Indian J. pure appl. Math., **34**(8) : 1177-1187, August 2003

THE α-SUN TOPOLOGY AND L-SUN TOPOLOGY IN THE PLANE

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(*Received* 2 *July* 2002; *accepted* 28 *November* 2002)

The sun topology, one of the fine topologies, was proved by Pyrih in 1999 to be strictly finer than 2-fine topology from the logarithmic potential theory in the plane. In this paper two new fine topologies in the plane named the α-sun topology with $\alpha \in (1, 2)$ and the L-sun topology respectively are introduced, and it is proved that (1) the α-sun topology is strictly finer than the α-fine topology from Riesz potential theory; (2) the α-sun topology is strictly finer than the α'-sun topology for any $\alpha, \alpha' \in (1, 2)$ satisfying that $\alpha < \alpha'$. (3) the L-sun topology is strictly finer than the 2-fine topology and strictly coarser than the α-sun topology. And so the following strictly conclusion relation formulas are valid :

(*i*) The sun topology $\supset$ the α-sun topology $\supset$ the α'-sun topology $\supset$ the α'-fine topology $\supset$ the 2-fine topology $\supset$ the Euclidean topology, and

(*ii*) The α-sun topology $\supset$ the L-sun topology $\supset$ the 2-fine topology.

Some separation properties of the α-sun topology and the L-sun topology are also studied.

Key Words : The Sun Topology; The Fine Topology; Riesz Potential Theory; Hausdorff Measure; Capacity

1. INTRODUCTION

It is known that there is a close connection between Riwsz capacities and Hausdorff measures which promotes the developments of not only the potential theory but also the fractal geometry in the recent 30 years[1, 3, 8, 11]

In this paper we investigate fine topologies by means of capacities and Hausdorff measures. It is known[5, 10, 11] that the 2-fine topology in $\mathbf{R}^N$ $(N \geq 2)$ (i.e. the fine topology from the logarithmic potential theory when $N = 2$ and the Newton potential theory when $N \geq 3$) is strictly finer than the Euclidean topology, and the α-fine topology from Riesz potential theory $(0 < \alpha < 2)$) strictly finer than the 2-fine topology; for $0 < \alpha' < \alpha < 2$, we have the α'-fine topology strictly finer than the α-fine topology.

In 1999, Pyrih[9] proved the sun topology is strictly finer than the 2-fine topology. The sun topology is introduced by G. Horbaczewska[4] and from now on we call it P-sun topology for the convenience of the comparison with the other topologies.

Definition 1 — A subset A of the complex plane C is called an open set of the P-sun topology or P-sun open set, if every point x in A satisfies the following condition: there is a Borel set $F = F_x \subset [0, 2\pi]$ with linear Lebesgue measure zero such that for each $\beta \in [0, 2\pi] \setminus F$ there exist a real number $t_\beta > 0$ satifying

$$\left\{ z \in C : z = x + t(\cos\beta + i\sin\beta), |t| < t_\beta \right\} \subseteq A. \quad \dots (1)$$

In other words, if A contains x, then A contains a segment on almost every line through x.

Remark : Definition 1 is essentially the definition of "sun topology" in[9], but here we omit the restriction that A is Lebesgue measurable, and add the hypothesis that F is Borel set in order to compare with the other fine topologies.

After having proved in[12] the P-sun topology is strictly finer than the α-fine topologies, we are faced with the problem: is there another topology which is coarser than the P-sun topology and finer than the α-fine topologies or the 2-fine topology?

By introducing the α-sun topology in section 2 and the L-sun topology in section 4, we solve this problem, furthermore and obtain the comparison relations for different topologies concerned below :

(1) The α-sun topologies are strictly finer than the α-fine topology $(1 < \alpha < 2)$ and strictly coarser than the P-sun topology;

(2) for $1 < \alpha' < \alpha < 2$, the α'-sun topology is strictly finer than the α'-sun topology;

(3) the L-sun topology is strictly finer than the 2-fine topology and strictly coarser than the α-sun topology.

i.e. we have the following strict conclusion relations:

The P-sun topology $\supset$ the α'-sun topology $\supset$ the α-sun topology $\supset$ the α-fine topology $\supset$ the 2-fine topology;

The P-sun topology $\supset$ the α-sun topology $\supset$ the L-sun topology $\supset$ the 2-fine topology.

In section 5 some properties such as separation and separable properties for the α-sun topology and the P-sun topology are also discussed.

At first, let us recall some basic notions and conclusions about Hausdorff measure and dimension.

Definition 2 — Suppose B is a subset of $\mathbf{R}^n$ and $\{U_j\}$ is a sequence of subsets of R^n $|U_j|$ is the diameter of U_j. If for $\delta > 0$ we have $\bigcup_{j=1}^{\infty} U_j \supseteq B$ and $|U_j| < \delta$ for all $j \in N := \{1, 2, \dots\}$, then $\{U_j\}$ is said to be a δ-cover of B. For a real number $s > 0$, set

$$H^s_\delta(B) = \inf \sum_{j=1}^{\infty} |U_j|^s,$$

where the infimum is taken over all δ-covers of B, and set

$$H^s(B) = \lim_{\delta \to 0} H^s_\delta(B),$$

then $H^s(B)$ is called the s-dimension Hausdorff measure of B.

It is easy to see[3] that (*i*) if $0 < s < t$ and $H^s(B) = 0$, then $H^t(B) = 0$;

(*ii*) Hausdorff measure is an outer measure, each Souslin set (including Borel set) is Hausdorff measurable.

(*iii*) $H^n(B)$ is the n dimension Lebesgue outer measure of B when $n \geq 1$ is an integer.

Definition 3[3] — Set $dim_H B := \inf\{s : H^s(B) = 0\} = \sup\{s : H^s(B) = \infty\}$ and call it the Hausdorff dimension of B.

Then we have $H^s(B) = \infty$ when $s < \dim_H B$ and $H^s(B) = 0$ when $s > \dim_H B$.

Definition 4[3] — Suppose E is a subset of R^N. A mapping $f: E \to R^N$ is said to be bi-Lipschitz mapping if there exist constants η and ρ with $0 < \eta \leq \rho < \infty$ such that

$$\eta\,|x - y| \leq |f(x) - f(y)| \leq \rho\,|x - y|, \quad \forall x, y \in E \qquad \text{... (2)}$$

Lemma 1[3] — For a bi-Lipschitz mapping f we have

$$\eta^s H^s(E) \leq H^s(f(E)) \leq \rho^s H^s(E),$$

which means $f(E)$ and E have the same Hausdorff dimension. In particular, for a similar mapping $f: C \to C : f(x) = rx, r \geq 0$, we have $H^s(f(E)) = r^s H^s(E)$, $0 < s < \infty$.

It is not difficult to verify that there exists a bi-Lipschitz mapping f from the segment $[0, 2\pi]$ to the unit circle $S = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$, therefore definition 1 is equivalent to the following Definition 1′.

Definition 1′ — A subset A of the complex plane $\mathbf{C}$ is called an open set of the P-sun topology or P-sun open set, if every point x in A satisfies the following condition : there is a Borel set $F = F_x$ having 1 dimension Hausdorff measure zero (i.e. $H^1(F) = 0$) such that for each $\xi \in S_x \setminus F$, there exists a real number $t_\xi > 0$ satisfying

$$\{z = x + t\,\xi \in \mathbf{C} : 0 \leq t < t_\xi\} \subseteq A. \qquad \text{... (1}a\text{)}$$

where $S_x := \{y \in \mathbf{C} : |y - x| = 1\}$.

Lemma 2 — For any $s : 0 < s < 1$, we can construct a generalized Cantor E which is a subset of segment $I := [0, 1]$ or a segment arc J such that

(*i*) $dim_H E = s$ and $0 < H^s(E) < \infty$;

(*ii*) E is a complete set with card $(E) = \aleph$ (the cardinal of continuum).

PROOF : By Lemma 1 we need only consider the case that I : [0, 1]. Fix an $s : 0 < s < 1$ and put $a = \left(\frac{1}{2}\right)^{1/s}$. Then $0 < a < 1/2$, $2a^s = 1$. Let $S_1(x) = ax$, $S_2(x) = 1 - a(1 - x)$, then both S_1 and S_2 are the contraction mapping on I, and so there exists a compact invariant set E by Theorem 9.1 of[3] such that

$$E = S_1(E) \cup S_2(E).$$

E is a self-similar generalized Cantor set and can be constructed as the classical Cantor set : At first one divides I into 3 intervals: I_1, I_2 and I_3, and let left interval I_1 and right interval I_3 both has a length a, and the mid interval I_2 with the length $(1 - 2a)$. Put $E_1 := I_1 \cup I_2$; and let I_1 and I_2 take the place of I respectively and the same procedure produces a set E_2 which is the union of 4 intervals, each of them is a subset of I_1 or I_2 and has a length a^2; by repeating the procedure we get E_n which is the union of 2^n intervals (each of them has a length a^n and the same property), $n \in \mathbf{N}$. Then the set $E := \bigcap_{n=1}^{\infty} E_n$ is desired. Indeed, the Hausdorff dimension of E $\dim_H E = s$ and $0 < H^s(E) < \infty$ by Theorem 9.3 of [3], and one can verify E is a complete set by using the same method as that for the classical Cantor set.

2. DEFINITION OF THE α-SUN TOPOLOGY AND SOME LEMMAS FOR THE α–CAPACITY

Definition 5 — Suppose μ is a Radon measure on C, $0 < \alpha < 2$, the integral

$$U^{\mu}_{\alpha}(y) = \int \frac{d\mu(x)}{|x - y|^{2-\alpha}}, \quad y \in C$$

is called the α-potential of measure μ and

$$I_{\alpha}(\mu) = \int U^{\mu}_{\alpha}(y)\, d\mu(y) = K_{\alpha} \int \frac{d\mu(x)\, d\mu(y)}{|x - y|^{2-\alpha}}$$

is said to be the α -energy of measure μ. For a compact subset F of C, set

$$W_{\alpha}(F) := \inf\{I_{\alpha}(\mu) : \text{supp}(\mu) \subset F \text{ and total mass of } \mu \text{ is } 1\}.$$

And $C_{\alpha}(F) := (W_{\alpha}(F))^{-1}$ is called the α-capacity of F if $W_{\alpha}(F) < \infty$, and F is called a set with α-capacity zero and denoted with $C_{\alpha}(F) = 0$ if $W_{\alpha}(F) = \infty$. The α-capacity of an open set G is defined as $C_{\alpha}(G) = \sup\{C_{\alpha}(F) : F \text{ is a subset of } G\}$; The α-outer capacity of a general set E is $C^*_{\alpha}(E) = \inf\{C_{\alpha}(G) : G \text{ is open and } G \supset E\}$; A set E is said to be capacitable if

$$C^*_{\alpha}(E) = \sup\{C_{\alpha}(F) : F \text{ is a compact subset of } E\}.$$

For a capacitable set E, The α-capacity of E is defined to be its α-outer capacity, i.e. $C_{\alpha}(E) := C^*_{\alpha}(E)$.

It is known[2] that any analytic set, including Borel set is capacitable.

By means of the set α-capacity zero we introduce the concept the α-sun topology:

Definition 6 — Suppose $1 < \alpha < 2$. A subset G of the plane $\mathbf{C}$ is called an open set of the α -sun topology or α-sun open set, if every point x in G satisfies the following condition : there is a Borel set $F = F_x$ with α-capacity zero (i.e. $C_\alpha(F) = 0$) such that for each $\xi \in S_x \setminus F$, there exists a real number $t_\xi > 0$ satisfying

$$\left\{ z = x + t\,\xi \in \mathbf{C} : 0 \le t < t_\xi \right\} \subseteq G. \qquad \text{... } (1b)$$

where $\qquad S_x := \{y \in \mathbf{C} : |y - x| = 1\}.$

It is easy to verify that the all α-sun open sets construct a topology in $\mathbf{C}$, and we call it the α-sun topology.

Definition 7[5] — The coarsest topology on $\mathbf{C}$ in which all the α-potentials are continuous is called the α-fine topology $(0 < \alpha < 2)$.

Lemma 3[2] — A subset G of the place $\mathbf{C}$ is an α-fine open set, i.e. the open set of the α-fine topology when and only when $E := \mathbf{C} \setminus G$ is α-thin at every point in G, i.e. for any $x_0 \in G$ and any $q \in (0, 1)$ we have

$$\sum_{k=1}^{\infty} \frac{C_\alpha^*(E_k)}{q^{k(2-\alpha)}} < \infty,$$

where $C_\alpha^*(F)$ is the outer capacity of F, $E_k := E \bigcap \left\{ z \in \mathbf{C} : q^k \le |z - x_0| < q^{k-1} \right\}, k \in \mathbf{N}$.

Lemma 4[5] — For any $z \in \mathbf{C}$, there is a neighborhood base $\boldsymbol{B}$ of the α-fine topology $(0 < \alpha < 2)$ such that each member in $\boldsymbol{B}$ is compact in the Euclidean topology.

Lemma 5[5,6] — The α-capacity satisfies that

(*i*) monotonicity;

(*ii*) sub-additivity;

(*iii*) invariance under translation and rotation;

(*iv*) contraction principle : Suppose E is a subset of $\boldsymbol{C}, f : E \to \boldsymbol{C}$ is a contraction mapping: i.e. for any $x, y \in E$ we have: $|f(x) - f(y)| \le k\,|x - y|$, where $0 < k \le 1$ is a constant, then $C_\alpha(f(E)) \le C_\alpha(E)$.

(*v*) If $0 < \alpha' < \alpha < 2$ and $C_\alpha(E) = 0$, then $C_{\alpha'}(E) = 0$.

Lemma 6 — Suppose $1 < \alpha < 2$, E is a subset of $\boldsymbol{C}, f : E \to \boldsymbol{C}$ is bi-Lipschitz mapping, i.e. (2) is valid, then

$$\eta^{2-\alpha} C_\alpha(E) \le C_\alpha(f(E)) \le \rho^{2-\alpha} C_\alpha(E) \qquad \text{... } (4)$$

PROOF : By the definition of the α-capacity, we need only verify (4) is valid for any compact set E. At first, we verify that for the similar mapping $g : \mathbf{C} \to \mathbf{C} : g(x) = \rho\, x, \rho \ge 0$, we have

$$C_\alpha(g(E)) = \rho^{2-\alpha} C_\alpha(E). \qquad \text{... } (5)$$

Set $x' = g(x), E' = g(E)$, then for any unit positive measure μ on E there is a unit positive measure μ' on E' such that $\mu'(F') = \mu(g^{-1}(F'))$, $\forall F' \subseteq E'$; conversely, for any unit positive measure μ' on E', by setting $\mu(F) := \mu'(g(F))$, $\forall F \subseteq E$, we get a unit positive measure μ on E. Thus there is a correspondence between the set of unit positive measures μ on E and the set of unit positive measures μ' on E', hence

$$
\begin{aligned}
I_\alpha(\mu') &= \int_{E' \times E'} |x-y|^{\alpha-2} d\mu'(x') \, d\mu'(y') = \int_{E \times E} |g(x)-g(y)|^{\alpha-2} d\mu(x) \, d\mu(y) \\
&= \rho^{\alpha-2} \int_{E \times E} |x-y|^{\alpha-2} d\mu(x) \, d\mu(y) \\
&= \rho^{\alpha-2} I_\alpha(\mu),
\end{aligned}
$$

which implies

$$W_\alpha(E') = \rho^{\alpha-2} W_\alpha(E), \; C_\alpha(E') = \rho^{2-\alpha} C_\alpha(E).$$

Now suppose $f: E \to \mathbf{C}$ is a bi-Lipschitz mapping such (2) is valid. Since f is one-to-one, it follows from (2),

$$|f(x)-f(y)| \le |g(x)-g(y)|, \;\; \forall x, y \in E$$

$$\text{hence } |f \circ g^{-1}(x') - f \circ g^{-1}(y')| \le |x'-y'|, \;\; \forall x', y' \in E'$$

which means that the composed mappings $f \circ g^{-1}$ is a contraction mapping from E' to $f(E)$, by Lemma 5 we have

$$C_\alpha(f(E)) \le C_\alpha(E') = \rho^{2-\alpha} C_\alpha(E).$$

Replacing f by f^{-1}, we can obtain the left inequality in (4) and the proof is complete.

Lemma 7[3] — Suppose E is a Souslin set, $1 < \alpha < 2$.

(*i*) If $C_\alpha(E) = 0$, then $H^s(E) = 0$ for any $s > 2 - \alpha$, in particular, $H^1(E) = 0$.

(*ii*) If $H^s(E) < \infty$ for some $s \in (0, 1)$, then $C_{2-s}(E) = 0$.

Lemma 8[12] — Suppose $1 < \alpha < 2$, E is a line segment or segmental arc with length l, then there exists a constant $k > 0$ depending only on α such that $C_\alpha(E) \ge k l^{2-\alpha}$.

3. Main Results on the α-Sun Topology

***Theorem* 1** — *Suppose $1 < \alpha < 2$, then the α-sun topology is strictly finer than the α-fine topology.*

PROOF : At first, we verify that the α-sun topology is finer than α-fine topology.

Let G be an α-fine open set, $x \in G$. Without loss of the generality, we may suppose x is the origin, i.e. $x = 0$ since the α-capacity is invariant under translation and rotation by Lemma 5.

By Lemma 4, there exists an α -fine neighborhood K which is compact in Euclidean topology such that $K \subset G$. By Definition 6, we have to prove that there exists a Borel set F satisfying $F \subseteq S := \{ z \in \mathbf{C} : |z| = 1 \}$, and $C_\alpha(F) = 0$, and that for any $\xi \in S \setminus F$ there is $t_\xi > 0$ such that

$$\left\{ z = t\,\xi \in \mathbf{C} : 0 \le t < t_\xi \right\} \subseteq K. \qquad \text{... (1c)}$$

To do this, put $E_n := T_n \setminus K = T_n \cap (\mathbf{C} \setminus K)$, where $T_n := \left\{ z \in \mathbf{C} : \frac{1}{2^n} \le |z| < \frac{1}{2^{n-1}} \right\}$; $L_n := \left\{ z \in \mathbf{C} : |z| = \frac{1}{2^n} \right\}$; $B_n := \{ z \in L_n$: there is some $r > 0$ such that $rz \in E_n \}$, for all $n \in \mathbf{N}$.

Then E_n and B_n are Borel sets, and B_n is the imagine of E_n under the radial projection which is a contraction mapping. By Lemma 5 we have

$$C_\alpha(B_n) \le C_\alpha(E_n), \quad n \in \mathbf{N}. \qquad \text{... (6)}$$

On the other hand, we have by Lemma 3

$$\sum_{n=1}^{\infty} \frac{C_\alpha(E_n)}{2^{-n(2-\alpha)}} < \infty. \qquad \text{... (7)}$$

Now the similar mapping $g : \mathbf{C} \to \mathbf{C} : g(x) = 2^n x$, maps $B_n \subset L_n$ into the unit circle S, it follows from Lemma 6

$$C_\alpha(g(B_n)) = 2^{n(2-\alpha)}\, C_\alpha(B_n). \qquad \text{... (8)}$$

And (6), (7) and (8) imply

$$\sum_{k=1}^{\infty} C_\alpha(g(B_n)) < \infty. \qquad \text{... (9)}$$

By the definitions of E_n, B_n and $g(B_n)$, it is easy to see that

$$\left(\left\{ e^{i\theta} \mid e^{i\theta} \in S \setminus g(B_n), r > 0 \right\} \cap T_n \right) \subset K, \qquad \text{... (10)}$$

Put $A_m := \bigcup_{n=m}^{\infty} g(B_n), \quad F := \bigcap_{m=1}^{\infty} A_m$,

Then $\{A_m\}$ is a decreasing sequence of Borel sets and F is a Borel set, and by (9) and Lemma 5 (*i-ii*) we obtain

$$C_\alpha(F) \le \lim_{m \to \infty} C_\alpha(A_m) \le \lim_{m \to \infty} \sum_{n=m}^{\infty} C_\alpha(g(B_n)) = 0.$$

which implies $C_\alpha(F) = 0$. Obviously, for any $\xi \in S \setminus F$, there exists $m = m_\xi \in \mathbf{N}$ such that $\xi \notin A_m$, i.e. $\xi \notin g(B_n)$ for all $n \geq m$ which implies $\{z = t\,\zeta \in \mathbf{C} : t \geq 0\} \cap T_n = K$ for all $n \geq m$ since (10) is valid, hence we have

$$\{z = t\,\xi \in \mathbf{C} : t \geq 0\} \cap \left\{ z \in Q : |z| < \frac{1}{2^{m+1}} \right\} \subseteq K.$$

Therefore, if we take $t_\xi := \frac{1}{2^{m-1}}$, $m = m_\xi$, then (1c) is valid.

Secondly, we have an example to show there exists an α-sun open set which is not an α-fine open set. In fact, put

$$D := \{z = t\,e^{i\theta} : 0 \leq t < 1\} \setminus E, \text{ where } E := \{z = t; 0 < t < 1\}.$$

Then it is easy to verify that D is an α-sun open set. Set

$$E_n := \left\{ z = t \in E, \frac{1}{2^n} \leq t < \frac{1}{2^{n-1}} \right\},$$

then the segment E_n has a length 2^{-n}, by Lemma 8, there is a constant $k > 0$ depending on only α such that $C_\alpha(E_n) \geq k\,(2^{-n})^{2-\alpha}$, hence we have

$$\sum_{n=1}^{\infty} \frac{C_\alpha(E_n)}{2^{-n(2-\alpha)}} \geq k \sum_{n=1}^{\infty} \frac{(2^{-n})^{2-\alpha}}{2^{-n(2-\alpha)}} = \infty.$$

which means by Lemma 3 that D is not an α-fine open set. Thus the proof is complete.

***Theorem* 2** — *If $1 < \alpha' < \alpha < 2$, then the α'-sun topology is strictly finer than the α-sun topology.*

PROOF : Since $1 < \alpha' < \alpha < 2$, $C_\alpha(F) = 0$ implies $C_{\alpha'}(F) = 0$ by Lemma 5(v), hence an α-sun open set is an α'-fine open set by Definition 6, i.e. the α'-sun topology is finer than the α-sun topology.

On the other hand, for any fixed real number $s : 2 < -\alpha < s < 1 - \alpha'$, there exists a compact set F such that $F \subset S := \{z \in \mathbf{C} : |z| = 1\}$, $\dim_H F = s$ and $0 < H^s(F) < \infty$ by Lemma 2; then $H^{2-\alpha'}(F) = 0$, but $H^{2-\alpha}(F) = \infty$, which imply $C_{\alpha'}(F) = 0$ and $C_\alpha(F) > 0$ by Lemma 7. Put

$$G := \{z = t\,e^{i\beta} \in \mathbf{C} : t \geq 0, e^{i\beta} \in S \setminus F\},$$

Then G is an α'-sun open set, but not an α-sun open set. Therefore the α'-sun topology is strictly finer than the α-sun topology.

***Theorem* 3** — *The P-sun topology strictly finer than the α-sun topology* $(1 < \alpha < 2)$.

PROOF : Suppose $F \subset S$, F is a Borel set and $C_\alpha(F) = 0$, then $H^1(F) = 0$ by Lemma 7. Hence an α-sun open set is an P-sun open set by Definition 6 and Definition 1′, i.e. the P-sun topology is finer than the α-sun topology.

On the other hand, for any fixed real number $s : 2-\alpha < s < 1$, there exists a compact set F such that $F \subset S\left(S := \left\{e^{i\theta} : 0 \le \theta \le 2\pi\right\}\right)$, $\dim_H F = s$ and $0 < H^s(F) < \infty$ by Lemma 2; then $H^1(F) = 0$ and $C_\alpha(F) > 0$ by Lemma 7. Now we can verify that the set

$$G := \left\{z = t\, e^{i\beta} \in \mathbf{C} : t \ge 0, e^{i\beta} \in S \setminus F\right\},$$

is a P-sun open set, but not an α-sun open set. Therefore the P-sun topology is strictly finer than the α-sun topology.

From the above 3 Theorems and some well-known facts we obtain

Corollary — The P-sun topology $\supset$ the α'-sun topology $\supset$ the α-sun topology $\supset$ the α-fine topology $\supset$ the 2-fine topology $\supset$ the Euclidean topology for $0 < \alpha' < \alpha < 2$, where "$\supset$" means strictly inclusion.

4. L-SUN TOPOLOGY

Definition 8 — Suppose μ is a Radon measure on $\mathbf{C}$, the integration

$$U_2^\mu(y) := \int \log \frac{d\mu}{|x-y|}, \quad y \in \mathbf{C}$$

is called the logarithmic potential of μ and

$$I_2(\mu) := \int U_2^\mu(y)\, d\mu(y) = \int\int \log \frac{1}{|x-y|}\, d\mu(x)\, d\mu(y), \qquad \text{... (10)}$$

is said to be the energy of μ. For a compact subset F of $\mathbf{C}$, set

$$W_\alpha(F) := \inf\ \{I_2(\mu) : \text{Supp}\ (\mu) \subset F \text{ and total mass of } \mu \text{ is } 1\}.$$

And $C_\alpha(F) = (W_2(F))^{-1}$ is called the Wiener capacity of F if $W_2(F) < \infty$, and F is called a set with Wiener capacity zero if $W_\alpha(F) = \infty$. Furthermore, $C_l(F) = \exp(-W_2(F))^{-1})$ is called the logarithmic capacity of E. The logarithmic capacity of an open set G is defined as $C_l(G)$ = sup $\{C_l(F) : F$ is a subset of $G\}$; The logarithmic outer capacity of a general set E is $C_l^*(E)$ = inf $\{C_l(G) : G$ is open and $G \supset E\}$; A set E is said to be capacitiable if

$$C_l(E) = \sup\ \{C_l(F) : F \text{ is a compact subset of } E\}.$$

By replacing the α-capacity with the logarithmic capacity we get

Definition 9 — A subset G of the plane C is called an open set of L-sun topology or L-sun open set, if every point x in G satisfies the following condition : there is a Borel set $F = F_x$ having logarithmic capacity zero such that for each $\xi \in S_x \setminus F$, there exists a real number $t_\xi > 0$ satisfying

$$\left\{z = x + t\,\xi \in \mathbf{C} : 0 \le t < t_\xi\right\} \subseteq G. \qquad \text{... (1}d\text{)}$$

Definition 10[5] — The coarsest topology on **C** in which all logarithmic potentials are continuous is called 2-fine topology. The open set of 2-fine topology is called 2-fine open set.

Lemma 9[2] — A subset E of **C** is a 2-fine open set when and only when $E := \mathbf{C} \setminus G$ is thin at each point $x_0 \in G$, i.e. for any $q \in (0, 1)$ we have

$$\sum_{k=1}^{\infty} \frac{-k}{\log C_l^* (E_k)} < \infty,$$

where $C_l^* (F)$ is the outer logarithmic capacity of F,

$$E_k := E \cap \left\{ z \in \mathbf{C} : q^k \leq | z - x_0 | < q^{k-1} \right\}, \quad k \in \mathbf{N}.$$

Lemma 10[6] — For any $z \in \mathbf{C}$, there is a neighborhood base $\boldsymbol{B}$ of the 2-fine topology such that each member in $\boldsymbol{B}$ is compact in the Euclidean topology.

Lemma 11[6] — Suppose E is a Borel subset of C and $C_l (E) = 0$, then $C_\alpha (E) = 0$.

Theorem 4 — *L-sun topology is strictly coarser than the* α*-sun topology* $(1 < \alpha < 2)$.

PROOF : Suppose F is a Borel subset of S and $C_l (F) = 0$, then $C_\alpha (F) = 0$ by Lemma 11. Hence by Definition 6 and 9, any L-sun open set is an α-sun open set, which means the α-sun topology is finer than the L-sun topology.

For any $1 < \alpha < 2$, there is an α' such that $0 < \alpha < \alpha' < 2$. By Lemma 2, one can construct a Borel subset F of S such that

$$C_\alpha (F) = 0 \text{ and } C_{\alpha'} (F) > 0.$$

Then we have $C_l (F) > 0$ by Lemma 11. Then one can construct an α-sun open set which is not an L-sun open set as we do in the proof of Theorem 2. Therefore the L-sun topology is strictly coarser than the α-sun topology.

Lemma 12 — Suppose E is a Borel subset of $\boldsymbol{C}$, a mapping $f : E \to \boldsymbol{C}$ is bi-Lipschitz, i.e. (2) is valid then

$$\eta \, C_l (E) \leq C_l (f (E)) \leq \rho \, C_l (E)$$

especially for a similar mapping $g : \boldsymbol{C} \to \boldsymbol{C} : g (x) = \rho x, \; \rho \geq 0$, we have

$$C_l (g (E)) = \rho \, C_l (E).$$

The proof is similar to Lemma 6 and omitted.

By Lemmas 9 and 12 we can prove the following theorem similarly to Theorem 1.

Theorem 5 — *The L-sun topology is strictly finer than the 2-fine topology.*

5. Some Properties of the α-Sun Topology and L-Sun Topology

Definition 11[6] — A topology T on **C** is said to satisfy the essential radius condition if for any x on C and any neighbourhood U of x, there exists an essential radius $r(x, U_x) > 0$ such that

$$|x-y| \leq \min\{r(x, U_x), r(y, U_y)\} \Rightarrow U_x \cap U_y \neq \varnothing,$$

for every neighbourhood U_x, U_y of x and y in **C**, where $|x-y|$ is the Euclidean distance between x and y.

***Theorem* 6** — *The α-sun topology and L-sun topology both have the following properties:*

(1) Satisfy the essential radius conditon;

(2) Have the Euclidean G_δ-insertion property: i.e. for any α-(or L-) sun open set G and α-(or L-) sun closed set F with $G \subseteq F$, there exists a set D of type Euclidian G_δ such that $G \subseteq D \subseteq F$;

(3) Both the α-sun topology and L-sun topology are not separable;

(4) Both the α-sun topology and L-sun topology are locally connected.

The proof is similar to that in [9] and omitted. We can also conclude the following results.

***Theorem* 7** — *The α-sun topology is not normal, where* $\alpha \in (1, 2)$.

Proof : By Lemma 2, Lemma 7, Theorem 6 and Theorem 2.2 of [10, p 347], a method similar to [9] leads to the conclusion.

References

1. H. Aikawa and M. Essen, *Potential theory-selected topics*, Springer, Berlin, 1996
2. J. Bliedtner and W. Hanser, *Potential theory — an approach of analysis and probability*, Springer, Berlin, 1986.
3. K. Faliconer, *Fractal Geometry-mathematical foundations and applications*, Wiley and Sons, New York, 1990.
4. G. Horbaczewska, *Real Anal. Exchange*, **20**(2) (1994/95), 416-17.
5. N. S. Landkof, *Foundations of modern potential theory*, Springer, Berlin, 1970.
6. Li. Yong, *Journal of Xiamen University*, **25**(4) (1986), 400-7.
7. J. Lukes, J. Maly, J. Zajicek, *Fine Topology methods in real analysis and potential theory*, Berlin: Springer, 1986.
8. Manfred Denken and Hiroshi Sato. *Potential Analysis*, **14** (2001), 211-32
9. P. Pyrih, *Glusnik Mathematics*, **34**(54) (1999), 91-96.
10. P. Pyrih, *Real Anal. Exchange*, **21**(1) (1995/96), 345-48.
11. Wu Jiongqi, *Introduction to modern potential theory*. Xiamen University Publishing House, Xiamen, 1998.
12. Wu Jiongqi, *Journal of Zhangzhou Teachers College*, **4**(3) (2001), 1-5.