Indian J. pure appl. Math., **34**(8) : 1153-1163, August 2003

SHAPE OPERATOR FOR SLANT SUBMANIFOLDS IN GENERALIZED COMPLEX SPACE FORMS

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(*Received* 8 *May* 2002; *accepted* 28 *November* 2002)

Some relationships of the shape operator with the sectional curvature and the k-Ricci curvature for slant sumbanifolds in generalized complex space forms are established.

Key Words: ***k*-Ricci Curvature; Generalized Complex Space Form; *RK*-Manifold; Complex Space Form; Slant Submanifold; Totally Real Sub-Manifold; Invariant Submanifold**

1. INTRODUCTION

Nash's immersion theorem guarantees for every n-dimensional Reimannian manifold to admit an isometric immersion into Euclidean space $\mathbb{E}^{\frac{n(n+1)(3n+11)}{2}}$. Thus, one is able to consider any Riemannian manifold as a submanifold of Euclidean space; and this provides a natural motivation for the study of submanifolds of Riemannian manifolds. To find simple relationships between the main intrinsic invariants and the main extrinsic invariants of a submanifold is one of the basic interests of study in the submanifold theory. Gauss-Bonnet Theorem, Isoperimetric inequality and Chern-Lashof Theorem provide relations between extrinsic and extrinsic invariants for a submanifold in a Euclidean space.

In[2], Chen establishes a relationship between sectional curvature and the shape operator for submanifolds in a real space form. He also establishes sharp relationship between the k-Ricci curvature and the shape oprator for a submanifold in a real space form[3].

On the other hand, Gray introduced the notion of constant type for a nearly Kähler manifold[5], which led to definitions of RK-manifolds $\tilde{M}(c, \alpha)$ of constant holomorphic sectional curvature c and constant type α [10] and generalized complex space forms $\tilde{M}(f_1, f_2)$ [7]. We have the inclusion relation $\tilde{M}(c) \subset \tilde{M}(c, \alpha) \subset \tilde{M}(f_1, f_2)$, where $\tilde{M}(c)$ is the complex space form of constant holomorphic sectional curvature c.

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Thus it is worthwhile to study relationships between intrinsic and extrinsic invariants of submanifolds in a generalized space form. In this paper, we establish several such relationships for slant, totally real and invariant submanifolds in generalized complex space forms, complex space forms and RK-manifolds. The paper is organized as follows. Section 2 is preliminary in nature. It contains necessary details about generalized complex space form and its submanifolds. In Section 3, we establish a relation of the shape operator with the sectional curvature for slant sumbanifolds in generalized complex space form. As an application, we list particular results for different kind of submanifolds in a table. In the last section, a relation between the shape operator and the k-Ricci curvature for slant submanifolds in generalized complex space form is established.

2. PRELIMINARIES

Let $\tilde{M}$ be an almost Hermitian manifold with an almost Hermitian structure $(J, \langle , \rangle)$. An almost Hermitian manifold becomes a nearly Kähler manifold[5] if $\left(\tilde{\nabla}_X J\right)X = 0$, and becomes a Kähler manifold if $\tilde{\nabla} J = 0$ for all $X \in T\tilde{M}$, where $\tilde{\nabla}$ is the Levi-Civita connection of the Riemannian metric $\langle , \rangle$. An almost Hermitian manifold with J-invariant Riemannian curvature tensor $\tilde{R}$, that is,

$$\tilde{R}(JX, JY, JZ, JW) = \tilde{R}(X, Y, Z, W), \quad X, Y, Z, W \in T\tilde{M},$$

is called an RK-manifold[10]. All nearly Kähler manifolds belong to the class of RK-manifolds.

The notion of constant type was first introduced by Gray for a nearly Kähler manifold[5]. An almost Hermitian manifold $\tilde{M}$ is said to have (point-wise) constant type if for each $p \in \tilde{M}$ and for all $X, Y, Z \in T_p\tilde{M}$ such that

$$\langle X, Y\rangle = \langle X, Z\rangle = \langle X, JY\rangle = \langle X, JZ\rangle = 0, \ \langle Y, Y\rangle = 1 = \langle Z, Z\rangle$$

we have

$$\tilde{R}(X, Y, X, Y) - \tilde{R}(X, Y, JX, JY) = \tilde{R}(X, Z, X, Z) - \tilde{R}(X, Z, JX, JZ).$$

An RK-manifld $\tilde{M}$ has (pointwise) constant type if and onlyif there is a differentiable function α on $\tilde{M}$ satisfying [10].

$$\tilde{R}(X, Y, X, Y) - \tilde{R}(X, Y, JX, JY) = \alpha\{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2 - \langle X, JY\rangle^2\}$$

for all $X, Y \in T\tilde{M}$. Furthermore, $\tilde{M}$ has global constant type if α is constant. The function α is called the constant type of $\tilde{M}$. An RK-manifold of constant holomorphic sectional curvature c and constant type α is denoted by $\tilde{M}(c, \alpha)$. For $\tilde{M}(c, \alpha)$ it is known that[10]

$$4\tilde{R}(X, Y)Z = (c + 3\alpha)\ \{\langle Y, Z\rangle X - \langle X, Z\rangle Y\}$$
$$+ (c - \alpha)\ \{\langle X, JZ\rangle - JY - \langle Y, JZ\rangle JX + 2\langle X, JY\rangle JZ\}$$

for all $X, Y, Z \in T\tilde{M}$. If $c = \alpha$ then $\tilde{M}(c, \alpha)$ is a space of constant curvature. A complex space form $\tilde{M}(c)$ (a Kähler manifold of constant holomorphic sectional curvature c) belongs to the class of almost Hermitian manifolds $\tilde{M}(c, \alpha)$ (with the constant type zero).

An almost Hermitian manifold $\tilde{M}$ is called a generalized complex space form $\tilde{M}(f_1, f_2)$ [7] if its Riemannian curvature tensor $\tilde{R}$ satisfies

$$\tilde{R}(X,Y)Z = f_1\ \{\langle Y,Z\rangle X - \langle X,Z\rangle Y\}$$
$$+ f_2\ \{\langle X,JZ\rangle JY - \langle Y,JZ\rangle JX + 2\langle X,JY\rangle JZ\} \qquad \text{... (1)}$$

for all $X, Y, Z \in T\tilde{M}$, where f_1 and f_2 are smooth functions on $\tilde{M}$.

The Riemannian invariants are the intrinsic characteristics of a Riemannian manifold. Here, we recall a number of Riemannian invariants in a Riemanninan manifold[4]. Let M be a Riemannian manifold and L be a k-plane section of T_pM. For a unit vector U, L, we choose an orthonormal basis $\{e_1, ..., e_k\}$ of L such that $e_1 = U$. Let K_{ij} mean the sectional curvature of the plane section spanned by e_i and e_j at $p \in M$. Then the Ricci curvature Ric_L of L at U is given by

$$\mathrm{Ric}_L(U) = K_{12} + K_{13} + \ldots + K_{1k}. \qquad \text{... (2)}$$

$\mathrm{Ric}_L(U)$ is called a k-Ricci curvature. The scalar curvature τ of the k-plane section L is given by

$$\tau(L) = \sum_{1 \le i \le j \le k} K_{ij}. \qquad \text{... (3)}$$

Given an orthonormal basis $\{e_1, ..., e_n\}$ for T_pM, $\tau_{1 \ldots k}$ will denote the scalar curvature of the k-plane section spanned by $e_1, ..., e_k$. For each integer k, $2 \le k \le n$, the Riemannian invariant θ_k on an n-dimensional Riemannian manifold M is defined by

$$\theta_k(p) = \left(\frac{1}{k-1}\right) \inf_{L,X} \mathrm{Ric}_L(X), \quad p \in M, \qquad \text{... (4)}$$

where L runs over all k-plane sections in T_pM and X runs over all unit vectors in L.

Let M be an n-dimensional submanifold in a manifold $\tilde{M}$ equipped with a Riemannian metric $\langle , \rangle$. The Gauss and Weingarten formulae are given respectively by $\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y)$ and $\tilde{\nabla}_X N = -A_N X + \nabla_X^{\perp} N$ for all $X, Y \in TM$ and $N \in T^{\perp} M$, where $\tilde{\nabla}$, ∇ and $\nabla^{\perp}$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\tilde{M}$, M and the normal bundle $T^{\perp} M$ of M respectively, and σ is the second fundamental form related to the shape operator A_N in the direction of N by $\langle \sigma(X,Y), N\rangle = \langle A_N X, Y\rangle$. The mean curvature vector H is expressed by $nH = \mathrm{trace}\ \sigma$. The submanifold M is totally geodesic in $\tilde{M}$ if $\sigma = 0$. The relative null space of M at a point $p \in M$ is defined by

$$\mathcal{N}_p = \{X \in T_pM \mid \sigma(X,Y) = 0 \text{ for all } Y \in T_pM\}.$$

In a submanifold M of an almost Hermitian manifold, for a vector $0 \ne X_p \in T_pM$, the angle $\theta(X_p)$ between JX_p and the tangent space T_pM is called the Wirtinger angle of X_p. If the Wirtinger angle is independent of $p \in M$ and $X_p \in T_pM$, then M is called a slant submanifold[1]. We put $JX = PX + FX$ for $X \in TM$, where PX (resp. FX) is the projection of JX on TM (resp. $T^{\perp}M$). Slant submanifolds of almost Hermitian manifolds are characterized by the condition $P^2 + \lambda^2 I = 0$ for some real number $\lambda \in [0, 1]$. Invariant and anti-invariant submanifolds are slant submanifolds with $\theta = 0$ ($F = 0$) and $\theta = \pi/2$ ($P = 0$) respectively. For more details about slant submanifolds we refer to [1].

3. Shape Operator and Sectional Curvature

Let M be an n-dimensional θ-slant submanifold of a $2m$-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. Let $p \in M$ and a number of $f_p > f_1 + 3f_2 \cos^2 \theta$ such that the sectional curvature $K \geq f_p$ at p. We choose an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{2m}\}$ at p such that e_{n+1} is parallel to the mean curvature vector H, and $e_1, ..., e_n$ diagonalize the shape operator A_{n+1}. Then we have

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}, \qquad \text{... (5)}$$

$$\text{trace } A_r = \sum_{i=1}^{n} \sigma_{ii}^r = 0, \quad A_r = \left(\sigma_{ij}^r\right) i, j = 1, ..., n; \quad r = n+2, ..., 2m. \qquad \text{... (6)}$$

For $i \neq j$, we put

$$u_{ij} \equiv a_i a_j. \qquad \text{... (7)}$$

Moreover, Gauss equation is given by[8]

$$\begin{aligned} R(X, Y Z W) = {} & f_1 \{\langle Y, Z\rangle \langle X, W\rangle - \langle X, Z\rangle \langle Y, W\rangle\} \\ & + f_2 \{\langle X, PZ\rangle \langle PY, W\rangle - \langle Y, PZ\rangle \langle PX, W\rangle + 2\langle X, PY\rangle \langle PZ, W\rangle\} \\ & + \langle \sigma(X, W), \sigma(Y, Z)\rangle - \langle \sigma(X, Z), \sigma(Y, W)\rangle \end{aligned} \qquad \text{... (8)}$$

for all $X, Y, Z, W \in TM$, where R is the curvature tensors of M. In view of eq. (8), for $X = Z = e_i$, $Y = W = e_j$, we have

$$u_{ij} \geq f_p - f_1 - 3f_2 \langle e_i, Pe_j\rangle^2 - \sum_{r=n+2}^{2m} \left(\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2 \right). \qquad \text{... (9)}$$

Now, we prove the following Lemma.

Lemma 3.1 — For u_{ij} we have the following properties:

(1) For any fixed $i \in \{1, ..., n\}$, we have

$$\sum_{i \neq j} u_{ij} \geq (n-1)(f_p - f_1 - 3f_2 \cos^2 \theta).$$

(2) For distinct $i, j, k \in \{1, ..., n\}$ it follows that $a_i^2 = u_{ij}\, u_{ik}/u_{jk}$.

(3) For a fixed k, $1 \leq k \leq \left[\frac{n}{2}\right]$ and for each $B \in S_k \equiv \{B \subset \{1, ..., n\} : |B| = k\}$, we have

$$\sum_{j \in B} \sum_{t \in \overline{B}} u_{jt} \geq k(n-k)(f_p - f_1 - 3f_2 \cos^2 \theta).$$

where $\overline{B}$ is the complement of B in $\{1, ..., n\}$.

(4) For distinct $i, j \in \{1, ..., n\}$, it follows that $u_{ij} > 0$.

PROOF : (1) From (6), (7) and (9), we obtain

$$\sum_{i \neq j} u_{ij} \geq (n-1)(f_p - f_1 - 3f_2 \cos^2 \theta) - \sum_{r=n+2}^{2m} \left(\sigma_{ii}^r \left(\sum_{j \neq i} \sigma_{jj}^r \right) - \sum_{j \neq i} (\sigma_{ij}^r)^2 \right)$$

$$= (n-1(f_p - f_1 - 3f_2 \cos^2 \theta) - \sum_{r=n+2}^{2m} \left(\sigma_{ii}^r (-\sigma_{ii}^r) - \sum_{j \neq i} (\sigma_{ij}^r)^2 \right)$$

$$= (n-1)(f_p - f_1 - 3f_2 \cos^2 \theta) + \sum_{r=n+2}^{2m} \sum_{j=1}^{n} (\sigma_{ij}^r)^2$$

$$\geq (n-1)(f_p - f_1 - 3f_2 \cos^2 \theta) > 0.$$

(2) We have $u_{ij}\, u_{ik}/u_{jk} = a_i a_j a_i a_k / a_j a_k = a_i^2$.

(3) Let $B = \{1, ..., k\}$ and $\overline{B} = \{k+1, ..., n\}$. Then

$$\sum_{j \in B} \sum_{t \in \overline{B}} u_{jt} \geq k(n-k)(f_p - f_1 - 3f_2 \cos^2 \theta)$$

$$- \sum_{r=n+2}^{2m} \left(\sum_{j=1}^{k} \sum_{t=k+1}^{n} \left[\sigma_{jj}^r \sigma_{tt}^r - (\sigma_{jt}^r)^2 \right] \right)$$

$$= k(n-k)(f_p - f_1 - 3f_2 \cos^2 \theta)$$

$$+ \sum_{r=n+2}^{2m} \left(\sum_{j=1}^{k} \sum_{t=k+1}^{n} (\sigma_{jt}^r)^2 + \sum_{j=1}^{k} (\sigma_{jj}^r) \right)$$

$$\geq k(n-k)(f_p - f_1 - 3f_2 \cos^2 \theta).$$

(4) For $i \neq j$, if $u_{ij} = 0$ then $a_i = 0$ or $a_j = 0$. The statement $a_i = 0$ implies that $u_{il} = a_i a_l = 0$ for all $l \in \{1, ..., n\}$, $l \neq i$. Then, we get

$$\sum_{j \neq i} u_{ij} = 0,$$

which is a contradiction with (1). Thus, for $i \neq j$, it follows that $u_{ij} \neq 0$. We assume that $u_{1n} < 0$. From (2), for $1 < i < n$, we get $u_{1i} u_{in} < 0$. Without loss of generality, we may assume

$$u_{12}, ..., u_{1l}, u_{(l+1)n}, ..., u_{(n-1)n} > 0$$

$$u_{1(l+1)}, ..., u_{1n}, u_{2n}, ..., u_{ln} < 0, \quad ... (10)$$

for some $\left[\frac{n+1}{2}\right] \leq l \leq n-1$. If $l = n-1$, then $u_{1n} + u_{2n} + \ldots + u_{(n-1)n} < 0$, which contradicts to (1). Thus $l < n-1$. From (2), we get:

$$a_n^2 = \frac{u_{in} u_{tn}}{u_{it}} > 0, \quad ... (11)$$

where $2 \leq i \leq l, l+1 \leq t \leq n-1$. By (10) and (11), we obtain $u_{it} < 0$ which implies that

$$\sum_{i=1}^{l} \sum_{t=l+1}^{n} u_{it} = \sum_{i=2}^{l} \sum_{t=l+1}^{n-1} u_{it} + \sum_{i=1}^{l} u_{in} + \sum_{t=l+1}^{n} u_{1t} < 0,$$

which is a contradiction to (3). Thus (4) is proved. □

Chen established a sharp relationship between the shape operator and the sectional curvature for submanifolds in real space forms[2]. A similar inequality for a slant submanifold in a complex space form is proved in[6]. As a natural generalization to the above two kinds of results, we establish a similar inequality between the shape operator and the sectional curvature for slant submanifolds in a generalized complex space form in the following theorem.

***Theorem* 3.2** — *Let $i : M \to \tilde{M}(f_1, f_2)$ be an isometric immersion of an n-dimensional θ-slant submanifold into a 2m-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. If at a point $p \in M$ there exists a number $f_p > f_1 + 3f_2 \cos^2 \theta$ such that the sectional curvature $K \geq f_p$ at p, then the shape operator A_H at the mean curvature vector satisfies*

$$A_H > \frac{n-1}{n}(f_p - f_1 - 2f_2 \cos^2 \theta)\, I_n, \quad at\ p, \quad ... (12)$$

where I_n is the identity map.

PROOF :Let $p \in M$ and a number $f_p > f_1 + 3f_2 \cos^2 \theta$ such that the sectional curvature $K \geq f_p$ at p. Choose an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{2m}\}$ at p such that e_{n+1} is parallel to the mean curvature vector H, and $e_1, ..., e_n$ diagonalize the shape operator A_{n+1}. Now, from Lemma 3.1 it follows that $a_1, ..., a_n$ have the same sign. We assume that $a_j > 0$ for all $j \in \{1, ..., n\}$.

Then

$$\sum_{j \neq i} u_{ij} = a_i (a_1 + \ldots + a_n) - a_i^2 \geq (n-1)(f_p - f_1 - 3f_2 \cos^2 \theta).$$

From (5) and the above relation, we have

$$a_i n \| H \| \geq (n-1)(f_p - f_1 - 3f_2 \cos^2 \theta) + a_i^2$$

$$> (n-1)(f_p - f_1 - 3f_2 \cos^2 \theta),$$

which implies that

$$a_i \| H \| > \frac{n-1}{n}(f_p - f_1\ 3f_2 \cos^2 \theta),$$

and consequently we get (12). □

As an application, from the above theorem we are able to state the following corollary

Corollary 3.3 — We have the following table:

Manifold	Submanifold	$K \geq f_p$	Inequality
$\tilde{M}(f_1, f_2)$	totally real	$f_p > f_1$	$nA_H > (n-1)(f_p - f_1) I_n$
$\tilde{M}(f_1, f_2)$	invariant	$f_p > f_1 + 3f_2$	$nA_H > (n-1)(f_p - f_1 - 3f_2) I_n$
$\tilde{M}(c, \alpha)$	θ-slant	$f_p > 3\alpha \sin^2 \theta + (1 + 3\cos^2 \theta) c$	$nA_H > (n-1)\ \{f_p - 3\alpha \sin^2 \theta - (1 + 3\cos^2 \theta)c\} I_n$
$\tilde{M}(c, \alpha)$	totally real	$f_p > c + 3\alpha$	$nA_H > (n-1)(f_p - c - 3\alpha) I_n$
$\tilde{M}(c, \alpha)$	invariant	$f_p > 4c$	$nA_H > (n-1)\ (f_p - 4c) I_n$
$\tilde{M}(4c)$	θ-slant	$f_p > c(1 + 3\cos^2 \theta)$	$nA_H > (n-1)\ (f_p - c - 3c\cos^2 \theta) I_n$
$\tilde{M}(4c)$	totally real	$f_p > c$	$nA_H > (n-1)\ (f_p - c) I_n$
$\tilde{M}(4c)$	invariant	$f_p > 4c$	$nA_H > (n-1)\ (f_p - 4c) I_n$

4. Shape Operator and k-Ricci Curvature

Chen established a relationship between the shape operator and the k-Ricci curvature for a submanifold with arbitrary codimension[3]. A corresponding inequality for a slant submanifold of a complex space form is established in[6]. In this section, we prove a relationship between the shape operator and the k-Ricci curvature for an n-dimensional slant submanifold M of a $2m$-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$.

First we recall the Theorem 5.2 of [9] in the following Lemma.

Lemma 4.1 — Let M be an n-dimensional submanifold of a generalized complex space form $\tilde{M}(f_1, f_2)$. Then, we have

$$\| H \|^2 (p) \geq \theta_k(p) - f_1 - \frac{3f_2 \| P \|^2}{n(n-1)}. \quad \text{... (13)}$$

Now, we prove the theorem.

***Theorem* 4.1** — *Let M be an n-dimensional θ-slant submanifold isometrically immersed in a $2m$-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. Then, for any integer k, $2 \leq k \leq n$, and any point $p \in M$, we have*

(1) *If* $\theta_k(p) \neq f_1 + \frac{3f_2 \cos^2 \theta}{n-1}$, *then the shape operator at the mean curvature satisfies*

$$A_H > \frac{n-1}{n} \left(\theta_k(p) - f_1 - \frac{3f_2 \cos^2 \theta}{n-1} \right) I_n, \quad at\ p. \quad \text{... (14)}$$

(2) *If* $\theta_k(p) = f_1 + \frac{3f_2 \cos^2 \theta}{n-1}$, *then* $A_H \geq 0$ *at* p.

(3) *For a unit vector* $X \in T_pM$ *it follows that*

$$A_H = \frac{n-1}{n} \left(\theta_k(p) - f_1 - \frac{3f_2 \cos^2 \theta}{n-1} \right) X. \quad \text{... (15)}$$

if and only if $\theta_k(p) = f_1 + \frac{3f_2 \cos^2 \theta}{n-1}$ *and* $X \in \mathcal{N}_p$.

(4) $A_H > \frac{n-1}{n} \left(\theta_k(p) - f_1 - \frac{3f_2 \cos^2 \theta}{n-1} \right) I_n$ *at* p *if and only if* p *is a totally geodesic point, that is, the second fundamental form vanishes identically at* p.

PROOF : In view of Lemma 4.1, it follows that $H(p)$ vanishes only when $\theta_k(p) \leq f_1 + \frac{3f_2 \cos^2 \theta}{n-1}$. Consequently if $H(p) = 0$, the statements (1) and (2) are correct. Now, we assume that $H(p) \neq 0$. From the equation (5) we obtain

$$a_i a_j = K_{ij} - f_1 - 3f_2 \langle e_i, Je_j \rangle - \sum_{r=n+2}^{2m} \left(\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2 \right), \quad \text{... (16)}$$

which implies that

$$a_1(a_{i_2} + ... + a_{i_k}) = \mathrm{Ric}_{L_{1i_2 \cdots i_k}}(e_1) - (k-1)f_1 - 3f_2 \sum_{j=2}^{k} \langle e_1, Je_{i_j} \rangle$$

$$- \sum_{r=n+2}^{2m} \sum_{j=2}^{k} \left(\sigma_{11}^{r} \sigma_{i_j i_j}^{r} - (\sigma_{1 i_j}^{r})^2 \right). \quad \ldots (17)$$

From the above equation we get

$$a_1(a_2 + \ldots + a_n) = \frac{1}{C_{k-2}^{n-2}} \sum_{2 \le i_2 < \ldots < i_k \le n} \mathrm{Ric}_{L_{1 i_2 \ldots i_k}}(e_1) - (n-1) f_1$$

$$- 3f_1 \sum_{j=2}^{n} \langle e_1, Je_j \rangle + \sum_{r=n+2}^{2m} \sum_{j=1}^{n} (\sigma_{1j}^{r})^2. \quad \ldots (18)$$

From (4) and (18) we obtain

$$a_1(a_1 + a_2 + \ldots + a_n) \ge a_1(a_2 + \ldots + a_n)$$

$$\ge (n-1) \left(\theta_k(p) - f_1 - \frac{3f_2 \cos^2 \theta}{n-1} \right). \quad \ldots (19)$$

Since $n \| H \| = a_1 + \ldots + a_n$, the above equation implies

$$A_H \ge \frac{n-1}{n} \left(\theta_k(p) - f_1 - \frac{3_2 \cos^2 \theta}{n-1} \right) I_n.$$

The equality does not hold because in our case $H(p) \neq 0$.

The statement (2) is obvious.

Now, we prove the statement (3). Let $X \in T_pM$ be a unit vector satisfying (15). By (19) and (18) we get $a_1 = 0$ and $\sigma_{1j}^{r} = 0$ for all $j \in \{1, \ldots, n\}$ and $r \in \{n+2, \ldots, 2m\}$. Thus the above conditions imply that $\theta_k(p) = f_1 + \dfrac{3f_2 \cos^2 \theta}{n-1}$ and $X \in \mathcal{N}_p$. The converse is straightforward.

The equality (15) is true for $X \in T_pM$ if and only if $\mathcal{N}_p = T_pM$, that is, p is a totally geodesic point. This proves the statement (4). □

Now, we are able to state the following corollaries.

Corollary 4.3 — Ler M be an n-dimensional totally real submanifold isometrically immersed in a $2m$-dimensional generalized complex space from $\tilde{M}(f_1, f_2)$. Then, for any integer $k, 2 \le k \le n$, and any point $p \in M$, we have:

(1) If $\theta_k(p) \neq f_1$, then the shape operator at the mean curvature satisfies

$$A_H > \frac{n-1}{n} (\theta_k(p) - f_1) I_n, \qquad \text{at } p$$

where I_n denotes the identity map of T_pM.

(2) If $\theta_k(p) = f_1$, then $A_H \geq 0$ at p.

(3) A unit vector $X \in T_pM$ satisfies

$$A_H X = \frac{n-1}{n}(\theta_k(p) - f_1)X$$

if and only if $\theta_k(p) \neq f_1$ and $X \in \mathcal{N}_p$.

(4) For $p \in M, A_H = \frac{n-1}{n}(\theta_k(p) - f_1) I_n$ at p if and only if p is a totally geodesic point.

Corollary 4.4. — Let M be an n-dimensional θ-slant submanifold isometrically immersed in a $2m$-dimensional RK-manifold $\tilde{M}(c, \alpha)$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have:

(1) If $\theta_k(p) \neq (c + 3\alpha) + \frac{3(c-\alpha)\cos^2\theta}{n-1}$, then the shape operator at the mean curvature satisfies

$$A_H > \frac{n-1}{n}\left(\theta_k(p) - (c+3\alpha) - \frac{3(c-\alpha)\cos^2\theta}{n-1}\right)I_n, \quad \text{at } p,$$

where I_n denotes the identity map of T_pM.

(2) If $\theta_k(p) = (c\ 3\alpha) + \frac{3(c-\alpha)\cos^2\theta}{n-1}$, then $A_H \geq 0$ at p.

(3) A unit vector $X \in T_pM$ satisfies

$$A_H X = \frac{n-1}{n}\left(\theta_k(p) - (c+3\alpha) - \frac{3(c-\alpha)\cos^2\theta}{n-1}\right)X$$

if and only if $\theta_k(p) \neq (c+3\alpha) + \frac{3(c-\alpha)}{n-1}$ and $X \in \mathcal{N}_p$.

(4) For $p \in M$, $A_H = \frac{n-1}{n}\left(\theta_k(p) - (c+3\alpha) - \frac{3(c-\alpha)\cos^2\theta}{n-1}\right)I_n$ at p

if and only if p is a totally geodesic point.

Corollary 4.5 — Let M be an n-dimensional totally real submanifold isometrically immersed in a $2m$-dimensional RK-manifold $\tilde{M}(c, \alpha)$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have:

(1) If $\theta_k(p) \neq c + 3\alpha$, then the shape operator at the mean curvature satisfies

$$A_H > \frac{n-1}{n}(\theta_k(p) - c - 3\alpha) I_n, \quad \text{at } p,$$

where I_n denotes the identity map of T_pM.

(2) If $\theta_k(p) = c + 3\alpha$, then $A_H \geq 0$ at p.

(3) A unit vector $X \in T_pM$ satisfies

$$A_H X = \frac{n-1}{n}(\theta_k(p) - c - 3\alpha) X$$

if and only if $\theta_k(p) \neq c + 3\alpha$ and $X \in \mathcal{N}_p$.

(4) For $p \in M$, $A_H = \frac{n-1}{n}(\theta_k(p) - c - 3\alpha) I_n$ at p if and only if p is a totally geodesic point.

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