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COVARIANT COMPLETELY POSITIVE MAPS

TIANZHOU XU

Department of Applied Mathematics, Beijing Institute of Technology, Beijing 100081,
People's Republic of China
e-mail: xutianzhou@bit.edu.cn

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Let A, B are unital C^*-algebras, M is a left-cancellative semigroup with unit, θ is a fixed 2-cocycle of M. $\alpha : x \to \alpha_x$ is a homomorphism from M into the group $Aut\ (A)$ of automorphisms of A, W is an isometric projective homomorphism from M to B, J is a closed two-sidedi deal of B, τ is a unital *-homomorphism from $C^*_\theta(M)$ to B/J. An abstract covariant version of Stinespring's dilation theorem is given. It is shown that a W-covariant completely positive map φ from A to B extends to a completely positive map on the (twisted) crossed product $C^*_\theta(A, M, \alpha)$. These results are used to give necessary and sufficient conditions for τ to have a completely positive lifting.

Key Words : C^*-dynamical System; Covariant Completely Positive Map; Stinespring's Theorem

1. INTRODUCTION

The theory of crossed products of C^*-algebras by groups of automorphisms is a well-developed area of the theory of operator algebras. Given the importance and the success of that theory, it is natural to attempt to extend it to a more general situation by, for example, developing a theory of crossed products of C^*-algebras by semigroups of automorphisms, or even of endomorphisms. Indeed, in recent years a number of papers have appeared that are concerned with such non-classical theories of covariance algebras, see, for instance[6-11].

In recent papers[10-11] Murphy introduced one such non-classical theory. There are many aspects of the extended theory developed by Murphy which are analogous to the results of the original classical theory. Nevertheless, there are significant differences, in fact, perhaps rather more than one might at first expect. These differences manifest themselves not only in the kind of results that can be obtained, but also in proofs and methods: the analogue of a classical result may turn out to be false, or if true, may require a new proof (and often, one that is rather more difficult). For further details see[10-11].

Let (A, M, α) be a C^*-dynamical system relative to the 2-cocycle θ, where M be a semigroup, with unit denoted by e and A be a unital C^*-algebra. Given an isometric projective homomorphism W from M to a unital C^*-algebra B, a linear map $\varphi : A \to B$ is called W-covariant if $\varphi(\alpha_x(a))\, W_x = \varphi(a)\, W_x$ for each $a \in A$ and and $x \in M$.

Motivated mainly by the works[1, 4, 5, 10, 11, 12, 14], in this paper we shall consider an abstract covariant version of Stinespring's dilation theorem, and explore the natural relation of covariant completely positive maps to (twisted) crossed products. We shall show that a covariant completely positive map $\varphi : A \to B$ extends to a completely positive map on the (twisted) crossed products $C^*_\theta(A, M, \alpha)$. This is done using the analog of the covariant version of Stinespring's theorem[12] for bounded operators on Hilbert modules. As an application, we consider the completely positive lifting problem for unital *-homomorphism $\tau : C^*_\theta(M) \to B/J$.

2. COVARIANCE ALGEBRAS AND HILBERT C^*-MODULES

In this section, we shall collect from[10-11] basic definitions and properties of the covariance algebra of the C^*-dynamical system (A, M, α) relative to a 2-cocycle θ.

Let M be a semigroup, with unit denoted by e, and let A be a unital *-algebra. We say that a map

$$W : M \to A, x \to W_x,$$

is an isometric projective homomorphism from M to A if all the elements of W_x are isometries, if $W_e = 1$, and if

$$W_{xy} = \theta_{x,y} W_x W_y, \quad x, y \in M,$$

where the $\theta_{x,y}$ are complex numbers of modulus 1. It follows from the equations

$$W_e = W_{ee} = \theta_{e,e} W_e E_e,$$

$$W_{(xy)z} = \theta_{xy,z} W_{xy} W_z = \theta_{x,y} \theta_{xy,z} W_x W_y W_z,$$

and $$W_{x(yz)} = \theta_{x,yz} W_x W_{yz} = \theta_{x,yz} \theta_{y,z} W_x W_y W_z,$$

that we have $$\theta_{e,e} = 1, \quad \theta_{x,y} \theta_{xy,z} = \theta_{x,yz} \theta_{y,z}. \qquad \text{... (2.1)}$$

We say that a function $\theta : M^2 \to \mathbf{T}, (x, y) \to \theta_{x,y}$ ($\mathbf{T}$ is the unit circle in $\mathbf{C}$) is a 2-cocycle of M if the eq. (2.1) holds. If $A = B(H)$ for a Hilbert space H then we call (H, W) an isometric projective representation of M on H.

If M is left-cancellative, then isometric projective representations exist. To be specific, let H be an arbitrary non-zero Hilbert space and put $\tilde{H} = l^2(M, H)$, the Hilbert space of all norm square-summable maps f from M to H (that is $\sum_{x \in M} \|f(x)\|^2 < +\infty$) with the norm and scalar product given by

$$\|f\| = \left(\sum_{x \in M} \|f(x)\|^2 \right)^{1/2}, \quad \langle f, g \rangle = \sum_{x \in M} \langle f(x), g(x) \rangle.$$

For each $x \in M$ we define an isometry W_x on $\tilde{H}$ by setting for each element of $f \in \tilde{H}$,

$$W_x f(z) = \begin{cases} \bar{\theta}_{x,y} f(y), & \text{if } z = xy, \text{ for some } y \in M, \\ 0, & \text{if } z \notin xM. \end{cases}$$

The map $W : M \to B(\tilde{H}), x \to W_x$, is an isometric projective representation. We call $(\tilde{H}, W)$ the regular isometric projective representation of M on $\tilde{H}$ (having θ as associated 2-cocycle).

In the sequel, M will always denote a left-cancellative semigroup with unit e and θ will denote a fixed 2-cocycle of M. All isometric projective hommorphisms of M considered will be understood to have θ as associated 2-cocycle, unless the contrary is indicated in a particular context.

We call a triple (A, M, α) a C^*-dynamical system if A is a C^*-algebra and α is a homomorphism from M into the group $Aut\,(A)$ of automorphisms of A. If B is a unital C^*-algebra, a covariant projective homomorphism (relative to θ) from (A, M, α) to B is a pair (φ, W), where $\varphi : A \rightarrow B$ is a *-homomorphism and $W : M \rightarrow B$ is an isometric projective homomorphism, and φ, W interact via the following equation :

$$\varphi(\alpha_x(a))\, W_x = W_x\, \varphi(a), \quad a \in A, x \in M.$$

If $B = B(H)$ for a Hilbert space H, then we call (H, φ, W) a covariant projective representation of (A, M, α). Murphy[11] has shown the following theorem:

***Theorem* 2.1** — *If (A, M, α) is a C^*-dynamical system, then there exists a C^*-algebra $C^*_\theta(A, M, \alpha)$ and a covariant projective homomorphism (ψ, V) (relative to θ) from (A, M, α) to $M(C^*_\theta(A, M, \alpha))$ having the following universal property: For each unital C^*-algebra B and covariant projective homomorphism (φ, W) (relative to θ) from (A, M, α) to B, there exists a unique *-homomorphism $\varphi \times W : C^*_\theta(A, M, \alpha) \rightarrow B$ such that*

$$\varphi \times W(\psi(a)\, V_x) = \varphi(a)\, W_x, \quad a \in A, x \in M.$$

Moreover, $C^*_\theta(A, M, \alpha)$ is generated by the elements $\psi(a)\, V_x \quad \forall a \in A, x \in M$. Up to isomorphism, these conditions uniquely determine $C^*_\theta(A, M, \alpha)$.

We call the C^*-algebra $C^*_\theta(A, M, \alpha)$ constructed in Theorem 2.1 the (twisted) crossed product of A by the semigroup M under the action α (relative to the cocycle θ), or the covariance algebra of the C^*-dynamical system (A, M, α) relative to the 2-cocycle θ.

Recall that each covariant projective homomorphism (φ, W) (relative to θ) from (A, M, α) to B canonically induces a *-homomorphism $\varphi \times W$ from the $C^*_\theta(A, M, \alpha)$ to B. Suppose now that (A, M, α) is a C^*-dynamical system and let (H, φ) is a faithful non-degenrate representation of A with $H \neq \{0\}$. Let $\tilde{H} = l^2(M, H)$, we get a representation $\tilde{\varphi}$ of A on $\tilde{H}$ is defined by $(\tilde{\varphi}(a) f)(x) = \varphi(\alpha_x^{-1}(a))(f(x))$ for all $a \in A, f \in \tilde{H}$ and $x \in M$. If $(\tilde{H}, W)$ is the regular isometric projective representation of M on $\tilde{H}$, it is easily checked that the triple $(\tilde{H}, \tilde{\varphi}, W)$ is a covariant projective representation of (A, M, α). We call $(\tilde{H}, \tilde{\varphi}, W)$ the covariant projective representation of (A, M, α) induced by the representation (H, φ) of A. If (H, φ) is a faithful representation of A and $(\tilde{H}, \tilde{\varphi}, W)$ is the induced covariant projective representation of (A, M, α), then $\tilde{\varphi}$ is injective. If $\pi = \tilde{\varphi} \times W$, by the equation $\pi\psi = \tilde{\varphi}$, it is easily seen that ψ to be injective. Therefore, we can regard ψ as an isometric embedding of A in $C^*_\theta(A, M, \alpha)$. Thus, we regard A as a C^*-subalgebra of

$C^*_\theta(A, M, \alpha)$. If $V_x = V_y$ then $\tilde{\varphi}(a)W_x = (\tilde{\varphi} \times W)\ (\psi(a)V_x) = (\tilde{\varphi} \times W)(\psi(a)V_y) = \tilde{\varphi}(a)W_y$, for all $a \in A$, so by non-degeneracy of the representation $(\tilde{H}, \tilde{\varphi})$ of A we have $W_x = W_y$, and therefore $x = y$. Thus V is injective.

Let (A, M, α) is a C^*-dynamical system, and (ψ, V) (relative to θ) is the canonical isometric projective homomorphism from M into $M(C^*_\theta(A, M, \alpha))$. The (twisted) crossed product $C^*_\theta(A, M, \alpha)$ of (A, M, α) is generated by the elements aV_x, for all $a \in A$ and $x \in M$; moreover, $C^*_\theta(A, M, \alpha)$ is, in fact, the closed linear span of all elements of the form $aV_{x_1} V^*_{y_1} \ldots V_{x_n} V^*_{y_n}$ for all $a \in A$ and $x_1, y_1, \ldots, x_n, y_n \in M$. A consequence is that any approximate unit for A is also one for $C^*_\theta(A, M, \alpha)$. In particular, if A is unital so is $C^*_\theta(A, M, \alpha)$. If $A = C$, then the action of M on A is trivial, we write $C^*_\theta(M)$ for $C^*_\theta(A, M, \alpha)$. Moreover, if θ is trivial also, we write $C^*(M)$ for $C^*_\theta(M)$.

Our next goal is to recall the definition of Hilbert C^*-modules.

Definition 2.2 — Let B be a C^*-algebra with the norm $\|\cdot\|$. A complex vector space E is called a pre-Hilbert B-module if E is a right B-module equipped with a B-valued mapping $\langle \cdot, \cdot \rangle : E \times E \to B$ which is linear in the second variable with the properties:

(*i*) $\langle x, y \rangle = \langle y, x \rangle^*$;

(*ii*) $\langle x, y \cdot b \rangle = \langle x, y \rangle\, b$;

(*iii*) $\langle x, x \rangle \geq 0$;

(*iv*) $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.

The mapping $\langle \cdot, \cdot \rangle$ is called a B-valued inner product on E. If, in addition, E is complete with respect to the norm $\| x \|_E = \| \langle x, x \rangle \|^{1/2}$, then E is called a Hilbert B-module.

Let E_1 and E_2 be Hilbert B-modules. We denote by $L(E_1, E_2)$ the space of all B-module homorphisms from E_1 into E_2, such that for each $T \in L(E_1, E_2)$ there exists an B-module homomorphisms $T^* : E_2 \to E_1$, called the adjoint of T, satisfying

$$\langle Tx, y \rangle = \langle x, T^*y \rangle, \quad x \in E_1,\ y \in E_2.$$

By the Banach-Steinhaus theorem, $T \in L(E_1, E_2)$ is bounded operator with respect to the operator norm. In particular, we write $L(E)$ for $L(E, E)$, which becomes a unital C^*-algebra with the operator norm [4, Lemma 1.1.7]. By a representation of a C^*-algebra A on a Hilbert B-module E, we mean a *-homomorphism $\pi : A \to L(E)$. In the special case, when a unital C^*-algebra B is regarded as Hilbert module over itself with respect to the inner produce $\langle x, y \rangle = x^* y$, $L(B)$ is the C^*-algebra of all left multiplication operators on B.

3. COVARIANT COMPLETELY POSITIVE MAPS

In this section, M will always denote a left-cancellative semigroup with unit e and θ will denote a fixed 2-cocycle of M. All isometric projective homomorphisms of M considered will be understood to have θ as associated 2-cocycle, unless the contrary is indicated in a particular context.

Let (A, M, α) is a C^*-dynamical system, $W : M \to B$ be an isometric projective homomorphism of M into a unital C^*-algebra B, and let $\varphi : A \to B$ is a linear map, then we call φ is W-covariant if $\varphi(\alpha_x(a))W_x = W_x \varphi(a)$ for all $a \in A$ and $x \in M$.

The following is an abstract covariant version of Stinespring theorem.

Theorem 3.1. — *Let (A, M, α) be a unital C^*-dynamical system, and let $W : M \to B$ be an isometric projective homomorphism of M into a unital C^*-algebra B. If (φ, W) is a W-covariant completely positive map (relative to θ) from (A, M, α) to B, then there exist :*

(i) a Hilbert B-module E;

(ii) a covariant projective representation $(\tilde{\varphi}, \tilde{W})$ of $(A M, \alpha)$ into $L(E)$;

(iii) an element $V \in L(B, E)$, such that

(1) $\varphi(a)\xi = V^* \tilde{\varphi}(a) V \xi$ for all $a \in A$ and $\xi \in B$.

(2) $\tilde{W}_x V \xi = V W_x \xi$ and $\tilde{W}_x^* V \xi = V W_x^* \xi$ for all $x \in M$ and $\xi \in B$.

PROOF : Let $A \otimes B$ be the algebraic tensor product of A and B, then $A \otimes B$ is a right B-module in the natural way. Define a B-module positive semi-definite inner product on $A \otimes B$ by

$$\Big\langle \sum_i a_i \otimes x_i, \sum_j c_j \otimes y_j \Big\rangle_{A \otimes B} = \sum_{i,j} x_i^* \varphi(a_i^* c_j) y_j, \quad a_i, c_j \in A, x_i, y_j \in B.$$

The Hillbert B-module E is obtained as the completion of the quotient of $A \otimes B$ by the kernel N of $\langle\, , \rangle_{A \otimes B}$ with respect to the inner product given $\langle \xi + N, \eta + N \rangle = \langle \xi, \eta \rangle_{A \otimes B}$, $\xi, \eta \in A \otimes B$. The covariant projective representation $(E, \tilde{\varphi}, \tilde{W})$ is defined by

$$\tilde{\varphi}(a)(c \otimes y + N) = (ac) \otimes y + N, \quad \tilde{W}_x(c \otimes y + N) = \alpha_x(c) \otimes (W_x y) + N.$$

The B-module homomorphism $V : B \to E$ is determined by $V(x) = 1_A \otimes x + N$, here 1_A denotes the unit of A. It is straightforward to check that $V^* : E \to B$ is given by $V^*(a \otimes x + N) = \varphi(a)x$. In particular, $V \in L(B, E)$, and $V^* \tilde{\varphi}(a) V \in L(B)$ is a left multiplication operator on B for each $a \in A$. Since φ is a W-covariant completely positive map, we have

$$\langle \tilde{W}_x(a \otimes \xi + N), c \otimes \eta + N \rangle = \langle \alpha_x(a) \otimes (W_x \xi) + N, c \otimes \eta + N \rangle$$

$$= (W_x \xi)^* \varphi(\alpha_x(a^*) c) \eta = \xi^* [\varphi(\alpha_x(\alpha_x^{-1}(c^*) a)) W_x]^* \eta$$

$$= \xi^* \varphi(a^* \alpha_x^{-1}(c)) W_x^* \eta$$

$$= \langle a \otimes \xi + N, \alpha_x^{-1}(c) \otimes (W_x^* \eta) + N \rangle,$$

for all $a, c \in A$, $\xi, \eta \in B$, and $x \in M$. Thus $\tilde{W}_x^* (c \otimes \eta + N) = \alpha_x^{-1}(c) \otimes (W_x^* \eta) + N$. Moreover, we have

$$\tilde{W}_x^* V \xi = \tilde{W}_x^* (1_A \otimes \xi + N) = 1_A \otimes (W_x^* \xi) + N = V W_x^* \xi$$

for all $\xi \in B$ and $x \in M$. The rest is routine.

***Theorem* 3.2** — *Let (A, M, α) be a unital C^*-dynamical system, (ψ, V) is the canonical covariant projective homomorphism from (A, M, α) to $M(C_\theta^*(A, M, \alpha))$ and $W : M \to B$ is an isometric projective homomorphism of M into a unital C^*-algebra B. If $\varphi : A \to B$ is a W-covariant completely positive map, then there is a completely positive map $\tilde{\varphi} : C_\theta^*(A, M, \alpha) \to B$ uniquely defined by*

$$\tilde{\varphi}(\psi(a) V_{x_1} V_{y_1}^* \cdots V_{x_n} V_{y_n}^*) = \varphi(a) W_{x_1} W_{y_1}^* \cdots W_{x_n} W_{y_n}^*,$$

where $a \in A$ and $x_1, y_1, \ldots, x_n, y_n \in M$.

PROOF : By Theorem 3.1 there exists a covariant projective representation $(\varphi_1, \tilde{W})$ of (A, M, α) into $L(E)$ and an element $V \in L(B, E)$ satisfying:

$$\varphi(a) \xi = V^* \varphi_1(a) V \xi, \qquad \tilde{W}_x V \xi = V W_x \xi, \; V W_x^* \xi = \tilde{W}_x^* V \xi, \qquad \ldots (3.1)$$

for all $a \in A$, $\xi \in B$, and $x \in M$. From the universal property of $C_\theta^*(A, M, \alpha, (\varphi_1, \tilde{W})$ gives rise to a unique *-homomorphism $\varphi_1 \times \tilde{W} : C_\theta^*(A, M, \alpha) \to L(E)$ such that

$$\varphi_1 \times \tilde{W}(\psi(a) V_{x_1} V_{y_1}^* \cdots V_{x_n} V_{y_n}^*) = \varphi_1(a) \tilde{W}_{x_1} \tilde{W}_{y_1}^* \cdots \tilde{W}_{x_n} \tilde{W}_{y_n}^*,$$

for all $a \in A$ and $x_1, y_1, \ldots, x_n, y_n \in M$.

Consider a completely positive map $\varphi_2 : C_\theta^*(A, M, \alpha) \to L(B)$ defined by

$$\varphi_2(\psi(a) V_{x_1} V_{y_1}^* \cdots V_{x_n} V_{y_n}^*) \xi = V^* (\varphi_1 \times \tilde{W}(a V_{x_1} V_{y_1}^* \cdots V_{x_n} V_{y_n}^*)) V \xi,$$

for all $a \in A$, $\xi \in B$ and $x_1, y_1, \ldots, x_n, y_n \in M$. By the eqs. (3.1), we have

$$\varphi_2(\psi(a) V_{x_1} V_{y_1}^* \cdots V_{x_n} V_{y_n}^*) \xi = V^* \varphi_1(a) \tilde{W}_{x_1} \tilde{W}_{y_1}^* \cdots \tilde{W}_{x_n} \tilde{W}_{y_n}^* V \xi$$

$$= V^* \varphi_1(a) V W_{x_1} W_{y_1}^* \cdots W_{x_n} W_{y_n}^* \xi$$

$$= \varphi(a)\, W_{x_1} W^*_{y_1} \dots W_{x_n} W^*_{y_n} \xi,$$

for all $a \in A$, $\xi \in B$ and $x_1, y_1, \dots, x_n, y_n \in M$. If μ be the natural isomorphism from $L(B)$ to B, set $\tilde{\varphi} = \mu \circ \varphi_2$, we obtain the required completely positive map $\tilde{\varphi} : C^*_\theta(A, M, \alpha) \to B$.

Corollary 3.3 — Let (A, M, α) and (B, M, β) are unital C^*-dynamical system, and let (ψ, V) and $(\tilde{\psi}, \tilde{V})$ are the canonical covariant projective homomorphisms from (A, M, α) ((B, M, β), respectively) to $M(C^*_{\theta_1}(A, M, \alpha))$ ($M(C^*_{\theta_2}(B, M, \beta))$, respectively. If $\varphi : A \to B$ is a completely positive map satisfying $\varphi(\alpha_x(a)) = \beta_x(\varphi(a))$ for all $a \in A$ and $x \in M$, then there is a completely positive map $\tilde{\varphi} : C^*_{\theta_1}(A, M, \alpha) \to C^*_{\theta_2}(B, M, \beta)$ given by

$$s\tilde{\varphi}(\psi(a)\, V_{x_1} V^*_{y_1} \dots V_{x_n} V^*_{y_n}) = \tilde{\psi}(\varphi(a))\, \tilde{V}_{x_1} \tilde{V}^*_{y_1} \dots \tilde{V}_{x_n} \tilde{V}^*_{y_n} .$$

PROOF : Since $(\tilde{\psi}, \tilde{V})$ is the canonical covariant projective homomorphism from (B, M, β) to $M(C^*_{\theta_2}(B, M, \beta))$, we have the following equation

$$\tilde{\psi}(\beta_x(b))\, \tilde{V}_x = \tilde{V}_x\, \tilde{\psi}(b), \quad b \in B, x \in M.$$

Thus, we have

$$\tilde{\psi}(\varphi(\alpha_x(a)))\, \tilde{V}_x = \tilde{\psi}(\beta_x(\varphi(a)))\, \tilde{V}_x = \tilde{V}_x\, \tilde{\psi}(\varphi(a)),$$

for each $a \in A$ and $x \in M$; and so $\tilde{\psi} \circ \varphi$ is a $\tilde{V}$-covariant projective completely positive map from A into $M(C^*_{\theta_2}(B, M\, \beta))$. Since B is unital, so $C^*_{\theta_2}(B, M, \beta)$ is unital. Hence $M(C^*_{\theta_2}(B, M, \beta))$ = $C^*_{\theta_2}(B, M, \beta)$, and $\tilde{\psi} \circ \varphi$ is a $\tilde{V}$-covariant projective completely positive map from A into $C^*_{\theta_2}(B, M, \beta)$. The required conclusion follows directly from Theorem 3.2.

Corollary 3.4 — Let (A, M, α) be a unital C^*-dynamical system and (ψ, V) is the canonical covariant projective homomorphism from (A, M, α) to $M(C^*_{\theta_1}(A, M, \alpha))$. Suppose B is a unital C^*-algebra, $\tau : C^*_\theta(M) \to B$ is a unital *-homomorphism, and let $W : M \to B$ be the isometric projective homomorphism defined by

$$\hat{W}_x = \tau(\tilde{V}_x), \quad x \in M,$$

where $\tilde{V}$ is the canonical isometric projective homomorphism from M into $C^*_\theta(M)$. By the universal property of $C^*_\theta(M)$, there exists a unique *-homomorphism ζ from $C^*_\theta(M)$ to $M(C^*_\theta(A, M, \alpha))$ (= $C^*_\theta(A, M, \alpha)$, since $C^*_\theta(A, M, \alpha)$ is unital) such that $\zeta(\tilde{V}) = V_x$ for all elements

$x \in M$. If $\varphi : A \to B$ is a unital $\hat{W}$-covariant completely positive map, then there exists a unital completely positive map $\tilde{\varphi} : C^*_\theta(A, M, \alpha) \to B$ such that

$$\tau = \tilde{\varphi} \circ \zeta, \qquad \varphi = \tilde{\varphi} \circ \psi.$$

PROOF : By Theorem 3.2 there exists a completely positive map $\tilde{\varphi} : C^*_{\theta_1}(A, M, \alpha) \to B$ uniquely defined by

$$\tilde{\varphi}(\psi(a) V_{x_1} V^*_{y_1} \ldots V_{x_n} V^*_{y_n}) = \varphi(a) \hat{W}_{x_1} \hat{W}^*_{y_1} \cdots \hat{W}_{x_n} \hat{W}^*_{y_n} = \varphi(a) \; \tau(\tilde{V}_{x_1} \tilde{V}^*_{y_1} \cdots \tilde{V}_{x_n} \tilde{V}^*_{y_n}).$$

Since φ and τ are unital, thus we have

$$\tilde{\varphi} \circ \zeta(\tilde{V}_{x_1} \tilde{V}^*_{y_1} \cdots \tilde{V}_{x_n} \tilde{V}^*_{y_n}) = \tilde{\varphi}(V_{x_1} V^*_{y_1} \cdots V_{x_n} V^*_{y_n}) = \tau(\tilde{V}_{x_1} \tilde{V}^*_{y_1} \cdots \tilde{V}_{x_n} \tilde{V}^*_{y_n}),$$

and $$\tilde{\varphi} \circ \psi(a) = \tilde{\varphi}(\psi(a)) = \varphi(a),$$

for all $a \in A$ and $x_1, y_1, \ldots, x_n, y_n \in M$. Note that $C^*_\theta(M)$ is the closed linear span of all elements of the form $\tilde{V}_{x_1} \tilde{V}^*_{y_1} \cdots \tilde{V}_{x_n} \tilde{V}^*_{y_n}$, where $x_1, y_1, \ldots, x_n, y_n \in M$. It follows that $\tilde{\varphi} \circ \zeta = \tau$ and $\tilde{\varphi} \circ \psi = \varphi$.

Recall the following definition and theorem.

Definition 3.5[13] — A C^*-algebra A is called injective if for any two C^*-algebra B and C with $B \subset C$ and any completely positive contraction (e.g. a *-homomorphism) $\varphi : B \to A$ there is a completely positive contraction $\tilde{\varphi} : C \to A$ extending φ.

Lemma 3.6 — ([2, Theorem 15.8.3]) — Let A be a nuclear C^*-algebra, B a unital C^*-algebra, J a closed two-sided ideal in B, $\pi : B \to B/J$ the quotient map. Let $\psi : A \to B/J$ be a unital completely positive map. Then there is a unital completely positive map $\varphi : A \to B$ with $\psi = \pi \circ \varphi$.

We will need the following elementary lemma:

Lemma 3.7[3] — Suppose that $\varphi : A \to B$ is a completely positive map between C^*-algebras, with $\varphi(1_A) = 1_B$. Suppose that there is a C^*-subalgebra N of A with $1_A \in N$, such that $\pi = \varphi \mid N$ is a *-homomorphism. Then φ is an N-bimodule map. That is,

$$\varphi(an) = \varphi(a)\, \pi(n), \quad \varphi(na) = \pi(n)\, \varphi(a)$$

for all $a \in A, n \in N$.

***Theorem* 3.8** — *Let (A, M, α) be a unital C^*-dynamical system such that $C^*_\theta(A, M, \alpha)$ is nuclear, and let B an injective unital C^*-algebra, J a closed two-sided ideal of B, $\pi : B \to B/J$ the quotient map. Suppose $\tau : C^*_\theta(M) \to B/J$ is a unital *-homomorphism, and let (ψ, V) is the canonical covariant projective homomorphism from (A, M, α) to $M(C^*_\theta(A, M, \alpha))$. Let $\hat{W} : M \to B/J$ be the isometric projective homomorphism defined by*

$$\hat{W}_x = \tau(\tilde{V}_x), \quad x \in M,$$

where $\tilde{V}$ is the canonical isometric projective homomorphism from M into $C^*_\theta(M)$. Then τ has a completely positive lifting if and only if there exists a unital $\hat{W}$-covariant completely positive map $\varphi : A \to B/J$.

PROOF : By the universal property of $C^*_\theta(M)$, there exists a unique *-homomorphism ζ from $C^*_\theta(M)$ to $M(C^*_\theta(A, M, \alpha))$ ($= C^*_\theta(A, M, \alpha)$, since $C^*_\theta(A, M, \alpha)$ is unital) such that $\zeta(\tilde{V}_x) = V_x$ for all elements $x \in M$. If $\zeta(\tilde{V}_x) = \zeta(\tilde{V}_y)$, then $V_x = V_y$, so by injectivity of V we have $x = y$, and therefore $\tilde{V}_x = \tilde{V}_y$. Note that $\tilde{V}_M$ generates $C^*_\theta(M)$, thus ζ is injective. By [13, Theorem 1.5.7], $\zeta(C^*_\theta(M))$ is a C^*-subalgebra of $C^*_\theta(A, M, \alpha)$, and ζ is isometric. Thus, ζ is an isomorphism from the C^*-algebra $C^*_\theta(M)$ into $C^*_\theta(A, M, \alpha)$, with image $\zeta(C^*_\theta(M))$ and $1_{C^*_\theta(A, M, \alpha)} \in \zeta(C^*_\theta(M))$.

Suppose $\varphi : A \to B/J$ is a unital, $\hat{W}$-covariant completely positive map. By Corollary 3.4, there is a unital completely positive map $\tilde{\varphi} : C^*_\theta(A, M, \alpha) \to B/J$ such that $\tau = \tilde{\varphi} \circ \zeta$ and $\varphi = \tilde{\varphi} \circ \psi$. Since $C^*_\theta(A, M, \alpha)$ is nuclear, from Lemma 3.6 $\tilde{\varphi}$ has a completely positive lifting $\rho : C^*_\theta(A, M, \alpha) \to B$. and hence $\tilde{\varphi} = \pi \circ \rho$. Furthermore, we set $\tilde{\rho} = \rho \circ \zeta$, and therefore $\tilde{\rho}$ is a completely positive lifting of τ.

Conversely, let $\sigma : C^*_\theta(M) \to B$ be a completely positive lifting of τ. We can assume that σ is unital. In fact, since $\pi \circ \sigma(1_{C^*_\theta(M)}) = \tau(1_{C^*_\theta(M)}) = 1_{B/J}$, $\sigma(1_{C^*_\theta(M)})$ must have the form $\sigma(1_{C^*_\theta(M)}) = 1_B + j$ where j is a self-adjoint element of J. Let $j = j_1 - j_2$ be the decomposition of j, where j_1 and j_2 are positive elements of J. Choose any normalized state ω of $C^*_\theta(M)$ and define

$$\sigma_1(x) = (1_B + j_1)^{-1/2} (\sigma(x) + \omega(x) j_2) (1_B + j_1)^{-1/2}, \quad x \in C^*_\theta(M).$$

Then σ_1 is a unital completely positive lifting of τ, and so is $\sigma \circ \zeta^{-1}$. Since B is injective, there is a completely positive contractive map $\tilde{\sigma} : C^*_\theta(A, M, \alpha) \to B$ such that $\tilde{\sigma}|_{\zeta(C^*_\theta(M))} = \sigma \circ \zeta^{-1}|_{\zeta(C^*_\theta(M))}$. We set $\rho = \pi \circ \tilde{\sigma}$. It is easily checked that $\rho : C^*_\theta(A, M, \alpha) \to B/J$ is a unital completely positive map such that $\rho \circ \zeta = \tau$. Let $\varphi = \rho \circ \psi$, since $\rho \circ \zeta = \tau$ and $\tau : C^*_\theta(M) \to B/J$ is a unital *-homomorphism, by Lemma 3.7, ρ is an $\zeta(C^*_\theta(M))$-bimodule map. Thus, by $\zeta(\tilde{V}_x) = V_x$ and (ψ, V) is the canonical covariant projective homomorphism from (A, M, α) to $M(C^*_\theta(A, M, \alpha))$, we have

$$\varphi(\alpha_x(a))\,\hat{W}_x = \rho(\psi(\alpha_x(a)))\,\tau(\tilde{V}_x) = \rho(\psi(\alpha_x(a))\,\rho\circ\zeta(\tilde{V}_x)$$
$$= \rho(\psi(\alpha_x(a))\,\zeta(\tilde{V}_x)) = \rho(\psi(\alpha_x(a))\,V_x) = \rho(V_x\,\psi(a))$$
$$= \rho(\zeta(\tilde{V}_x)\,\psi(a)) = \tau(\tilde{V}_x)\,\rho(\psi(a)) = \tau(\tilde{V}_x)\,\varphi(a) = \hat{W}_x\,\varphi(a)$$

for all $a \in A$ and $x \in M$. Therefore, φ is a unital $\hat{W}$-covariant completely positive map.

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