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SOME EXISTENCE THEOREMS FOR EVOLUTION HEMIVARIATIONAL INEQUALITIES

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We prove some existence theorems for a class of evolution hemivariational inequalities by using a surjectivity result for multivalued (S_+) type mappings, which generalize and extend previously known theorems.

Key Words : Hemivariational Inequalities; Multivalued Mappings; Existence Results

1. INTRODUCTION

Let H be a separable Hilbert space and B be a dense subspace of H carrying the structure of a separable reflexive Banach space. We assume that V compactly imbeds into H. Identifying H with its dual, we obtain $V \subseteq H \subseteq V'$ (V' is the dual space of V) which forms an evolution triple. The norm of any Banach space B is denoted by $\|\cdot\|_B$. The pairing between B and its dual space B' is denoted by $\langle \cdot, \cdot \rangle_B$. Let $2 \le p < +\infty, 0 < T < +\infty$, $1/p + 1/p' = 1$. For each $r \ge 1$, we denote by $L^r(I, B)$ the space of strongly measurable B-valued functions $b : I \equiv [0, T] \to B$ such that $\int_I \|(t)\|_B^r \, dt < +\infty$. Let $X = L^p(I, V), Z = L^p(I, H)$ and X', Z' the dual spaces of X, Z respectively, i.e., $X' = L^{p'}(I, V'), Z' = L^{p'}(I, H)$ (see Zeidler[1] p. 412). We may assume without loss of generality that X and X' are locally uniformly convex (see[1], for example). The norm convergence in X and X' is denoted by $\to$, and the weak convergence by $\rightharpoonup$.

Let $A : X \to X'$ be an operator and $g^0(t, \cdot, \cdot)$ stand for the Clarke's directional derivative (see below, for the definition). We shall study the existence of solutions to the following evolution hemivariational inequalities

Problem P — Find $u \in X$ such that $u(0) = u_0$ and

$$\langle \dot{u}, v \rangle_X + \langle Au, v \rangle_X + \int_I g^0(t, u, v)\, dt \ge \langle f, v \rangle_X \quad \forall v \in X,$$

where $f \in X'$ is given and $\dot{u}$ stands for the generalized derivative of u.

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The concept of a hemivariational inequality is introduced by Panagiotopoulos in[2]. The background of these problems are in physics, especially in Solid Mechanics, where nonmonotone, multivalued constitutive laws lead to hemivariational inequalities. We refer to Carl[3], Li[4, 5], Miettinen[6], Naniewicz and Panagiotopoulos[7] and references therein to see the applications of hemivariational inequalities. More specific, Liu[5] and Naniewicz and Panagiotopoulos[7] dealt with the stationary Problem P. For Problem P, Carl[3] used the method of upper and lower solutions and Miettinen[6] used a regularized approximating method, both of them got the existence results of Problem P with A being a linear elliptic operator of second order, while the author[4] studied the existence of solutions for Problem P, if the operator A is of class (S_+). For both mathematics and applications, we need to introduce more general assumptions on A. In this paper we shall assume that the operator A is only pseudomonotone and give a global existence theorem which generalized the corresponding ones in[2-7]

2. PRELIMINARIES

For any Banach space B, let $h : B \to R$ be Lipschitz near a given point $u \in B$, and let υ be any other vector in Banach space B. The generalized directional derivative of h at u in the direction υ, denoted by $h^0(u, \upsilon)$, is defined as follows :

$$h^0(u, \upsilon) = \limsup_{w \to u,\ \lambda \to 0^+} \lambda^{-1} [h(w + \lambda \upsilon) - h(w)],$$

where, of course, $w \in B$ and λ is a positive scalar (cf. [8, 9]), by means of which the Clarke's generalized gradient of h at u, denoted by $\partial h(u)$, is the subset of B' (the dual space of B) given by

$$\partial h(u) = \left\{ w \in B' : h^0(u, \upsilon) \geq \langle w, \upsilon \rangle_B \text{ for } all\ \upsilon \in B \right\}.$$

Concerning Problem P we deal with functional $G : L^p(I, H) \to R$ of the type

$$G(u) = \int_I g(t, u(t))\, dt, \text{ for any } u \in L^p(I, H), \qquad \text{... (2.1)}$$

where $g(t, u) : I \times H \to R$ is measurable in $t \in I$, locally Lipschitz in H and $g(t, 0) = 0$ for a.e. $t \in I$.

As in[4], we impose upon g the following growth condition :

$$(H_1)\ \ \| w \|_H \leq a_1 (\| \upsilon \|_H^{p-1} + 1)$$

for $w \in \partial g(t, \upsilon), t \in I, \upsilon \in H$, for some positive constant a_1 independent of $t \in I$ and $\upsilon \in H$. Here ∂g stands for the Clarke's generalized gradient of locally Lipschitz function g.

Hypothesis (H_1) and Lebourg's mean value theorem (see [9 p.41]) show that there exists a point u belonging to the segment $[0, v]$, $w \in \partial g(t, u)$ such that

$$|g(t, v)| = |\langle w, v \rangle_H| \leq a_1' (\| v \|_H^p + 1)$$

for all $v \in H$ and a.e., $t \in I$, with some constant $a_1' \geq 0$. It follcws that the functional $G : L^p(I, H) \to R$ in (2.1) is well-defined and Lipschitz continuous on the bounded subsets of

$L^p(I, H)$. The generalized gradient $\partial G(u)$ of G at u on $L^p(I, H)$ is the subset of $L^{p'}(I, H)$ given by

$$\partial(G \mid L^p)(u) = \{w \in L^{p'}(I, H) : G^0(u, v) \geq \langle w, v \rangle_{L^p} \text{ for all } v \in L^p(I, H)\}$$

where $$G^0(u, v) = \limsup_{w \to u,\, \lambda \to 0+} \lambda^{-1} [G(w + \lambda v) - G(w)].$$

We shall be dealing with multivalued mappings T acting from a subset $D(T)$ in X to $2^{X'}$. T is said to be bounded, if it takes bounded sets of X to bounded sets of X'. We also need the classes of mappings of monotone type. A mapping $T : D(T) \subseteq X \to 2^{X'}$ is called

— monotone (we denote $T \in (MON)$) if $\langle u^* - v^*, u - v \rangle_X \geq 0$ for all $u, v \in D(T)$ and $u^* \in T(u)$, $v^* \in T(v)$.

— quasimonotone ($T \in (QM)$) if for any sequence $\{u_n\}$ in $D(T)$ and $u_n^* \in T(u_n)$ with $u_n \to u$, we have $\limsup \langle u_n^*, u_n - u \rangle_X \geq 0$.

— pseudomonotone ($T \in (PM)$) if for any sequence $\{u_n\}$ in $D(T$ with $u_n \to u$ and for $u_n^* \in T(u_n), n \geq 1$, satisfying $u_n^* \to u^*$ and $\limsup \langle u_n^*, u_n - u \rangle_X \leq 0$, we obtain $u^* \in T(u)$ and $\langle u_n^*, u_n \rangle_X \to \langle u^*, u \rangle_X$ as $n \to \infty$.

— of class (S_+) ($T \in (S_+)$) if for any sequence $\{u_n\}$ in $D(T)$ with $u_n \to u$ and for $u_n^* \in T(u_n), n \geq 1$, satisfying $\limsup \langle u_n^*, u_n - u \rangle_X \leq 0$, we have $u_n \to u$ and there exists a subsequence $\{u_{n_k}^*\} \subset \{u_n^*\}$ such that $u_{n_k}^* \to u^* \in T(u)$.

— maximal monotone ($T \in (MM)$) if $T \in (MON)$ and its graph

$$G(T) = \{(u^*, u) \in X' \times X \mid u \in D(T), u^* \in T(u)\}$$

is not a proper subset of any monotone set in $X' \times X$.

It is well-known that the conditions

$$\| J(u) \|_{X'} = \| u \|_X, \langle J(u), u \rangle_X = \| u \|_X^2, \quad \text{for all } u \in X$$

determine a unique map J from X to X', which is called the duality map. In our case it is bijective bicontinuous strictly monotone and of class (S_+). For more details we refer to [1].

Define $L : D(L) \subset X \to X'$ by $Lu = \dot{u}$, where $D(L) = \{v \in X \mid \dot{v} \in X', v(0) = 0\}$, $\dot{u}$ stands for the generalized derivative of u, *i.e.*, $\int_I \dot{u}(t)\, \phi(t) dt = -\int_I u(t) \dot{\phi}(t) dt$ *for all* $\phi \in C_0^\infty(I)$. *Then* $\langle Lu, v \rangle_X = \int_I \langle \dot{u}(t), v(t) \rangle_V \, dt$ *for any* $u \in D(L)$ *and* $v \in X$.

Since L is a closed densely linear maximal monotone map (see Zeidler[1], p. 855 and p. 897], therefore the graph of L is a closed set in $X \times X'$. So $Y \equiv D(L)$ equipped with the graph norm

$$\| u \|_Y = \| u \|_X + \| Lu \|_{X'}, \qquad \text{for any } u \in Y,$$

becomes a real reflexive Banach space, which is completely imbedded into $Z(= L^p(I, H))$ (see Zeidler[1] p. 450).

In our study we deal with mappings of the form $F = L + S$ where L is a given linear densely defined maximal monotone map from $D(L) \subseteq X$ to X' and S is a bounded multivalued map of montone type from X to $2^{X'}$ satisfying one of the monotonicity conditions with respect to the graph norm topology of $D(L)$. Thus, for instance, we call S pseudomonotone with respect fo $D(L)$, if for any sequence $\{u_n\}$ in $D(L) \cap D(S)$ with $u_n \to u$ and $Lu_n \to Lu$ and for $u_n^* \in S(u_n)$, $n \geq 1$, satisfying $u_n^* \to u^*$ and $\lim\sup \langle u_n^*, u_n - u \rangle_X \leq 0$, we obtain $u^* \in S(u)$ and $\langle u_n^*, u_n \rangle_X \to \langle u^*, u \rangle_X$ as $n \to \infty$. Analogous definition applies for mappings of class (S_+) with respect to $D(L)$.

In the sequel, we assume that

(H_2) The mapping $A(t) : V \to V'$ satisfies the following conditions:

(i) $A(t)$ is pseudomonotone for every $t \in I$.

(ii) The function $t \to \langle A(t)u, v \rangle_V$ is measurable on I for all $u, v \in V$ and $t \to A(t)u(t) \in V'$ is measurable on I for all $u(t) \in X$.

(iii) There exist positive constants c_1, c_2 and $k_1(t) \in L^{p'}(0, T)$, $k_2(t) \in L^1(0, T)$ such that

$$\| A(t)\, u \|_{V'} \leq c_1 \| u \|_V^{p-1} - k_1(t) \text{ for each } u \in V \text{ and } t \in I$$

and
$$\langle A(t)\, u, u \rangle_V \geq c_2 \| u \|_V^p - k_2(t) \text{ for each } u \in V \text{ and } t \in I.$$

Then $A(t)$ generates a map $A : X \to X'$ by

$$A(u)(t) = A(t)\, u(t),\ t \in I.$$

It is easy to see that (ii) and (iii) imply that A is bounded with

$$\| Au \|_{X'} \leq C_1 \| u \|_X^{p-1} + C_2 \quad \forall u \in X, \qquad \text{... (2.2)}$$

$$\langle Au, u \rangle_X \geq c_2 \| u \|_X^p - C_3 \quad \forall u \in X, \qquad \text{... (2.3)}$$

with some constants $C_1, C_2 > 0$ depending on c_1, p and $\| k_1 \|_{L^{p'}}$ and C_3 depending on c_2 and $\| k_2 \|_{L^1}$.

Applying Theorem 3.4 in Chang [8 p.09] (cf. Clarke[9]), we have

$$\partial (G|_X)(u) = \partial (G|_{L^p}(u), \quad \forall u \in X. \qquad \text{... (2.4)}$$

Lemma 2.1 — (Berkovits and Mustonen[10]) — Assume that the condition (H_2) holds. Then $A : X \to X'$ is pseudomonotone with respect to $D(L)$.

Lemma 2.2 — (Liu[4]) — Under the assumption (H_1), the following inequalities hold :

$$G^0(u, v) \le c_3 (1 + \| u \|_{L^p}^{p-1}) \| v \|_{L^p} \text{ for } u, v \in L^p(I, H)$$

and $$\| u^* \|_{L^{p'}} \le c_3 (1 + \| u \|_{L^p}^{p-1}) \text{ for } u^* \in \partial (G|_{L^p})(u), u \in L^p(I, H) \quad \text{... (2.5)}$$

where c_3 is a positive constant and $\| \cdot \|_{L^p}$ is the norm of $L^p(I, H)$.

The main tool we use in this paper is a surjectivity result for multivalued (S_+) type mappings proved by the present author and Zhang in[11], for the convenience of the reader we also include it here.

Lemma 2.3 — If X is a reflexive Banach space, $L : D(L) \subseteq X \to X'$ is a closed densely linear maximal monotone operator, and $S : X \to 2^{X'}$ is of class (S_+) with respect to $D(L)$ with bounded closed convex values, and coercive (i.e., $\inf \left\{ \langle u^*, u \rangle_X : u^* \in Su \right\} / \| u \|_X \to \infty$ as $\| u \|_X \to \infty$), then $R(L + S) = X'$ (i.e., the operator $(L + S)(\cdot)$ is surjective).

3. Main Results

In the first part of this section, we consider the solvability of the following inclusions with generalized gradients

$$f \in Lu + Au + \partial (G|_X)(u), u \in D(L), \quad \text{... (3.1)}$$

which satisfies the assumptions $(H_1) - (H_2)$. This kind of problems themselves are also very interesting.

Since V is compactly imbedded into H, there exists ρ such that

$$\| u \|_H \le \rho \| u \|_V \quad \forall u \in V,$$

which implies $$\| u \|_{L^p} \le \rho \| u \|_X \quad \forall u \in X. \quad \text{... (3.2)}$$

By (2.4), for any $w \in \partial (G|_X)(u)$ we have

$$\langle w, v \rangle_X = \langle w, v \rangle_{L^p} \quad \forall v \in X. \quad \text{... (3.3)}$$

Theorem 3.1 — *Assume that the assumptions (H_1) and (H_2) hold and $c_2 - c_3 \rho^p > 0$, where c_2 and c_3 are the positive constants in eq. (2.3) and (2.5), respectively. Then for any $f \in X'$ there exists at least one solution of the inclusion (3.1)*

Proof : For each $\varepsilon > 0$, we define

$$S_\varepsilon = A + \varepsilon J + \partial (G|_X).$$

By the definition of the generalized gradient $\partial (G|_X)(u)$, it is easy to show that $S_\varepsilon(u)$ is a nonempty, bounded, closed and convex for each $u \in X$.

Now we show that S_ε is of class (S_+) with respect to $D(L)$ for each $\varepsilon > 0$.

For any sequence $\{u_n\}$ in $D(L)$ with $u_n \rightharpoonup u$ and $Lu_n \rightharpoonup Lu$ and for $u_n^* \in S_\varepsilon(u_n), n \geq 1$, satisfying lim sup $\langle u_n^*, u_n - u \rangle_X \leq 0$, which implies that there exist $w_n \in \partial(G|_X)(u_n)$ with $u_n^* = Au_n + \varepsilon J(u_n) + w_n$ such that

$$\limsup \langle Au_n + \varepsilon J(u_n) + w_n, u_n - u \rangle_X \leq 0. \quad \text{... (3.4)}$$

We first show that

$$\liminf \langle Au_n, u_n - u \rangle_X \geq 0. \quad \text{... (3.5)}$$

We prove (3.5) by contradiction. Let us assume

$$\liminf \langle Au_n, u_n - u \rangle_X < 0.$$

By the definition of lim inf, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\lim \langle Au_{n_k}, u_{n_k} - u \rangle_X = \liminf \langle Au_n, u_n - u \rangle_X < 0. \quad \text{... (3.6)}$$

In virtue of the weak convergence of $\{u_n\}$, $\{Lu_n\}$, and the boundedness of operator A, we may assume that $u_{n_k} \rightharpoonup u, Lu_{n_k} \rightharpoonup Lu$ and $Au_{n_k} \rightharpoonup u^*$. Then by Lemma 2.1, and the definition of pseudomonotonicity with respect to $D(L)$. We obtain

$$\lim \langle Au_{n_k}, u_{n_k} - u \rangle_X = 0,$$

which contradicts (3.6) and, thereby, proves (3.5).

Since $Y \equiv D(L)$ is compactly imbedded into $L^p(I, H)$, we infer that $u_n \to u$ in $L^p(I, H)$. Therefore, we obtain

$$\liminf \langle w_n, u_n - u \rangle_X = \lim \langle w_n, u_n - u \rangle_{L^p} = 0. \quad \text{... (3.7)}$$

We have from (3.4), (3.5) and (3.7) that

$$\limsup \langle J(u_n), u_n - u \rangle_X \leq 0.$$

By the (S_+) property and continuity of J, we obtain that

$$u_n \to u \text{ in } X, \quad \text{... (3.8)}$$

$$J(u_n) \to J(u) \text{ in } X'. \quad \text{... (3.9)}$$

Now we have from (3.4), (3.7) and (3.9) that

$$\limsup \langle Au_n, u_n - u \rangle_X \leq 0.$$

Since $A : X \to X'$ is a single-valued mapping, in virtue of the definition of pseudomonotonicity with respect to $D(L)$, passing to a subsequence if necessary, we may assume that

$$Au_n \rightharpoonup Au \text{ in } X'. \quad \text{... (3.10)}$$

Using Proposition 2.1.5 in [9 p. 29], we may also assume that

$$w_n \rightharpoonup w \in \partial (G|_X)(u). \quad \text{... (3.11)}$$

Therefore, from (3.9)-(3.11) we obtain

$$Au_n + \varepsilon J(u_n) + w_n \to Au + \varepsilon J(u) + w \in S_\varepsilon(u),$$

which proves that S_ε is of class (S_+) with respect to $D(L)$.

In the following, we shall show that the operator S_ε is coercive.

$$\forall w \in \partial (G|_X)(u), u \in X,$$

we have from (2.3), (2.5), (3.2) and (3.3).

$$\begin{aligned} \langle Au + \varepsilon J(u) + w, u \rangle_X &\geq c_2 \| u \|_X^p - C_3 + \varepsilon \| u \|_X^2 + \langle w, u \rangle_{L^p} \\ &\geq c_2 \| u \|_X^p - C_3 + \varepsilon \| u \|_X^2 - \| w \|_{L^{p'}} \| u \|_{L^p} \\ &\geq c_2 \| u \|_X^p - C_3 + \varepsilon \| u \|_X^2 - c_3 (1 + \| u \|_{L^p}^{p-1}) \| u \|_{L^p} \quad \text{... (3.12)} \\ &\geq (c_2 - c_3 \rho^p) \| u \|_X^p - C_3 + \varepsilon \| u \|_X^2 - c_3 \rho \| u \|_X. \end{aligned}$$

Since $p \geq 2$ and $c_2 - c_3 \rho^2 > 0$, by eq. (3.12) we obtain

$$\inf \{\langle u^*, u \rangle_X : u^* \in S_\varepsilon(u)\} \;/\; \| u \|_X \to \infty \text{ as } \| u \|_X \to \infty,$$

which proves the coercivitiy of S_ε for all $\varepsilon > 0$.

From Lemma 2.3, for each $f \in X'$, there exists $u_\varepsilon \in X$ such that

$$f \in Lu_\varepsilon + Au_\varepsilon + \varepsilon J(u_\varepsilon) + \partial (G|_X)(u_\varepsilon). \quad \text{... (3.13)}$$

Letting $\varepsilon \to 0_+$, we shall prove that $u_\varepsilon \rightharpoonup u$, and u is a solution of inclusion (3.1). To the end, by (3.13), there exists $w_\varepsilon \in \partial (G|_X)(u_\varepsilon)$ such that

$$f = Lu_\varepsilon + Au_\varepsilon + \varepsilon J(u_\varepsilon) + w_\varepsilon, \quad \text{... (3.14)}$$

Therefore, we have

$$\langle f, u_\varepsilon \rangle_X = \langle Lu_\varepsilon, u_\varepsilon \rangle_X + \langle Au_\varepsilon, u_\varepsilon \rangle_X + \varepsilon \| u_\varepsilon \|_X^2 + \langle w_\varepsilon, u_\varepsilon \rangle_X.$$

As in (3.12) and using the monotonicity of the operator L, we easily verify

$$\| f \|_{X'} \| u_\varepsilon \|_X \geq (c_2 - c_3 \rho^p) \| u_\varepsilon \|_X^p - C_3 + \varepsilon \| u_\varepsilon \|_X^2 - c_3 \rho \| u_\varepsilon \|_X,$$

which implies that

$$\| u_\varepsilon \|_X \leq \text{Const.} \qquad \text{... (3.15)}$$

We have from (2.2), (2.5) and (3.14) that

$$\| Lu_\varepsilon \|_{X'} \leq \text{Const.} \qquad \text{... (3.16)}$$

Therefore, we may choose a sequence $\varepsilon_n \to 0_+$ such that

$$u_n \to u \text{ in } X,$$

$$Lu_n \to Lu \text{ in } X', \qquad \text{... (3.17)}$$

$$u_n \to u \text{ in } L^p(I, H),$$

where we have denoted u_{ε_n} by u_n, and w_{ε_n} by w_n in the following.

By (2.2) and (2.5), we may assume that

$$Au_n \to u^* \text{ in } X'.$$

$$w_n \to w \text{ in } X'.$$

Consequently,

$$\limsup \langle Au_n, u_n - u \rangle_X = \limsup \{\langle f, u_n - u \rangle_X - \langle Lu_n, u_n - u \rangle_X$$
$$- \varepsilon_n \langle J(u_n), u_n - u \rangle_X - \langle w_n, u_n - u \rangle_{L^p}\}$$
$$= \limsup \{-\langle Lu_n - Lu, u_n - u \rangle_X\} \leq 0.$$

Since A is pseudomonotone with respect to $D(L)$ and single-valued, we obtain that

$$Au_n \to Au, \text{ and } \langle Au_n, u_n \rangle_X \to \langle Au, u \rangle_X \text{ as } n \to \infty. \qquad \text{... (3.18)}$$

Using proposition 2.1.5 in [9 p. 29], we also have

$$w_n \to w \in \partial(G|_{L^p})(u) = \partial(G|_X)(u). \qquad \text{... (3.19)}$$

Letting $n \to \infty$, i.e., $\varepsilon_n \to 0$ in (3.14), we have from (3.17) - (3.19) that

$$f = Lu + Au + w \in Lu + Au + \partial(G|_X)(u).$$

The proof is complete.

Now we turn to the non-zero initial-valued inclusion (3.1), that is, we consider the solvability of the inclusion

$$\begin{cases} f \in \dot{u} = Au + \partial (G\,|_X)(u) \\ u(0) = u_0. \end{cases} \qquad \ldots (3.20)$$

Assume that $u_0 \in V$. Let $\overline{A}\,u = A(u + u_0)\,\overline{G}\,u = G(u + u_0)$. Then it is easy to see that $\overline{G}$ and $\overline{A}$ also satisfy the conditions (H_1), (H_2), respectively. But the constants in (H_1) and (H_2) may depend on the initial value u_0.

By virtue of the operators L, $\overline{A}$ and $\partial\overline{G}$, one can easily verify that the problem (3.20) is equivalent to the following problem :

Find $v \in D(L)$ such that

$$f \in L\,v + \overline{A}\,v + \partial(\overline{G}\,|_X)(v).$$

Therefore, by Theorem 3.1, we have

***Theorem* 3.2** — *Assume that the conditions (H_1) and (H_2) hold, and $u_0 \in V$, $c_2 - c_3\,\rho^p > 0$, where c_2 and c_3 are the positive constans in eq. (2.3) and (2.5), respectively. Then for any $f \in X'$ there exists at least one solution of the problem (3.20).*

PROOF : Let $u_0 \in H$. Since V is dense in H, we can find a sequence $\{u_{0n}\}$ in V such that $u_{0n} \to u_0$ in H. For each $n \geq 1$, by Theorem 3.2, there exists a solution of the problem

$$\begin{cases} f \in \dot{u}_n + Au_n + \partial(G\,|_X)(u_n) \\ u_n(0) = u_{0n}, \end{cases}$$

i.e., there exists $w_n \in \partial(G\,|_X)(u_n) = \partial(G\,|_{L^p})(u_n)$ such that

$$\begin{cases} \dot{u}_n + Au_n + w_n = f \\ u_n(0) = u_{0n}. \end{cases}$$

Therefore, we have from (3.3) that

$$\frac{1}{2}\| u_n(T) \|_H^2 - \frac{1}{2}\| u_n(0) \|_H^2 + \langle Au_n, u_n \rangle_X + \langle w_n\, u_n \rangle_{L^p} = \langle f, u_n \rangle_X.$$

In virtue of (2.3) and (2.5), we obtain

$$\frac{1}{2}\| u_n(T) \|_H^2 + (c_2 - c_3\,\rho^p)\| u_n \|_X^p - c_3\,\rho\,\| u_n \|_X - C_3 \leq \| f \|_{X'}\, \| u_n \| X + \frac{1}{2}\| u_n(0) \|_H^2.$$

Since $c_2 - c_3\,\rho^p > 0$, by using the Young inequality, we have

$$\| u_n(T) \|_H^2 + c\,\| u_n \|_X^p \leq C + \sup_n \left\{ \| u_{0n} \|_H^2 \right\}, \qquad \ldots (3.21)$$

where c and C are positive constants.

By $u_{0n} \to u_0$ in H, we have that $\sup_n \left\{ \| u_{0n} \|_H^2 \right\} < \infty$. Therefore,

$$\| u_n \|_X \leq \text{Const.}$$

Like (3.16), we easily obtain

$$\| \dot{u}_n \|_{X'} \leq \text{Const.}$$

By the compact imbedding theorem (see [1, p. 450]), we may assume that

$$u_n \rightarrow u \text{ in } X,\ \dot{u}_n \rightarrow \dot{u} \text{ in } X',\ u_n \rightarrow u \text{ in } L^p(I, H). \qquad \ldots (3.22)$$

By (2.2) and (2.5), we may also assume that

$$Au_n \rightarrow u^* \text{ in } X',\ w_n \rightarrow w \text{ in } L^p(I, H).$$

Consequently,

$$\begin{aligned} \limsup \langle Au_n, u_n - u \rangle_X &= \limsup \{\langle f, u_n - u \rangle_X - \langle \dot{u}_n, u_n - u \rangle_X - \langle w_n, u_n - u \rangle_{L^p}\} \\ &= \limsup \left\{ -\langle \dot{u}_n - \dot{u}, u_n - u \rangle_X \right\} \\ &= \limsup \left\{ -\| u_n(T) - u(T) \|_H^2 + \| u_{0n} - u_0 \|_H^2 \right\} \\ &\leq 0. \end{aligned}$$

Using a procedure similar to that given in the proof of the last part in Theorem 3.1, we obtain that

$$Au_n \rightarrow Au \text{ in } X',\ w_n \rightarrow w \in \partial(G|_{L^p}(u) = \partial(G|_X(u),$$

and therefore,

$$\begin{cases} f \in \dot{u} + Au + \partial(G|_X)(u) \\ u(0) = u_0 \end{cases}$$

The proof is complete. Now we are in a position to prove our main theorem.

***Theorem* 3.4** — *Assume that all the conditions of Theorem 3.3 hold. Then for any $f \in X'$ there exists a solution of the problem P.*

PROOF : In virtue of Theorem 3.3, there exists $u \in X$ and $w \in \partial(G|_X)(u) = \partial(G|_{L^p})(u)$ such that

$$\begin{cases} f = \dot{u} + Au + w \\ u(0) = u_0 \end{cases}$$

Therefore,

$$\langle \dot{u}\ v \rangle_X + \langle Au, v \rangle_X + \langle w, v \rangle_{L^p} = \langle f,. v \rangle_X \quad \forall v \in X. \qquad \ldots (3.23)$$

Since the generalized gradient of G at u, denoted by $\partial(G|_{L^p})(u)$, is the subset of $L^{p'}(I, H)$ given by[9]

$$\partial(G|_{L^p})(u) = \left\{ w \in L^{p'}(I, H) : G^0(u, v) \geq \langle w, v \rangle_{L^p} \text{ for all } v \in L^p(I, H) \right\}$$

and
$$G^0(u, v) \leq \int_I g^0(t, u, v)\, dt.$$

In virtue of eq. (3.23), we have that $u \in X$, $u(0) = u_0$ and

$$\langle \dot{u}, v \rangle_X + \langle Au, v \rangle_X + \int_I g^0(t, u, v)\, dt \geq \langle f, v \rangle_X \quad \forall v \in X,$$

which has proved our theorem.

4. APPLICATIONS

We conclude with a simple example of parabolic hemivariational inequalities which can be treated by the theory developed above. Let Ω be a bounded open subset of R^N. By $V (= W_0^{1,2}(\Omega))$ we denote the subspace of the usual Sobolev space $W^{1,2}(\Omega)$ whose elements have generalized homogeneous boundary value. $H = L^2(\Omega)$. Then obviously $V \subseteq H \subseteq V'$ (the dual space of V).

We deal with functional $G : L^2(I, H) \to R$ of the type

$$G(u) = \int_I g(t, u(t))\, dt, \text{ for any } u \in L^2(I, H),$$

$$g(t, u) = \int_\Omega h(x, t, u(x, t))\, dx$$

where $h(x, t, s) : \Omega \times I \times R \to R$ is measurable in $\Omega \times I$, locally Lipschitz in R and $h(x, t, 0) = 0$ for a.e. $\Omega \times I$. For the sake of simplicity we shall impose upon h the following growth condition :

$$|w| \leq a(|v| + 1) \text{ for } w \in \partial h(x, t, s), (x, t) \in \Omega \times I, s \in R,$$

for some positive constant a. Here ∂h stands for the Clarke's generalized gradient of locally Lipschitz function h with respect to s. Let $p = 2$. Then it is easy to show that G is well-defined and g satisfies the growth condition (H_1).

Let $X = L^2(I, V)$, $Z = L^2(I, H)$. Define $A : X \to X'$

$$\langle Au, v \rangle_X = \sum_{i=1}^{n} \int_I \int_\Omega a_i(x, t, u, \nabla u) \frac{\partial u}{\partial x_i}\, dx\, dt \text{ for all } u, v \in X.$$

The aim of this example is to show the existence of solutions for the following parabolic hemivariational inequality :

Find $u \in X$ such that $u(0) = u_0$ and

$$\langle \dot{u}, v \rangle_X + \langle Au, v \rangle_X + \int_I g^0(t, u, v)\, dt \geq \langle f, v \rangle_X \text{ for any } v \in X, \quad \ldots (4.1)$$

where g^0 is the generalized directional derivative of g at u in the direction v.

We impose the standard conditions of Leray-Lions type on the coefficient functions $a_i : \Omega \times R^{n+1} \to R$.

(A_1) Each $a_i(x, t, \xi)$ is a Carathéodory function, i.e. measurable in $(x, t) \in \Omega \times I$ for all $\xi = (\eta, \zeta) \in R^{N+1}$ and continuous in $\xi \in R^{N+1}$ for a.e. $(x, t) \in \Omega \times I$.

(A_2) There exists a constant $c_0 \geq 0$ and a function $k_0 \in L^1(\Omega \times I)$ such that

$$| a_i(x, t, \eta, \zeta) | \leq k_0(x, t) + c_0 (| \eta | + | \zeta |),$$

$$i = 1, \ldots, n \text{ for } a.e. (x, t) \in \Omega \times I, \ \forall \eta \in R, \zeta \in R^N.$$

$$(A_3) \sum_{i=1}^{N} (a_i(x, t, \eta, \zeta) - a_i(x, t, \eta, \zeta')) (\zeta_i - \zeta_i') \geq 0$$

for a.e. $(x, t) \in \Omega \times I \ \forall (\eta, \zeta), (\eta, \zeta') \in R^{N+1}$.

(A_4) There exists a positive constant c_1 and a function $k_1 \in L^1(\Omega \times I)$ such that

$$\sum_{i=1}^{N} a_i(x, t, \eta, \zeta) \zeta_i \geq c_1 (| \eta |^2 + | \zeta |^2) - k_1(x, t)$$

for a.e. $(x, t) \in \Omega \times I$ and for all $\xi = (\eta, \zeta) \in R^{N+1}$.

Then we can show that the assumption (H_2) with $p = 2$ holds (see [12]). Applying our Theorem 3.4 above, we easily obtain the existence results of the problem (4.1). However, assumptions (A_1)-(A_4) do not imply that the operator A defined above is of class (S_+). For more details we refer to Li[12]. So we can not treat this problem by the theory developed in[4].

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