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Δ^r-STRONGLY ALMOST SUMMABLE SEQUENCES DEFINED BY ORLICZ FUNCTIONS

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The purpose of this paper is to introduce the space of sequences that are Δ^r-strongly almost summable with respect to an Orlicz function. We give some relations related to these sequence spaces. It is also shown that if a sequence is Δ^r-strongly almost summable with respect to an Orlicz function, then it is Δ^r-almost statistically convergent.

Key Words : Difference Sequence; Statistical Convergence; Orlicz Function

1. INTRODUCTION

Let w be the set of all sequences of real or complex numbers and l_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\| x \| = \sup |x_k|$, where $k \in N = \{1, 2, ...\}$, the set of positive integers.

A sequence $x \in l_\infty$ is said to be almost convergent (Lorentz[13]) if all Banach limits of x coincide. Lorentz[13] proved that

$$\hat{c} = \left\{ x : \lim_n \frac{1}{n} \sum_{k=1}^{n} x_{k+m} \text{ exists, uniformly in } m \right\}.$$

Several authors including Lorentz[13], Duran[3] and King[9] have studied almost convergent sequences. Maddox[14, 15] has defined x to be strongly almost convergent to a number L if

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+m} - L| = 0, \text{ uniformly in } m.$$

By $[\hat{c}]$ we denote the space of all strongly almost convergent sequences. It is easy to see that $c \subset [\hat{c}] \subset \hat{c} \subset l_\infty$.

The space of strongly almost convergent sequences was generalized by Nanda[17]. Let $p = (p_k)$ be a sequence of strictly positive real numbers. Nanda[17] defined

$$[\hat{c}, p] = \left\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} | x_{k+m} - L |^{p_k} = 0, \quad \text{uniformly in } m \right\},$$

$$[\hat{c}, p]_0 = \left\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} | x_{k+m} |^{p_k} = 0, \quad \text{uniformly in } m \right\},$$

$$[\hat{c}, p]_\infty = \left\{ x = (x_k) : \sup_{n,m} \frac{1}{n} \sum_{k=1}^{n} | x_{k+m} |^{p_k} < \infty \right\}.$$

The idea of difference sequence spaces was introduced by Kizmaz. In 1981, Kizmaz[10] defined the sequence spaces

$$X(\Delta) = \left\{ x = (x_k) : (\Delta x_k) \in X \right\}$$

for $X = l_\infty, c$ or c_0, where $\Delta x = (x_k - x_{k+1})$.

The operators $\Delta^{(r)}, \Sigma^{(r)} : w \to w$

are defined by $(\Delta^1 x)_k = \Delta^1 x_k = x_k - x_{k+1}$, $(\Sigma^1 x)_k = \sum_{j=1}^{k-1} x_j$, $(k = 0, 1, \ldots)$,

$$\Delta^{(r)} = \Delta^{(1)} \circ \Delta^{r-1}, \quad \Sigma^r = \Sigma^1 \circ \Sigma^{r-1} \quad (r \geq 2)$$

and $\Sigma^r \circ \Delta^r = \Delta^r \circ \Sigma^r = id$, the identity on w.

Then Et and Colak[5] generalized the above sequence spaces to the sequence spaces

$$X(\Delta^r) = \left\{ x = (x_k) : (\Delta^r x_k) \in X \right\}$$

for $X = l_\infty$, c or c_0, where $r \in N$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$,

$\Delta^r x = (\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1})$, and so

$$\Delta^r x_k = \sum_{v=0}^{r} (-1)^v \binom{r}{v} x_{k+v}.$$

Recently Et and Basair[6] have extended the above sequence spaces to the sequence spaces $X(\Delta^r)$ for $X = l_\infty(p), c(p), c_0(p), [\hat{c}, p], [\hat{c}, p]_0$ and $[\hat{c}, p]_\infty$.

Lindenstrauss and Tzafriri[12] used the idea of Orlicz function and they defined the sequence space l_M as follows :

$$l_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{| x_k |}{\rho} \right) < \infty, \quad \text{for some } \rho > 0 \right\}.$$

The space l_M is a Banach space with the norm

$$\| x \| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho} \right) \leq 1 \right\}$$

and this space is called an Orlicz sequence space. Lindenstrauss and Tzafriri proved that every Orlicz sequence space l_M contains a subspace isomorphic to l_p for some $p \geq 1$. For $M(x) = x^p$, $1 \leq p < \infty$, the spaces l_M coincide with the classical sequence space l_p.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $K > 0$ such that $M(2u) \leq KM(u), u \geq 0$. The Δ_2-condition is equivalent to the inequality $M(lu) \leq KlM(u)$ for all values of u and for $l > 1$ being satisfied.

It is well known that if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Subsequently Orlicz sequence spaces have been studied by Bhardwaj and Singh[1], Nuray and Gülcü[19], Parashar and Choudhary[20].

Let $x \in w$ and $X, Y \subset w$. Then we shall write

$$M(X, Y) = \bigcap_{x \in X} x^{-1} * Y = \{a \in : ax \in Y \text{ for all } x \in X\}^{23}$$

The set $X^{\alpha} = M(X, l_1)$ is called Köthe-Toeplitz dual space or α-dual of X.

Let X be a sequence space. Then X is called

(*i*) Solid (or normal), if $(\alpha_k x_k) \in X$ whenever, $(x_k) \in X$ for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbf{N}$.

(*ii*) Symmetric, if $(x_k) \in X$ implies $(x_{\pi(k)}) \in X$, where $\pi(k)$ is a permutation of N.

(*iii*) Monotone provided X contains the canonical preimages of all its stepspace.

(*iv*) Perfect $X = X^{\alpha\alpha}$

It is well known that X is perfect $\Rightarrow$ X is normal $\Rightarrow$ X is monotone[8].

The following inequality will be used throughout this paper.

$$|a_k + b_k|^{p_k} \leq D \left\{ |a_k|^{p_k} + |b_k|^{p_k} \right\}, \qquad \dots (1)$$

where $a_k, b_k \in \mathbf{C}, 0 < p_k \leq \sup_k p_k = G, D = \max(1, 2^{G-1})$[16].

2. Some new Sequence Spaces Defined by an Orlicz Function

In the present paper we introduce the space of Δ^r-strongly almost summable sequences with respect to an Orlicz function and examine some properties of these spaces.

Definition 1 — Let M be an Orlicz function and $p = (p_k)$ be any sequence of strictly positive real numbers. We define the following sequence sets

$$[\hat{c}, M, p]\,(\Delta^r) = \left\{ \begin{array}{l} x = (x_k) : \lim_n \dfrac{1}{n} \displaystyle\sum_{k=1}^{n} \left[M\left(\dfrac{|\Delta^r x_{k+m} - L|}{\rho} \right) \right]^{p_k} = 0, \\ \text{uniformly in } m, \text{ for some } \rho > 0 \text{ and } L > 0 \end{array} \right\},$$

$$[\hat{c}, M, p]_0\,(\Delta^r) = \left\{ \begin{array}{l} x = (x_k) : \lim_n \dfrac{1}{n} \displaystyle\sum_{k=1}^{n} \left[M\left(\dfrac{|\Delta^r x_{k+m}|}{\rho} \right) \right]^{p_k} = 0, \\ \text{uniformly in } m, \text{ for some } \rho > 0 \end{array} \right\},$$

$$[\hat{c}, M, p]_\infty\,(\Delta^r) = \left\{ \begin{array}{l} x = (x_k) : \sup_{n,m} \dfrac{1}{n} \displaystyle\sum_{k=1}^{n} \left[M\left(\dfrac{|\Delta^r x_{k+m}|}{\rho} \right) \right]^{p_k} < \infty \\ \text{for some } \rho > 0 \end{array} \right\}.$$

We denote $[\hat{c}, M, p]\,(\Delta^r)$, $[\hat{c}, M, p]_0\,(\Delta^r)$ and $[\hat{c}, M, p]_\infty\,(\Delta^r)$ by $[\hat{c}, M]\,(\Delta^r)$, $[\hat{c}, M]_0\,(\Delta^r)$ and $[\hat{c}, M]_\infty\,(\Delta^r)$, respectively, when $p_k = 1$ for all k. If $x \in [\hat{c}, M]\,(\Delta^r)$ then we say that x is Δ^r-strongly almost summable with respect to the Orlicz function M.

***Theorem* 2.1** — *Let (p_k) be bounded. Then $[\hat{c}, M, p]\,(\Delta^r)$, $[\hat{c}, M, p]_0\,(\Delta^r)$ and $[\hat{c}, M, p]_\infty\,(\Delta^r)$ are linear space over the set of complex numbers* **C**.

PROOF : We shall prove for $[\hat{c}, M, p]_0\,(\Delta^r)$. The others can be proved similarly. Let $x, y \in [\hat{c}, M, p]_0\,(\Delta^r)$ and $\alpha, \beta \in$ **C**. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\frac{1}{n} \sum_{k=1}^{n} \left[M\left(\frac{|\Delta^r x_{k+m}|}{\rho_1} \right) \right]^{p_k} \to 0$$

and

$$\frac{1}{n} \sum_{k=1}^{n} \left[M\left(\frac{|\Delta^r y_{k+m}|}{\rho_2} \right) \right]^{p_k} \to 0 \text{ uniformly in } m.$$

Define $\rho_3 = \max\,(2\,|\alpha|\,\rho_1, 2\,|\beta|\,\rho_2)$. Since M is non-decreasing, convex and Δ^r-linear.

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(\frac{|\Delta^r(\alpha x_{k+m}+\beta y_{k+m})|}{\rho_3}\right)\right]^{p_k}$$

$$\leq \frac{1}{n}\sum_{k=1}^{n}\left[M\left(\frac{|\alpha\Delta^r x_{k+m}|}{\rho_3}+\frac{|\beta\Delta^r y_{k+m}|}{\rho_3}\right)\right]^{p_k}$$

$$\leq \frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^{p_k}}\left[M\left(\frac{|\Delta^r x_{k+m}|}{\rho_1}+\frac{|\Delta^r y_{k+m}|}{\rho_2}\right)\right]^{p_k}$$

$$\leq \frac{1}{n}\sum_{k=1}^{n}\left[M\left(\frac{|\Delta^r x_{k+m}|}{\rho_1}\right)+M\left(\frac{|\Delta^r y_{k+m}|}{\rho_2}\right)\right]^{p_k}$$

$$\leq D\frac{1}{n}\sum_{k=1}^{n}\left[M\left(\frac{D'|x_{k+m}|}{\rho_1}\right)\right]^{p_k}+D\frac{1}{n}\left[M\left(\frac{\Delta'|x_{k+m}|}{\rho_2}\right)\right]^{p_k}\to 0$$

as $n\to\infty$ uniformly in m. This proves that $[\hat{c}, M, p]_0(\Delta^r)$ is linear. ■

***Theorem* 2.2** — *$[\hat{c}, M, p]_0(\Delta^r)$ is paranormed (need not total paranorm) space with*

$$g(x)=\inf\left\{\rho^{p_n/H}:\left(\frac{1}{n}\sum_{k=1}^{n}\left[M\left(\frac{|\Delta^r x_{k+m}|}{\rho}\right)\right]^{p_k}\right)^{1/H}\leq 1,\quad n=1,2,\dots,\; m=1,2,\dots\right\}$$

where $H = \max(1, \sup p_k)$.

PROOF : Clearly, $g(x)=g(-x)$. The subadditivity of g follows from the proof of Theorem 2.1, taking $\alpha=1, \beta=1$. It is trivial that $\Delta^r x_{k+m}=0$ for $x=0$. Since $M(0)=0$, we get $\inf\{\rho^{pn/H}\}=0$ for $x=0$.

The continuity of scalar multiplication follows from the following equality :

$$g(\lambda x)=\inf\left\{\rho^{pn/H}:\left(\frac{1}{n}\sum_{k=1}^{n}\left[M\left(\frac{|\Delta^r x_{k+m}|}{\rho}\right)\right]^{p_k}\right)^{1/H}\leq 1,\quad n=1,2,\dots,\; m=1,2,\dots\right\}$$

$$= \inf \left\{ (|\lambda| s)^{pn/H} : \left(\frac{1}{n} \sum_{k=1}^{n} \left[M\left(\frac{|\Delta^r x_{k+m}|}{s} \right) \right]^{p_k} \right)^{1/H} \leq 1, \quad n = 1, 2, \ldots \quad m = 1, 2, \ldots \right\}$$

where $s = \rho / |\lambda|$. ■

***Theorem* 2.3** — *Let X stand for $[\hat{c}, M]_0$, $[\hat{c}, M]$ or $[\hat{c}, M]_\infty$ and $r \geq 1$. Then the inclusion $X(\Delta^{r-1}) \subset X(\Delta^r)$ is strict. In general $X(\Delta^i) \subset X(\Delta^r)$ for all i = 1, 2, ..., $r - 1$ and the inclusion is strict.*

PROOF : We give the proof for $[\hat{c}, M]_\infty$ only. It can be proved in a similar way for $[\hat{c}, M]$ and $[\hat{c}, M]_0$. Let $x \in [\hat{c}, M]_\infty(\Delta^{r-1})$. Then we have

$$\sup_{m,n} \frac{1}{n} \sum_{k=1}^{n} \left[M\left(\frac{|\Delta^{r-1} x_{k+m}|}{\rho} \right) \right] < \infty$$

for some $\rho > 0$. Since M is non-decreasing and convex function we have

$$\frac{1}{n} \sum_{k=1}^{n} \left[M\left(\frac{|\Delta^r x_{k+m}|}{2\rho} \right) \right] = \frac{1}{n} \sum_{k=1}^{n} \left[M\left(\frac{|\Delta^{r-1} x_{k+m} - \Delta^{r-1} x_{k+m+1}|}{2\rho} \right) \right]$$

$$\leq \frac{1}{n} \sum_{k=1}^{n} \left[\frac{1}{2} M\left(\frac{|\Delta^{r-1} x_{k+m}|}{\rho} \right) \right] + \frac{1}{n} \sum_{k=1}^{n} \left[\frac{1}{2} M\left(\frac{|\Delta^{r-1} x_{k+m+1}|}{\rho} \right) \right] < \infty$$

Thus $[\hat{c}, M]_\infty(\Delta^{r-1}) \subset [\hat{c}, M]_\infty(\Delta^r)$. Proceeding in this way one will have $[\hat{c}, M]_\infty(\Delta^i) \subset [\hat{c}, M]_\infty(\Delta^r)$ for i = 1, 2, ..., $r - 1$. The inclusion is strict the sequence $x = (k^r)$, for example, belongs to $[\hat{c}, M]_\infty(\Delta^r)$, but does not belong to $[\hat{c}, M]_\infty(\Delta^{r-1})$ for $M(x) = x$. (If $x = (k^r)$, then $\Delta^r x_k = (-1)^r r!$ and $\Delta^{r-1} x_k = (-1)^{r+1} r! \left(k + \frac{(r-1)}{2} \right)$ for all $k \in N$) ■

***Theorem* 2.4** — *Let M, M_1, M_2 be Orlicz functions those satisfy Δ_2-condition. Then we have*

(*i*) $[\hat{c}, M_1, p]_0(\Delta^r) \subset [\hat{c}, M \circ M_1, p]_0(\Delta^r)$,

(*ii*) $[\hat{c}, M_1, p]_0(\Delta^r) \cap [\hat{c}, M_2, p]_0(\Delta^r) \subset [\hat{c}, M_1 + M_2, p]_0(\Delta^r)$.

PROOF : (*i*) Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \leq t \leq \delta$. Write $y_{k+m} = M_1\left(\frac{|\Delta^r x_{k+m}|}{\rho} \right)$ and consider

$$\sum_{k=1}^{n} [M(y_{k+m})]^{p_k} = \sum_{1} [M(y_{k+m})]^{p_k} + \sum_{2} [M(y_{k+m})]^{p_k}$$

where the first summation is over $y_{k+m} \leq \delta$ and second summation is over $y_{k+m} > \delta$. Since M is continuous, we have

$$\sum_{1} [M(y_{k+m})]^{p_k} < n\,\varepsilon^{H} \qquad \ldots (2)$$

and for $y_{k+m} > \delta$, we use the fact that

$$y_{k+m} < \frac{y_{k+m}}{\delta} \leq 1 + \frac{y_{k+m}}{\delta}.$$

Since M is non-decreasing and convex, it follows that

$$M(y_{k+m}) < M\left(1 + \frac{y_{k+m}}{\delta}\right) < \frac{1}{2} M(2) + \frac{1}{2} M\left(2\frac{y_{k+m}}{\delta}\right).$$

Since M satisfies Δ_2-condition, therefore there exists $K \geq 1$ such that

$$M(y_{k+m}) < \frac{1}{2} K \frac{y_{k+m}}{\delta} M(2) + \frac{K}{2}\frac{y_{k+m}}{\delta} M(2) = K y_{k+m}\,\delta^{-1} M(2).$$

Hence $$\frac{1}{n}\sum_{2} [M(y_{k+m})]^{p_k} \leq \max\left(1, (K\delta^{-1} M(2)^{H})\right) \frac{1}{n}\sum_{k=1}^{n} y_{k+m}^{p_k}. \qquad \ldots (3)$$

From (2) and (3), we obtain

$$[\hat{c}, M, p]_0 (\Delta^r) \subset [\hat{c}, M \circ M_1, p]_0 (\Delta^r).$$

(*ii*) From (1) we have

$$\left[(M_1 + M_2)\left(\frac{|\Delta^r x_{k+m}|}{\rho}\right)\right]^{p_k} \leq D\left[M_1\left(\frac{|\Delta^r x_{k+m}|}{\rho}\right)\right]^{p_k} + D\left[M_2\left(\frac{|\Delta^r x_{k+m}|}{\rho}\right)\right]^{p_k}$$

Let $x \in [\hat{c}, M_1, p]_0 (\Delta^r) \cap [\hat{c}, M_2, p]_0 (\Delta^r)$. When adding the above inequality from $k = 1$ to n we get $x \in [\hat{c}, M_1 + M_2, p]_0 (\Delta^r)$. ∎

The proof of the following result is a routine work in view of the above Theorem.

Corollary 2.5 — Let M, M_1, M_2 be Orlicz functions those satisfy Δ_2-condition. Then we have

(*i*) $[\hat{c}, M_1, p] (\Delta^r) \subset [\hat{c}, M \circ M_1, p] (\Delta^r)$

(*ii*) $[\hat{c}, M_1, p] (\Delta^r) \cap [\hat{c}, M_2, p] (\Delta^r) \subset [\hat{c}, M_1 + M_2, p] (\Delta^r)$, (*iii*) $[\hat{c}, M_1, p]_\infty (\Delta^r) \subset [\hat{c}, M \circ M_1, p]_\infty (\Delta^r)$

(*iv*) $[\hat{c}, M_1, p]_\infty (\Delta^r) \cap [\hat{c}, M_2, p]_\infty (\Delta^r) \subset [\hat{c}, M_1 + M_2, p]_\infty (\Delta^r)$.

The proof of the following result is a consequence of Theorem 2.4. (*i*) and Corollary 2.5. (*i*) and (*iii*).

Proposition 2.6 — Let M be an Orlicz function which satisfies Δ_2-condition. Then

(*i*) $[\hat{c}, p] (\Delta^r) \subset [\hat{c}, M, p] (\Delta^r)$

(*ii*) $[\hat{c}, p]_0 (\Delta)^r \subset [\hat{c}, M, p]_0 (\Delta^r)$,

(*iii*) $[\hat{c}, p]_\infty (\Delta)^r \subset [\hat{c}, M, p]_\infty (\Delta^r)$.

Theorem 2.7 — *Let* $0 < p_k \le q_k$ *and* (q_k/p_k) *be bounded.*

Then $[\hat{c}, M, q] (\Delta^r) \subset [\hat{c}, M, p] (\Delta^r)$.

PROOF : If we take $w_{k,m} = \left[M \left(\frac{|\Delta^r x_{k+m} - L|}{\rho} \right) \right]^{q_k}$ for all k, then using the same technique of Theorem 2 of Nanda[18], it is easy to prove of the Theorem. ■

Theorem 2.8 *The sequence spaces* $[\hat{c}, M, p]_0$ *and* $[\hat{c}, M, p]_\infty$ *are solid.*

PROOF : We give the proof for $[\hat{c}, M, p]_0$. Let $(x_k) \in [\hat{c}, M, p]_0$ and (α_k) be sequences of scalars such that $|\alpha_k| \le 1$ for all $k \in \mathbf{N}$. Then we have

$$n^{-1} \sum_{k=1}^{n} \left[\left(\frac{|\alpha_{k+m} x_{k+m}|}{\rho} \right) \right]^{p_k} \le n^{-1} \sum_{k=1}^{n} \left[\left(\frac{|x_{k+m}|}{\rho} \right) \right]^{p_k} \to 0, \ (n \to \infty),$$

uniformly in m.

Hence $(\alpha_k x_k) \in [\hat{c}, M, p]_0$ for all sequences of scalars (α_k) with $|\alpha_k| \le 1$ for all $k \in \mathbf{N}$, whenever $(x_k) \in [\hat{c}, M, p]_0$.

Remark 1 : In general it is difficult to predict about the solidity of $[\hat{c}, M, p]_0 (\Delta^r)$ and $[\hat{c}, M, p]_\infty (\Delta^r)$. For this consider the following example.

Example 1 — Let $p_k = 1$ for all k, $r = 1$ and $M(x) = x$. Then $(x_k) = (k) \in [\hat{c}, M, p]_\infty (\Delta)$ but $(\alpha_k x_k) \notin [\hat{c}, M, p]_\infty (\Delta)$ when $\alpha_k = (-1)^k$ for all $k \in \mathbf{N}$. Hence $[\hat{c}, M, p]_\infty (\Delta)$ is not solid.

From the above theorem and Remark 1, we may give the following corollary.

Corollary 2.9 — *i*) The sequence spaces $[\hat{c}, M, p]_0$ and $[\hat{c}, M, p]_\infty$ are monotone.

(*ii*) The sequence spaces $[\hat{c}, M, p]_0 (\Delta^r)$ and $[\hat{c}, M, p]_\infty (\Delta^r)$ are not perfect.

***Theorem* 2.10** — *The sequence spaces* $[\hat{c}, M p](\Delta^r)$ *and* $[\hat{c}, M, p]_\infty(\Delta^r)$ *are not symmetric for* $r \geq 1$.

PROOF : To show that the spaces are not symmetric, consider the following example.

Example 2 — Let $p_k = 1$ for all k, $r = 1$ and $M(x) = x$. Then $(x_k) = (k) \in [\hat{c}, M, p]_\infty(\Delta^r)$.

Let (y_k) be a rearrangement of (x_k), which is defined as follows :

$$(y_k) = \left\{ x_1, x_2, x_4, x_3, x_9, x_5, x_{16} x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, \ldots \right\}.$$

Then $$(y_k) \notin [\hat{c}, M, p]_\infty(\Delta^r).$$

Remark 2 : The example 2 shows that the space $[\hat{c}, M, p]_0(\Delta^r)$, is not symmetric for $r \geq 2$.

3. Δ^r-ALMOST STATISTICAL CONVERGENCE

The notion of statistical convergence was introduced by Fast[4] and studied by various authors[2,7,11,21,22].

In this section we define Δ^r-almost staistically convergent sequences and give some inclusion relations between $\hat{s}(\Delta^r)$ nd $[\hat{c}](\Delta^r)$. It is also shown that if a sequence is Δ^r-strongly almost summable with respect to an Orlicz function, then it is Δ^r-almost statistically convergent.

Definition 2 — A sequence $x = (x_k)$ is said to be Δ^r-almost statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : |\Delta^r x_{k+m} - L| \geq \varepsilon \right\} \right| = 0, \text{ uniformly in } m$$

In this case we write $\hat{s}(\Delta^r) - \lim x = L$ or $x_k \to L s(\Delta^r)$.

Definition 3 — A sequence $x = (x_k)$ is said to be Δ^r-strongly almost convergent to the number L if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |\Delta^r x_{k+m} - L| = 0, \text{ uniformly in } m.$$

In this case we write $[\hat{c}(\Delta^r) - \lim x = L$ or $x_k \to L [\hat{c}](\Delta^r)$.

***Theorem* 3.1** — *(i)* $x_k \to L [\hat{c}](\Delta^r) \Rightarrow x_k \to L \hat{s}(\Delta^r)$,

(ii) If $x \in l_\infty(\Delta^r)$ *and* $x_k \to L \hat{s}(\Delta^r)$, *then* $x_k \to L [\hat{c}](\Delta^r)$,

(iii) $\hat{s}(\Delta^r) \cap l_\infty(\Delta^r) = [\hat{c}](\Delta^r) \cap l_\infty(\Delta^r)$.

PROOF : (*i*) Let $\varepsilon > 0$ and $x_k \to L [\hat{c}](\Delta^r)$. Then we have

$$\sum_{k=1}^{n} |\Delta^r x_{k+m} - L| \geq \varepsilon \left| \left\{ k \leq n : |\Delta\, x_{k+m} - L| \geq \varepsilon \right\} \right|.$$

Hence $\quad x_k \to L\, \hat{s}\,(\Delta^r)$.

(ii) Suppose that $x_k \to L\, \hat{s}\,(\Delta^r)$ and $x \in l_\infty(\Delta^r)$, say that $|\Delta^r x_{k+m} - L| \leq M$ for all k and m. Given $\varepsilon > 0$, we get

$$n^{-1} \sum_{k=1}^{n} |\Delta^r x_{k+m} - L| = n^{-1} \sum_{|\Delta^r x_{k+m} - L| \geq \varepsilon} |\Delta^r x_{k+m} - L|$$

$$+\, n^{-1} \sum_{|\Delta^r x_{k+m} - L| < \varepsilon} |\Delta^r x_{k+m} - L|$$

$$\leq \frac{K}{n} \left| \left\{ k \leq n : |\Delta^r x_{k+m} - L| \geq \varepsilon \right\} \right| + \varepsilon.$$

Hence $\quad x_k \to L\, [\hat{c}\,(\Delta^r)]$.

(iii) This immediately follows from (i) and (ii). ■

***Theorem* 3.2** — *Let M be an Orlicz function. Then $[\hat{c}, M]\,(\Delta^r) \subset \hat{s}\,(\Delta^r)$.*

PROOF : Let $x \in [\hat{c}, M]\,(\Delta^r)$ and $\varepsilon > 0$ be given. Then

$$\frac{1}{n} \sum_{k=1}^{n} \left[M\left(\frac{|\Delta^r x_{k+m} - L|}{\rho} \right) \right] \geq \frac{1}{n} \sum_{|\Delta^r x_{k+m} - L| \geq \varepsilon} M\left(\frac{|\Delta^r x_{k+m} - L|}{\rho} \right)$$

$$\geq M\left(\frac{\varepsilon}{\rho} \right) \frac{1}{n} \left| \left\{ k \leq n : |\Delta^r x_{k+m} - L| \geq \varepsilon \right\} \right|, \text{ uniformly in } m.$$

Hence $\quad x \in \hat{s}\,(\Delta^r)$. ■

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