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REAL INVERSION FORMULA FOR A HANKEL TYPE INTEGRAL TRANSFORMATION

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In this paper it is proposed to extend the Hankel type integral transformation

$$J(y) = y^{1+2\nu} \int_0^\infty (y\tau)^{-\nu} J_\mu (y\tau) j(\tau)\, d\tau \quad (0 < y < \infty)$$

to certain classes of generalized functions where $J_\mu(z)$ is the Bessel function of first kind of order μ. A real inversion formula is shown to be valid when the limiting operation in the formula is understood as weak convergence in the space $D'(I)$ of Schwartz distributions where I denotes the interval $0 < \tau < \infty$.

1. INTRODUCTION

Some generalizations of the classical Hankel transformations

$$J(y) = h_\lambda (j(\tau))(y) = \int_0^\infty \sqrt{y\tau}\, J_\lambda (y\tau) j(\tau)\, d\tau \quad (0 < y < \infty) \qquad \ldots (1.1)$$

and

$$J(y) = h_\lambda (j(\tau))(y) = \int_0^\infty \tau J_\lambda (y\tau) j(\tau)\, d\tau \quad (0 < y < \infty) \qquad \ldots (1.2)$$

were given by many authors from time to time and some of the aspects of these were studied in the conventional as well as in the distributional sense.

Recently a simple generalization of (1.1) and (1.2) which we may call as a Hankel type integral transformation defined by

$$J(y) = (F_{\mu,\nu} J(\tau))(y) = y^{1+2\nu} \int_0^\infty (y\tau)^{-\nu} J_\mu [(y\tau)] J(\tau)\, d\tau \qquad \ldots (1.3)$$

where $J_\mu(z)$ is the Bessel function of first kind of order μ, is extended to a certin class of generalized functions by kernel method and the method of adjoint by Malgonde[3]. Note that when $\nu = -1/2$, (1.3) reduces to (1.1) and when $\nu = -1$.(1.3) reduces to (1.2).

Another transformation which is especially interesting is the convolution transforms as it encompasses a number of specific transformations such as the one sided Laplace, Stieltjes and K-transforms [Hirschman and Widder[1] and Zemanian[11]]. In recent years the investigations into conventional convolution transformations have become an active and important part of research in integral transforms.

The conventional convolution of a suitably restricted function $f(t)$ and the Kernel $G(t)$, viewed as an integral transformation with its Kernel $G(x-t)$ is defined by

$$F(x) = \int_{-\infty}^{\infty} f(t)\, G(x-t)\, dt \qquad (-\infty < x < \infty). \qquad \text{... (1.4)}$$

The theory of transformation (1.4) was pioneered by Hirschman and Widder[1] and was extended to generalized functions by Zemanian[10] as

$$F(x) = \langle f(t), G(x-t) \rangle. \qquad \text{... (1.5)}$$

The transformation (1.3) is a special case of convolution transformation (1.4) through certain changes of variables. Using Kernel method is one means of generalizing (1.3) to certain generalized functions. Adjoint method provides the second method of generalizing (1.3). The third method is that of generalized convolution transformation.

The aim of present paper is to extend the transformation (1.3) to certain classes of generalized functions by and to derive its real inversion formula in a way similar to that of generalized convolution transformation due to Zemanian[10]

The notation and terminology of this work follow that of Hirschman and Widder[1] and Zemanian[10, 11] For the sake of simiplicity we denote the Kernel

$$y^{1+2\nu} (y\,\tau)^{-\nu} J_\mu (y\,\tau)$$

$$= y^{1+2\nu} 2^{-\nu} \; H_{0,2}^{1,0} \, \frac{[(y\,\tau)^2 \,|\, (\mu/2 - \nu/2.1), (-\nu/2 - \mu/2, 1)]}{4}$$

$$= y^{1+2\nu} 2^{-\nu} \; H_{0,2}^{1,0} \, \frac{[(y\,\tau)^2]}{4} = y^{1+2\nu}\, k\,(y\tau) \qquad \text{... (1.6)}$$

where

$$H_{p,q}^{m,n}[z] = H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right]$$ is Fox's H-function [see Mathai and Saxena[4]].

2. Real Inversion Formula for the Hankel Type Integral Transformation

Following the technique of Hirschman and Widder[1] we shall obtain an inversion operator which will transform the image function into an object function for the transformation (1.3) in the following theorem.

***Theorem* 2.1** — *If $\phi(t)$ is continuous and bounded on $(0, \infty)$, and if*

$$F(x) = x^{1+2\nu} \int_0^\infty (xt)^{-\nu} J_\mu (xt)\, \phi(t)\, dt \qquad \text{... (2.1)}$$

$$= x^{1+2\nu} \int_0^\infty k\,(xt)\, \phi(t)\, dt$$

then

$$\lim_{n \to \infty} (-1)^{n+1} n^{-\nu-1} x^{-\nu-\mu-2n} D^{-(n+1)} x^{2(n+\mu+1)}$$

$$D^{(n+1)} x^{(-\nu-\mu-2+1)} F(x) |_{x=\frac{n}{t}}$$

$$= t^{(-1-2\nu)} \phi(t). \quad \ldots (2.2)$$

holds for $0 < t < \infty$.

PROOF : From {Malgonde[3]] the result viz. "For $\mu \geq -1/2$. If

$$(i)\ J_\mu(x) \cong \sqrt{2/\pi}\ x^{-1/2} \cos\left[x - \left(\frac{\pi}{2}\right)(\mu + 1/2)\right] o(x)^{-3/2}, \text{as } x \to \infty$$

$$(ii)\ J_\mu(x) \cong (x/2)^\mu, \text{ as } x \to 0^+$$

(*iii*) $f(x)$ is a locally integrable function on $0 < x < \infty$ such that

$f(x) = O(x^\eta), x \to 0^+$ and

$= O(x^\zeta), x \to \infty$

then the integral $\int_0^\infty x^{-\nu} J_\mu(x) f(x)\, dx$, defining the transformation (1.3) is absolutely convergent according to (i) and (ii), when $\eta > \nu - \mu - 1$ and $\zeta \langle \nu - 1/2 \rangle$, the integral (2.1) converges for $0 < x < \infty$.

Using the exponential change of variables viz. $x = e^y, t = e^{-\tau}$, (2.1) becomes

$$e^y F(e^y) = e^{(1+2\nu)y} \int_0^\infty K(e^{y-\tau}) e^{y-\tau} \phi(e^{-\tau})\, d\tau$$

or
$$f(x) = e^x F(e^x) = \int_{-\infty}^\infty e^{(x-t)} e^{(1+2\nu)(x-t)} K(e^{x-t}) e^{(1+2\nu)t} \phi(\epsilon^{-t})\, dt$$

$$= G(x) * \psi(x),$$

where $G(x) = e^x e^{(1+2\nu)x} K(e^x) = e^{(2+\nu)x} J_\mu(e^x)$ and $\psi(x) = e^{(1+2\nu)x} \phi(e^{-x})$. From Hirschman and Widder[1], the bilateral Laplace transform of the Kernel $G(x)$ is

$$\frac{1}{E(s)} = \int_{-\infty}^\infty G(\tau) e^{-s\tau} d\tau = \int_{-\infty}^\infty e^{(2+\nu)\tau} J_\mu(e^T) e^{-s\tau} d\tau$$

$$= \int_{-\infty}^\infty e^{(2+\nu-s)\tau} J_\mu(e^\tau)\, d\tau = \int_0^\infty x^{(1+\nu-s)} J_\mu(x)\, dx$$

$$= 2^{(1+\nu-s)} \frac{\Gamma(1+\nu/2-s/2+\mu/2)}{\Gamma(-\nu/2+s/2+\mu/2)}$$

provided

$$Re\ (1+\nu/2-s/2+\mu/2)>0 \text{ and } Re\ (-\nu/2+s/2+\mu/2)>0.$$

Therefore

$$E(s) = 2^{-1-\nu+s} \frac{\Gamma(-\nu/2+s/2+\mu/2)}{\Gamma(1+\nu/2-s/2+\mu/2)} \quad \text{... (2.3)}$$

Now, define

$$E(D) = 2^{-1-\nu+D} \frac{\Gamma(-\nu/2+D/2+\mu/2)}{\Gamma(1+\nu/2-D/2+\mu/2)}$$

where $2^D f(x) = f(x+\log 2)$ or $n^D (f(e^x) = f(ne^x) = f(x+\log n)$.

This is the desired inversion operator for the transformation (2.1). In order to express Γs in terms of operators we use [see Rainville[7], p.11].

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\,n^z}{zz(z+1)(z+2)\ldots(z+2)z} = \lim_{n\to\infty} \frac{n!n^z}{\prod_{p=0}^{n}(z+p)} \quad \text{... (2.4)}$$

Using (2.4) we see that

$$\Gamma(D/2-\nu/2+\mu/2) = \lim_{n\to\infty} \frac{n\,!\,n^{(D/2-\nu/2+\mu/2)}}{\prod_{p+0}^{n}(D/2-\nu/2+\mu/2+p)} \quad \text{... (2.5)}$$

and

$$\Gamma(1+\nu/2-D/2+\mu/2) = \lim_{n\to\infty}$$

$$\frac{(-1)^{-(n+1)}\,n\,!\,n^{(1-\alpha/2\,\nu-D/2\,\nu+\mu/2)}}{\prod_{p=0}^{n}(-1+\alpha/2\,\nu+D/2\,\nu-\mu/2-p)} \quad \text{... (2.6)}$$

Using (2.5) and (2.6), (2.3) becomes

$$E(D) = \lim_{n\to\infty} (-1)^{(n+1)} n^{(-\nu-1+D)} \prod_{p=0}^{n} (D/2-\nu/2+\mu/2+p)^{-1}$$

$$(-1-\nu/2+D/2-\mu/2-p) = \lim_{n\to\infty} P_n(D). \quad \text{... (2.7)}$$

Therefore inversion formula $E(D)f(x) = \psi(x)$ in terms of original functions F and ϕ becomes

$$\lim_{n\to\infty} P_n(D)\,[e^x F(e^x)] = e^{(1+2\nu)x}\,\phi(e^{-x})$$

or

$$\lim_{n \to \infty} \left((-1)^{(n+1)} n^{(-\nu+D-1)} \prod_{p=0}^{n} (D/2 - \nu/2 + \mu/2 + p)^{-1} \right.$$

$$\left. (-1 - \nu/2 + D/2 - \mu/2 - p) \right) e^x F(e^x) |_{x = t + 2 \log n}$$

$$= e^{(1+2\nu)t} \phi(e^{-t}).$$

Now we record first the simple results concerning change of variable in the following lemmas.

Lemma 2.1 — If $F(x) \in C^n, 0 < x < \infty$, then

$$(a) \prod_{p=0}^{n} (D/2 + b - p) [e^{-2bx} F(e^{2x}) = e^{2(-b+n+1)} F^{(n+1)}(e^{2x})$$

for $-\infty < x < \infty$ ... (2.9)

and

$$(b) \prod_{p=0}^{n} (D/2 + a + p)^{-1} [e^{-2(a-1)x} F(e^{2x}) = e^{2(a+n)} F^{(-(n+1)}(e^{2x})$$

for $-\infty < x < \infty$ ... (2.10)

where $F^{(n+1)}(e^{2x})$ and $F^{-(n+1)}(e^{2x})$ denote the $(n+1)$th derivative and integral of $F(e^{2x})$ with respect to e^{2x} respectively.

PROOF : (*a*) Using the familiar fact that the order of application of the factor operators is immaterial when the coefficients are constants, we have

$$(D/2 + b)\left[e^{-2bx} F(e^{2x}) \right] = e^{2(-b+1)x} F(e^{2x})$$

$$(D/2 + b - 1)(D/2 + b) [e^{-2bx} F(e^{2bx})] = e^{2(-b+2)x} F(e^{2x})$$

$$\prod_{p=0}^{n} (D/2 + b - p) [e^{-2bx F(e^{2bx})}] = e^{2(-b+n+1)x} F^{(n+1)}(e^{2x})$$

which proves the result (*a*).

(*b*) By (*a*), we have

$$\prod_{p=0}^{n} (D/2 + a + p)\left[e^{-2(a+n)x} F(e^{2\nu x}) \right] = e^{-2(a-1)x} F^{(n+1)}(e^{2x})$$

and since the factors (or operators) obey commutative law of multiplication, if we apply the process in the reverse order we get $\prod_{p=0}^{n} (D/2 + a + p)^{-1}\left[e^{-2(a-1)x} F(e^{2x}) \right] = e^{-2(a+n)x} F^{-(n+1)}(e^{2x})$ which proves the result (*b*).

We consider

$$\prod_{p=0}^{n} (-1-\nu/2+D/2-\mu/2-p)\,[e^{x} F(e^{x})]$$

$$= \prod_{p=0}^{n} (D/2+b-p)\,[e^{-2bx} R(e^{2x})]$$

where $b = -\nu/2-\mu/2-1$ and $e^{x} F(e^{x}) = e^{-2bx} R(e^{2x})$

$$= e^{2[-b+n+a]x} R^{(n+1)} e^{2x}) \text{ (using (2.9))} \qquad \ldots (2.11)$$

where $R(e^{2x}) = e^{(2b+1)x} F(e^{x})$.

Now taking $\prod_{p=0}^{n} (-\nu/2+D/2+\mu/2+p)^{-1} \left\{ e^{2[-b+n+a]x} R^{(n+1)}(e^{2x}) \right\}$

$$= \prod_{p=0}^{n} (D/2+a+p)^{-1} \left[e^{-2(a-1)x} H(e^{2x}) \right]$$

where $a = -\nu/2+\mu/2$ and $H(e^{2x}) = e^{2[-b+n+a]x} R^{(n+1)}(e^{2x})$

$$= e^{-2[n+a]x} H^{-(n+1)}(e^{2x}) \qquad \text{(using (2.10))} \qquad \ldots (2.12)$$

Using (2.11) and (2.12) into (2.8) we get

$$\lim_{n\to\infty z} (-1)^{n+1} n^{D-\nu-1} e^{-2(a+n)x} D^{-(n+1)} e^{2(n+a-b)x} D^{(n+1)} e^{(2b+1)x} F(x)$$

$$= e^{(1+2\nu)x} \phi(e^{-x}). \qquad \ldots (2.13)$$

Again

$$n^{D} F(e^{x}) = e^{D\log n} F(e^{x}) = F(e^{x+\log n}) \text{ since } n^{D} f(x) = f(x+\log n).$$

Therefore to operate n^{D} upon $F(e^{x})$, we have to change x to $x + \log n$. Hence (2.13) becomes

$$\lim_{n\to\infty} (-1)^{(n+1)} n^{D-1} e^{-2(a+n)(x+\log n)} D^{-(n+1)} e^{2(n+a-b)(x+\log n)}$$

$$D^{(n+1)} e^{(2b+1)(x+\log n)} F(e^{x+\log n}) = e^{(1+2\nu)x} \phi(e^{-x})$$

or

$$\lim_{n\to\infty} (-1)^{n+1} n^{-\nu-1} x^{\nu-\mu-2n} D^{-(n+1)} x^{2(n+\mu+1)}$$

$$D^{(n+1)} x^{(-\nu-\mu-2+1)} F(x) \mid = \mid_{x=\frac{n}{t}}$$

$$= t^{(-1+2\nu)} \phi(t) \qquad \ldots (2.14)$$

which is the required real inversion formula (2.2) for the transform (2.1) or (1.3).

As an illustration of formula (2.2), take $\phi(t) = t$. From (2.1) we find

$$F(x) = x^{1+2\nu} \int_0^{\infty} (xt)^{-\nu} J_\mu(xt)\, t\, dt$$

$$= x^{1+\nu} \int_0^{\infty} t^{-\nu+1} J_\mu(xt)\, dt$$

$$= 2^{1-\nu} x^{2\nu-1} \frac{\Gamma(-\nu/2+1+\mu/2)}{\Gamma(\mu/2+\nu/2)} \qquad \dots (2.15)$$

provided $Re\,(-\nu/2+1+\mu/2) > 0$ and $Re\,(\mu/2+\nu/2) > 0$.

Therefore left hand side of (2.2) becomes

$$\lim_{n\to\infty} (-1)^{n+1} n^{-\nu-1} x^{\nu-\mu-2n} D^{-(n+1)} x^{2(n+\mu+1)}$$

$$D^{(n+1)} x^{(-\nu-\mu-1)} 2^{1-\nu} x^{-1+2\nu} \frac{\Gamma(-\nu/2+1+\mu/2)}{\Gamma(\mu/2+\nu/2)} \Bigg|_{x=\frac{n}{t}} \quad \text{(by using (2.15))}$$

$$= \lim_{n\to\infty} (-1)^{n+1} n^{-\nu-1} x^{\nu-\mu-2n} D^{-(n+1)} x^{2(n+\mu+1)}$$

$$D^{(n+1)} 2^{1-\nu} \frac{\Gamma(-\nu/2+1+\mu/2)}{\Gamma(\mu/2+\nu/2)} x^{-\mu+\nu-2} \Bigg|_{x=\frac{n}{t}}$$

$$= \lim_{n\to\infty} n^{-\nu-1} 2^{1-\nu} \frac{\Gamma(-\nu/2+1+\mu/2)\,\Gamma(\mu-\nu+g+n)\,\Gamma(\mu+n+\nu)}{\Gamma(\mu/2+\nu/2)\,\Gamma(\mu-\nu+2)\,\Gamma(\mu+2n+1+\nu)}$$

$$x^{2\nu} \Bigg|_{x=\frac{n}{t}} = t^{-2\nu}$$

by using the following results

(a) if $y = x^{-m}\ (m > 0)$ then $D^n y = \dfrac{d^n y}{dx^n} = (-1)^n \dfrac{\Gamma(m+n)}{\Gamma(m)} x^{-m-n}$

(b) the Pochmer's symbo $(\alpha)_r = \dfrac{\Gamma(\alpha+r)}{\Gamma(\alpha)} = \alpha(\alpha+1)\dots(\alpha+r-1)\ ;\ (\alpha)_0 = 1$

(c) $\displaystyle\lim_{n\to\infty} \frac{\Gamma(n)\, n^{z+(s-1)/m}}{\Gamma[z+n+(s-1)\ m]} = 1.$

This example shows that the condition imposed on $\phi(t)$ are stronger than needed. For, the inversion formula is valid even though $\phi(t) = t$ is not bounded.

The Hankel type integral transformation (1.3) is a special case of the convolution transformation (1.4) can be seen as follows

Choosing the Kernel

$$G(x-t) = e^{(2+\nu)(x-t)} J_{\mu}[e^{(x-t)}]$$

and applying the change of variables $y = e^{x}$, $\tau = e^{-\tau}$ to (1.4), we obtain

$$y^{-1} F(\log y) = y^{1+2\nu} \int_{0}^{\infty} r^{1+2\nu} f(-\log r)(y\tau)^{-\nu} J_{\mu}(y\tau)\, d\tau \quad (0 < y < \infty)$$

which can be identified with (1.3).

3. Test Function Spaces and Generalized Function Spaces

To extend the results of article 2 to generalized functions we use the spaces $M_{c,d}$ and $M'_{c,d}$ constructed in Malgonde[2]. For this we briefly review the definitions of test functions spaces $L_{c,d}$ and generalized functions spaces $L_{c,d}$ and paramount properties here even though it has been discussed in Pandey and Zemanian[6] and Zemanian[10]. The spaces $L_{c,d}$ and $L_{c,d}$; Let c and d be two fixed real numbers and $\lambda_{c,d}$ be a fixed smooth positive function defined on $(-\infty < t < \infty)$ R such that

$$\lambda_{c,d}(t) = \begin{cases} e^{ct} & 0 \le t < \infty \\ e^{dt} & -\infty < t < 0. \end{cases}$$

$L_{c,d}$ is defined as the linear spaces of all complex-valued smooth functions $\phi(t)$ on R such that for each k = 0, 1, 2 ...

$$\gamma_k(\phi) = \gamma_{c,d,k}(\phi) = \sup_{-\infty < t < \infty} \left| \lambda_{c,d}(t)\, \phi^{(k)}(t) \right| < \infty.$$

The spaces $M_{c,d}$ and $M'_{c,d}$: Given any two real numbers c and d, $M_{c,d}$ is the space of all smooth functions $\psi(\tau)$ on $(0 < \tau < \infty)$ 1 such that

$$\zeta_k[\psi(\tau)] = \zeta_{c,d,k}[\psi(\tau)] = \sup_{\tau \in I} \left| \lambda_{c,d}(-\log \tau)(-\tau D_{\tau})^{k} [\tau\, \psi(\tau)] \right| < \infty \quad \text{... (3.1)}$$

for k = 0, 1, 2, ... where $\lambda_{c,d}(-\log) = \begin{cases} \tau^{-c} & 0 < \tau \le 1 \\ \tau^{-d} & 1 < \tau < \infty \end{cases}$. ... (3.2)

The topology of $M_{c,d}$ is that generated by the multinorm $\left\{ \zeta_k \right\}_{k=0}^{\infty}$

As a consequence $M_{c,d}$ is a complete countably multinormed space and $M_{c,d}$ is the dual space of generalized functions.

The spaces $D(I)$ and $D(I)$ — The space $D(I)$ of all smooth complex-valued functions on I whose support is contained in a compact subset k of I equipped with the seminorm

$$\rho_n(\phi) = \sup_{\tau \in I} \left| D_\tau^n \phi(\tau) \right| \qquad \dots (3.3)$$

where $\phi \in D(I)$ is such that supp $\phi \subset k$ is a subspace of $M_{c,d}$. Further the restriction of any $f \in M_{c,d}$ to $D(I)$ belongs to $D(I)$. Following Malgonde[2] the following lemmas are immediate.

Lemma 3.1 — The mapping $\psi(\tau) = e^{-t}\,\psi(e^{-t}) = \phi(t)$ where $\psi(\tau) = \tau^{-1}\,\phi(-\log r)$ is an isomorphism from $M_{c,d}$ onto $L_{c,d}$. The inverse mapping is given by $\phi(t) \to \psi(\tau)$ where $\phi = e^{-t}\,\psi(e^{-t})$.

Next if $f(t) \in L_{c,d}$ we define $f(-\log \tau)$ as a functional on $M_{c,d}$ by

$$\langle f(-\log \tau), \tau^{-1}\,\phi(-\log \tau)\rangle = \langle f(t), \phi(t)\rangle, \phi \in L_{c,d}. \qquad \dots (3.4)$$

If $\psi(\tau)$ and $\phi(t)$ are as defined in Lemma 3.1, then it can be easily proved that the mapping $f(t) \to f(-\log r)$ defined by (3.4) is an isomorphism from $L_{c,d}$ onto $M'_{c,d}$. The inverse mapping $f(\tau) \to f(e^{-t})$ is defined by $\langle f(e^{-t}), \phi(t)\rangle = \langle f(\tau), \psi(\tau)\rangle$.

Lemma 3.2 — The mapping $\tau^{-1}\,\phi(-\log \tau) \to \phi(t)$ where $t = -\log \tau$ is an isomorphism from $D(I)$ onto D, the space of smooth functions on $-\infty < t < \infty$ having compact support. The inverse mapping is given by $e^{-t}\,\psi(e^{-1}) \to \psi(\tau)$.

Lemma 3.3 — The mapping $f(t) \to f(-\log \tau)$ is an isomorphism from D onto $D(I)$.

4. The Distributional Hankel Type Integral Transformation

Let $G(t) = e^{(2+\nu)t} J_\mu[e^t]$. Setting $y = e^x$, $\tau = e^{-t}$ and $\phi(t) = G(x-t)$ we get

$$\tau^{-1}\,\phi(-\log \tau) = e^t\,G(x-t) = e^t\left\{e^{(2+\nu)(x-t)}\,J_\mu[e^{(x-t)}]\right\}.$$

If we choose c and d such that $c < -\nu + \mu + 1$ and $d > 0$ then we may replace $\phi(t)$ by $G(x-t)$ in (3.4) to obtain

$$\langle f(-\log \tau), y\,\tau^{1+2\nu}\left\{y^{1+2\nu}(y\,\tau)^{-\nu} J_\mu\right\}\rangle = \langle f(t), G(x-t)\rangle = F(\log y).$$

Setting $J(y) = y^{-1}\,F(\log y)$ and $J(\tau) = \tau^{1+2\nu} f(-\log \tau)$, we finally obtain the new definition of distributional Hankel type integral transformation

$$J(y) = \langle j(\tau), y^{1+2\nu}(y\,\tau)^{-\nu} J_\mu(y\,\tau)\rangle$$

$$= \langle j(\tau), y^{1+2\nu} K(y\,\tau)\rangle \quad (0 < y < \infty). \qquad \dots (4.1)$$

(4.1) has meaning as the application of $j(\tau) \in M'_{c,d}$ to $y^{1+2\nu} K(y\,\tau) \in M_{c,d}$ for $c < -\nu + \mu + 1$ and $d > 0$ as it can be seen from the following lemma.

Lemma 4.1 — For any fixed real $y > 0$, $y^{1+2\nu} K(y\tau)$ is a member of $M_{c,d}$ for $c < -\nu + \mu + 1$ and $d > 0$.

PROOF : The kernel function $y^{1+2\nu} K(y\tau)$ is an analytic function of τ for $0 < y < \infty$ and hence a smooth function on I. It remain to show that

$$\zeta_{c,d,k}[y^{1+2\nu} K(y\tau)] < \infty \text{ for } k = 0, 1, 2, 3 \ldots$$

From (3.1), we have

$$\zeta_{c,d,k}[y^{1+2\nu} K(y\tau)] = \sup_{\tau \in I} \left| \lambda_{c,d}(-\log\tau)(-\tau D_\tau)^k [\tau y^{1+2\nu} k(y\tau)] \right|$$

$$= \sup_{\tau \in I} \left| \lambda_{c,d}(-\log\tau)\, y^{1+2\nu} (-1)^k \sum_{p=0}^{k} A_p \tau^{p+1} D_\tau^k [k(y\tau)] \right|$$

where A_p's are some constants.

Following Mathai and Saxena[4] we have

$$D_\tau^p K(y\tau) = \tau^{-p} H_{1,3}^{1,1}\left[\frac{(y\tau)^2}{4} \,\middle|\, \begin{matrix} (0,2) \\ (\mu/2 - \nu/2, 1), (-\nu/2 - \mu/2, 1), (p, 2) \end{matrix} \right]$$

$$= \tau^{-p} H_{1,3}^{1,1}\left[\frac{(y\tau)^2}{4} \right], \text{ say.}$$

Therefore, $\zeta_{c,d,k}[y^{1+2\nu} k(y\tau)]$

$$= \sup_{\tau \in I} \left| \lambda_{c,d}(-\log\tau)\, y^{1+2\nu} 2^\nu (-1)^k \sum_{p=0}^{k} A_p \tau H_{1,3}^{1,1}\left[\frac{(y\tau)^2}{4} \right] \right|$$

The general term of this supremum is

$$\lambda_{c,d}(-\log\tau) B_p \tau H_{i,g}^{i,i}\left[\frac{(y\tau)^2}{4} \,\middle|\, \begin{matrix} (0,2) \\ (\mu/2 - \nu/2, i)(-\nu/2 - \mu/2, 1), (\rho, 2) \end{matrix} \right]$$

for some constants B_p.

For large τ,

$$\sup_{1 < \tau < \infty} \left| \tau^{-d} B_p \tau H_{1,3}^{1,1}\left[\frac{(y\tau)^2}{4} \,\middle|\, \begin{matrix} (0,2) \\ (\mu/2 - \nu/2, 1), (-\nu/2 - \mu/2, 1), (p, 2) \end{matrix} \right] \right|$$

is finite if $d > 0$ and for small τ.

$$\sup_{1<\tau<\infty} \left| \tau^{-c} B_p \, \tau H_{1,3}^{1,1}\left[\frac{(y\,\tau)^2}{4} \left| \begin{matrix} (0,2) \\ (\mu/2-\nu/2,1),(-\nu/2-\mu/2,1),(p,2)] \end{matrix} \right. \right| < \infty$$

if $c < -\nu + \mu + 1$ by the asymptotic Epansion of H-function [see Malgonde[2]].

Remark : It can also be proved $y^{1+2\nu} K(y\,\tau) \in M_{c,d}$ for every d and $c < -\nu + \mu + 1$ by using properties of H-function [see Malgonde[2]].

By the arguments similar used in the proof of Lemma 4.1, we have

Lemma 4.2 — For any fied real $y > 0, D_\tau^k [y^{1+2\nu} K(y\,\tau)] \in M_{c,d}$ for every d and $c < -\nu + \mu + 1$. Further $D_y^k \left\{ y^{1+2\nu} k(y\,\tau) \right\} \in M_{c,d}$ and $J(y)$ be defined by (4.1) for $0 < y < \infty$. Then $J(y)$ is analytic in $0 < y < \infty$ and

$$D_k^k [J(y)] = \langle j(\tau), D_y^k \left\{ y^{1+2\nu} k(y\,\tau) \right\} \rangle \qquad \text{... (4.2)}$$

5. Real Inversion Formula for the Distributional Hankel Type Integral Transformation

In this section we shall obtain a real inversion formula for (4.1) from (2.8). The suъstitution of $\tau = e^{-1}$ into

$$P_n(D_t) F(t) = P_n(D_t) [e^t J(e^t)], \quad -\infty < t < \infty$$

yields after usign (2.2)

$$(-1)^{n+1} n^{-\nu-1} \left(\frac{n}{t} \right)^{\nu-\mu-2n} D^{-n+1} \left(\frac{n}{t} \right)^{2(n+\mu+1)}$$

$$D^{(n+1)} \left(\frac{n}{t} \right)^{(-\nu-\mu-1)} J\left(\frac{n}{t} \right)$$

for $0 < \tau < \infty$.

Moreover, $\tau^{-1} \phi(-\log \tau) \to \phi(t)$ is an isomorphism from $D(I)$ onto D by Lemma 3.2. where now I denotes the interval $0 < \tau < \infty$ and the change of variable we have used (viz. (3.4)) defines an isomorphism from D onto D' (I) by Lemma 3.3. Thus using the definition (3.4) again where now $\phi(t)$ is restricted to D we obtain the following real inversion formula for the distributional Hankel type integral transformation :

***Theorem* 5.1** — *If $j(\tau) \in M_{c,d}$ where $c < -\nu = \mu + 1$ and $d > 0$ is arbitrary, and if $J(y)$ is defined by (4.1), then*

$$\lim_{n \to \infty} (-1)^{n+1} n^{-\nu-1} \left(\frac{n}{t}\right)^{\nu-\mu-2n} D^{-(n+1)} \left(\frac{n}{t}\right) 2\,(n+\mu+1)$$

$$D^{(n+1)} \left(\frac{n}{t}\right)^{(-\nu-\mu-1)} J\left(\frac{n}{t}\right) = j\,(\tau) \qquad \ldots (5.1)$$

in the sense of convergence in $D'(I)$ where I is the interval $0 < \tau < \infty$.

Formula (5.1) is the classical real inversion formula given by (2.2); it now has a meaning for certain generalized functions $j\,(\tau)$.

PARTICULAR CASES : 1) Putting $\nu = -1/2$, the result (5.1) reduces to a real inversion formula

$$\lim_{n \to \infty} (-1)^{n+1} n^{-1/2} \left(\frac{n}{t}\right)^{-1/2-\mu-2n} D^{-(n+1)} \left(\frac{n}{t}\right)^{2\,(n+\mu+1)}$$

$$D^{(n+1)} \left(\frac{n}{t}\right)^{(1/2-\mu-1)} J\left(\frac{n}{t}\right) = j\,(\tau)$$

for the distributional Hankel transformation studied by Zemanian[11].

2. Putting $\nu = -1$, the formula (5.1) reduces to a real inversion formula

$$\lim_{n \to \infty} (-1)^{n+1} n^{-2} \left(\frac{n}{t}\right)^{-1-\mu-2n} D^{-(n+1)} \left(\frac{n}{t}\right)^{2\,(n+\mu+1)}$$

$$D^{(n+1)} \left(\frac{n}{t}\right)^{\mu} J\left(\frac{n}{t}\right) = j\,(\tau)$$

for the distributional Hankel transformation studied by Mendez[5].

3. In view of the general nature of the kernel involved in the transformation (1.3) on specializing the parameters we obtain real inversion formulae for the distributional Hankel transformations studied by Sneddon[8], Watson[9] amongst others.

Remark : It is proposed to develop a theory that (5.1) can be used for a still wider class of generalized functions than that described here in this paper.

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