(y)[x, z] + y[d(x), z] + [x, y]d(z) + y[x, d(z)] - y[x, z] = 0, \text{ for all } x, y, z \in I. \quad \dots (2)$$

Again replacing z by zx in (2) we obtain

$$F(y)[x,z]x + yz[d(x),x] + y[d(x),z]x + [x,y]d(z)x + [x,y]zd(x)$$
$$+ y[x,d(z)]x + yz[x,d(x)] + y[x,z]d(x) - y[x,z]x = 0, \text{ for all } x,y,z \in I,$$

which when compared with (2) yields

$$[x,y]zd(x) + y[x,z]d(x) = 0, \text{ for all } x,y,z \in I. \qquad \ldots (3)$$

Finally, replacing y by $y_1 y$ in (3) we obtain $[x, y_1]yzd(x) = 0$ for all $x, y, z \in I$, and hence $[x, y_1]yRId(x) = (0)$ for all $x, y, z \in I$. Thus, primeness of R forces that for each $x \in I$ either $Id(x) = (0)$ or $[x, y_1]y = 0$. The set of all $x \in I$ for which these two properties hold are additive subgroups of I whose union is I; therefore, $Id(x) = (0)$ for all $x \in I$ or $[x, y_1]y = 0$ for all $x, y_1, y \in I$. If $Id(x) = (0)$ for all $x \in I$, then by Lemma 1.1, $d = 0$, a contradiction. On the other hand if $[x, y_1]y = 0$ for all $y, y_1 \in I$, this gives $[x, y_1]RI = (0)$; and since R is prime and $I \neq (0)$, we get $[x, y_1] = 0$ for all $x, y_1 \in I$. Hence by application of Lemma 1.2, R is commutative.

A slight modification in the proof of the above theorem yields the following :

***Theorem* 2.2** — *Let R be a prime ring and I be a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F([x, y]) + [x, y] = 0$ for all $x, y \in I$, then R is commutative.*

If we replace the commutator by anticommutator in the above theorem, then also the result holds.

***Theorem* 2.3** — *Let R be a prime ring and I be a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F(x \circ y) = x \circ y$ for all $x, y \in I$, then R is commutative.*

PROOF : For any $x, y \in I$, we have $F(x \circ y) = x \circ y$. If $F = 0$, then $x \circ y = 0$, for all $x, y \in I$. Replacing y by yz and using the fact that $xy = -yx$, we find that $y[x, z] = 0$ for all $x, y, z \in I$ and hence $IR_{[x,\ z]} = (0)$ for all $x, z \in I$. Since $I \neq (0)$ and R is prime, we get $[x, z] = 0$ for all $x, z \in I$, and hence by Lemma 1.2, R is commutative. Henceforth we assume that $F \neq 0$. For any $x, y \in I$, we have $F(x \circ y) = x \circ y$. This can be written as

$$F(x)y + xd(y) + F(y)x + yd(x) - x \circ y = 0, \text{ for all } x, y \in I. \qquad \ldots (4)$$

Replacing y by yx in (4), we get

$$F(x)yx + xd(y)x + xyd(x) + F(y)x^2 + yd(x)x + yxd(x) - (x \circ y)x = 0 \text{ for all } x, y \in I,.$$

Application of (4) gives

$$(x \circ y)d(x) = 0, \text{ for } x, y \in I.$$

Replacing y by zy in the above expression, we get $[x, z]\, yd(x) = 0$ for all $x, y\, z, \in I$, hence $[x, z]\, IRd(x) = (0)$ for all $x, z \in I$ Thus, primeness of R forces that for each $x \in I$ either $d(x) = 0$ or $[x, z]\ I = (0)$ for all $z \in I$. The set of $x \in I$ for which these two properties hold are additive subgroups of I whose union is I, and therefore $d(x) = 0$ for all $x \in I$ or $[x, z]\ I = (0)$ for all $x, z \in I$. If $d(x) = 0$ for all $x \in I$, then by Lemma 1.1, $d = 0$, a contradiction. On the other hand if $[x, z]\ I = (0)$ for all $x, z \in I$, $[x, z]\ RI = (0)$. Since $I \neq (0)$, we find that $[x, z] = 0$ for all $x, z \in I$. By the application of Lemma 1.2, R is commutative.

Following on the same lines as above with necessary variations, we can prove the following:

***Theorem* 2.4** —*Let R be a prime ring and I be a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F(x \circ y) + x \circ y = 0$ holds for all $x, y \in I$, then R is commutative.*

The following example shows that in the hypothesis of Theorem 2.1, if we replace the prime ring by a semi-prime ring, then R may not be commutative, even for an ordinary derivation.

Example — Let R_1 be an integral domain admitting a nonzero derivation d_1 and R_2 be any non-commutative prime ring with nonzero derivation d_2; Let $R = R_1 \oplus R_2$ and $F = d_1 \oplus d_2$ is a derivation on R. The semi-prime ring R satisfies the hypothesis of the Theorem 2.1. However, R is not commutative.

ACKNOWLEDGEMENT

The authors are thankful to the referee of the paper for useful suggestions and critical remarks.

REFERENCES

1. H. E. Bell and M. N. Daif, *Canad. Math. Bull.*, **37** (1994), 443-47.
2. H. E. Bell and W. S. Martindale, *Canad. Math. Bull.*, **30** (1987), 92-101.
3. M. Bresar, *Trans. Amer. Math. Soc.*, **335** (1993), 525-46.
4. M. Bresar, *J. Algebra*, **156** (1993), 385-94.
5. M. Bresar, *Glasglow Math. J.*, **33** (1991), 89-93.
6. M. N. Daif and H. E. Bell, *Internal. J. Math. and Math. Sci.*, **15** (1992), 205-6.
7. Q. Deng and M. Ashraf, *Results in Math.*, **30** (1996), 259-63.
8. B. Havala, *Communication in Algebra*, **24**(4) (1998), 1147-66.
9. I. N. Herstein, *Topics in ring theory*, University Chicago press, Chicago 1969.
10. I. N. Herstein, *Rings with involution*, University Chicago press, Chicago, 1976.
11. J. H. Mayne, *Canad. Math. Bull.*, **27** (1984), 122-26.
12. E. C. Posner, *Proc. Amer. Math. Soc.*, **8** (1957), 1093-1100.