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# RANGE OF DIAMETERS OF A BIPARTITE GRAPH AND ITS GENERALIZED 2-PARTITE COMPLEMENT

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## 1. INTRODUCTION AND DEFINITIONS

Complements of graphs, more particularly self-complementary graphs have been extensively studied by many mathematicians; Sachs[9], Ringel[8], Clapham[1], Gibs[4], Rao[7], Gangopadhyay[2, 3] etc. being the more prominent ones. Many problems such as the Hamiltonian problem, the characterization of potentially and forcibly self-complementary degree sequences have been solved for this class of graphs (see references given in [7]). Similar problems have been studied for the class of bipartite/multipartite graphs. A new concept of generalized complement of a graph has been introduced by Sampathkumar[10]. The problem of determining the values assumed by the diameter of a bipartite self-complementary graph has been solved by Gangopadhyay and Hebbare[2]. Gangopadhyay has also given the range of diameters in a graph and its bipartite complement[3].

In this paper we study the diameters of a bipartite graph and its generalized 2-partite complement. Below we give the necessary notations and definitions. For terms not defined here we refer the reader to Harary[5].

By a graph we mean a finite undirected graph without loops or multiple edges. A graph $G$ is said to be bipartite if there exist two disjoint subsets $V_1, V_2$ of $V(G)$ such that $V(G) = V_1 \cup V_2$ and each $V_i$ is an independent set in $G$. Such a partition $V_1, V_2$ is called a bipartition of $G$. A bipartitioned graph is a pair $(G, P)$, where $G$ is a bipartite graph and $P$ is a bipartition of $G$. Let $Q = \{W_1, W_2\}$ be any 2-partition of $G$. Given a bipartitioned graph $(G, P)$ and a 2-partition $Q$ of $G$, the generalized 2-partite complement of $G$ is defined to be the graph $G_Q$ where $V(G_Q) = V(G)$ and $E(G_Q) = \{uv \mid u, v$ belong to different parts of $Q$ and $uv$ is not an edge of $G\} \cup \{uv \mid u, v$ belong to the same part of $Q$ and $uv$ is an edge of $G\}$.

Let $f$ be a graph theoretic parameter and $p$ a positive integer. The Nordhaus - Gaddum[6] problem for $f$ is to determine upper and lower bounds (preferably sharp) for $f(G)+f(\overline{G})$ and $f(g)\cdot f(\overline{G})$, where $G$ is a graph on $p$ vertices and $\overline{G}$ is its ordinary complement. One can also consider the problem of determining all the triples $(a, b, p)$ for which there exists a graph $G$ such that $|V(G)| = p, f(G) = a$ and $f(\overline{G}) = b$. In the class of bipartite graphs the corresponding problems are (i) to determine upper and lower bounds for $(f(G)+f(G_Q))$ and $f(G)\cdot(f(G_Q))$ where $G$ is a bipartite graph on $p$ vertices and $G_Q$ is its generalized 2-partite complement with respect to the

2-partition $Q$ and, (ii) to enumerate all triples $(a, b, p)$ for which there exists a bipartite graph $G(V_1, V_2)$ and a 2-partition $Q$ such that $|V(G)| = p, f(G) = a$ and $f(G_Q) = b$. A solution of the second problem necessarily provides a solution for the first problem. We solve problem (ii) when $f$ stands for the diameter $d$ of a graph. In this context we define the following :

*Definition* 1.1 — A triple $(a, b, p)$ is said to be $g_1$-realizable if there exists a bipartite graph $G(V_1, V_2)$ on $p$ vertices and a 2-partition $Q = \{W_1, W_2\}$ of $V_1 \cup V_2$ with $W_i \cap V_j = \phi$ for at least one $i$ and $j$; $i = 1, 2$; $j = 1, 2$ such that $d(G) = a$ and $d(G_Q) = b$. Such a $G(V_1, V_2)$ is called a $g_1$-realization (or when the context is clear just a realization) of $(a, b, p)$.

*Definition* 1.2 — A triple $(a, b, p)$ is said to be $g_2$-realizable if there exists a bipartite graph of $G(V_1, V_2)$ on $p$ vertices and a 2-partition $Q = \{W_1, W_2\}$ of $V_1 \cup V_2$ with $W_i \cap V_j \neq \phi$ for any $i$ and $j$; $i = 1, 2$; $j = 1, 2$ such that $d(G) = a$ and $d(G_Q) = b$. Such a $G(V_1, V_2)$ is called a $g_2$-realization (or simply a realization) of $(a, b, p)$.

*Observation* 1.1 — There cannot be a bipartite graph on $p$ vertices ($p > 2$) with diameter 1. The only bipartite graph on two vertices with diameter 1 is a one factor.

*Observation* 1.2 — A graph (so also a bipartite graph) with diameter $a$ must have at least $(a+1)$ vertices and if it has exactly $(a+1)$ vertices then it must be a path.

*Lemma* 1.1 — Let $\min(a, b) \geq 2$. If $(a, b, p)$ is $g_1$-realizable or $g_2$-realizable then so is $(a, b, p+1)$.

PROOF : Let $(a, b, p)$ be $g_1$-realizable (or $g_2$-realizable) and $G(P)$ together with $Q$ and be a $g_1$ (or $g_2$) -realization of $(a, b, p)$ such that $d(G_Q)$ is $b$. Fix a vertex $u$ in $V(G)$. Construct a graph $H$ from $G$ by adding a new vertex $u_0$ to $V(G)$ and joining it to all the vertices which are adjacent to $u$ in $G$. The 2-partition $P_0$ of $V(H)$ obtained by including $u_0$ in the set of $P$ which contains $u$ and the 2-partition $Q_0$ of $V(H)$ obtained by including $u_0$ in the set of $Q$ which contains $u$ gives a bipartite graph $H$ on $(p+1)$ vertices and the diameter of $H(P_0)$ is $a$ and that of $H_{Q_0}$ is $b$. Thus $H(P_0)$ is a $g_1$ (or $g_2$)-realization of $(a, b, p+1)$. □

Thus if we can find the smallest value of $p$ for which $(a, b, p)$ is realizable then by Lemma 1.1 $(a, b, p^*)$ will be realizable for all $p^* \geq p$.

## 2. ENUMERATION OF $g_1$-REALIZABLE TRIPLES

In this section we enumerate all $g_1$-realizable triples. Without loss of generality, let $W_1 \cap V_2 = \phi$ i.e., $W_1 \subset V_1$. If $W_1 = V_1$, then $W_2 = V_2$ and $G_Q = \overline{G}$. The range of diameters for $G$ and $\overline{G}$ is already known[3]. Hence we consider $W_1 \subsetneqq V_1$. Therefore, obviously $p \geq 3$.

Now, consider Fig. 2.1.

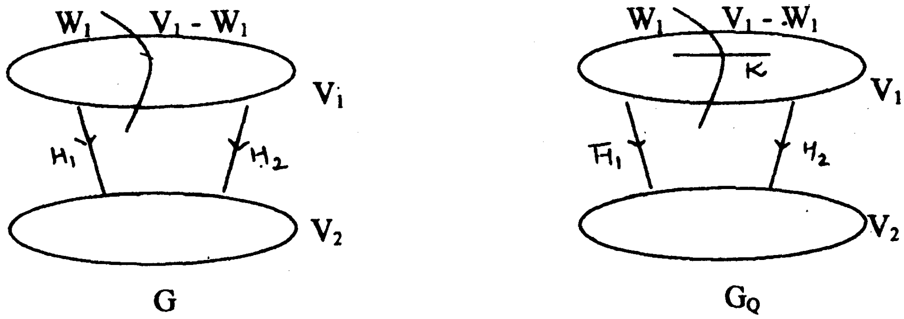



FIG. 2.1

If $G(W_1 \mid V_2)$ represents the induced subgraph of $G$ with vertex set $W_1 \cup V_2$ and if $G(W_1 \mid V_2) = H_1$ then $G_Q(W_1 \mid V_2) = \overline{H}_1$ and if $G(V_1 - W_1 \mid V_2) = H_2$ then $G_Q(V_1 - W_1 \mid V_2) = H_2$. Also $G_Q(V_1 - W_1 \mid V_2) = H_2$. Also $G_Q(W_1 \mid V_1 - W_1)$ is a complete bipartite graph $K$.

In all the figures below, encircled vertices represent $W_1$ and darkened vertices give a pair that is farthest in the graph.

*Observation* 2.1 — $(\infty, 1, 3)$ is the only $g_1$-realizable triple. (See Fig. 2.2)

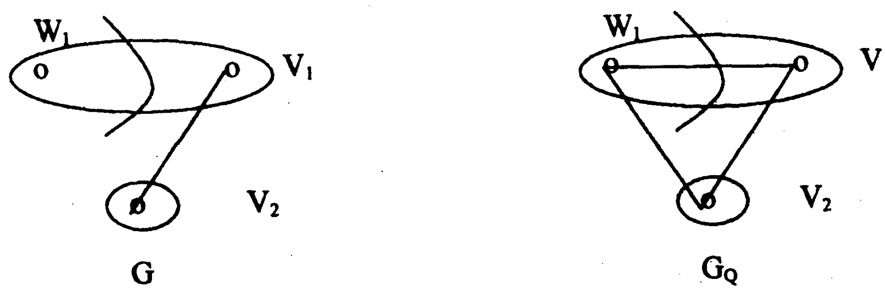



FIG. 2.2

***Theorem* 2.1** — *Let min $(a, b) \geq 2$. The smallest value of $p$, if it exists, for which $(a, b, p)$ is $g_1$-realizable is given in Table 2.1.*

PROOF : *Claim A* — If in Table 2.1, '-' corresponds to a pair $(a, b)$ then $(a, b, p)$ is not realizable for any $p$.

We prove this claim in 2 steps.

*Step* 1 — $b \leq 4$ or $b = \infty$.

If any vertex $u \in V_2$ is isolated in $G_Q$ then $b = \infty$. Otherwise the longest possible path between any two vertices $u$ and $v$ in $G_Q$ is of length (*i*) 1 if $u \in W_1$ and $v \in V_1 - W_1$, (*ii*) at most 2 if $u \in W_1$ and $v \in V_2$ or $u \in V_1 - W_1$ and $v \in V_2$ or $u, v \in W_1$ or $u, v \in V_1 - W_1$, (*iii*) at most 4 if $u, v \in V_2$. Hence $b \leq 4$ or $b = \infty$.

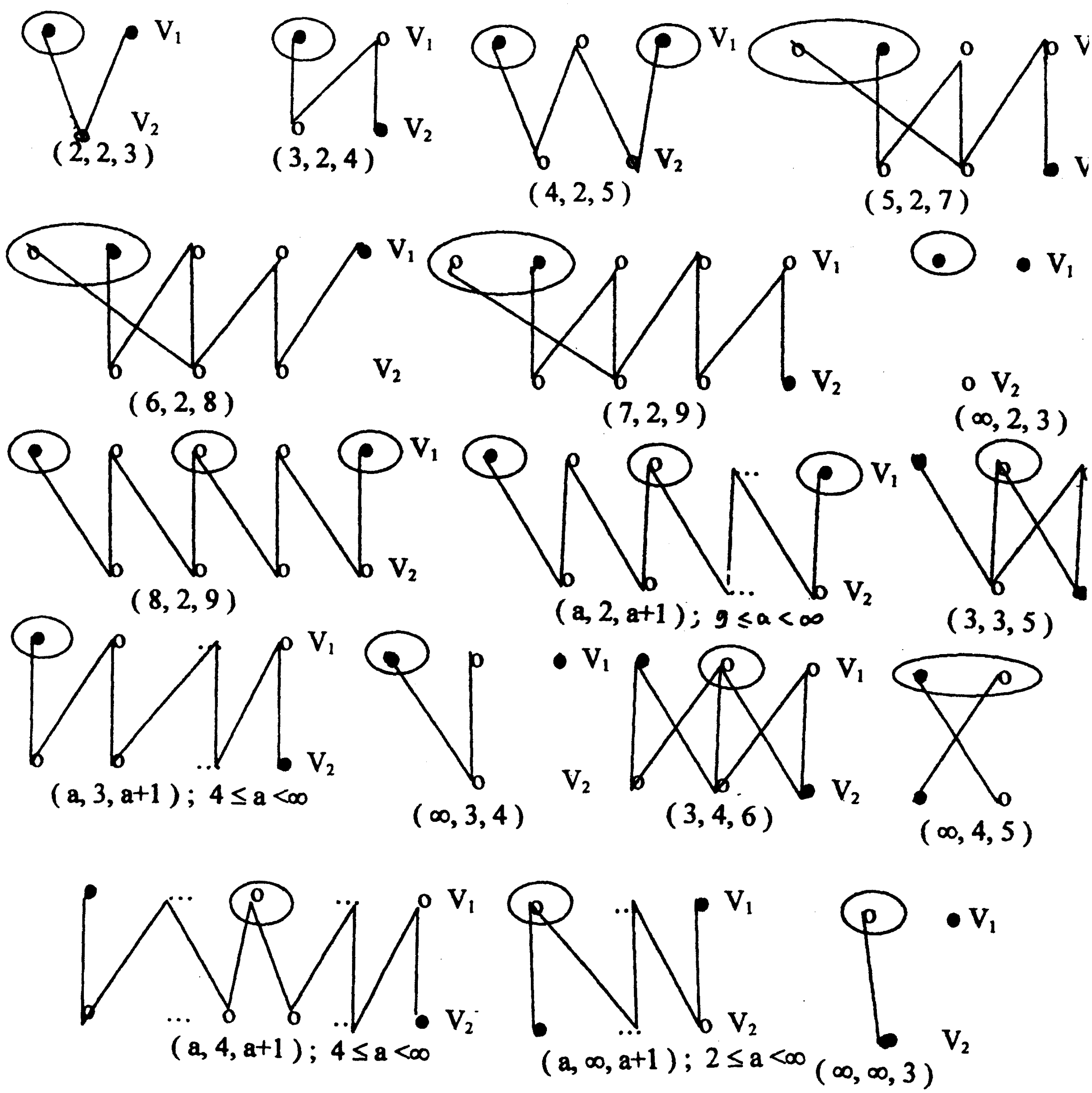

V1
V2
(2, 2, 3)
(3, 2, 4)
(4, 2, 5)
(5, 2, 7)
(6, 2, 8)
(7, 2, 9)
(∞, 2, 3)
(8, 2, 9)
(a, 2, a+1) ; 9 ≤ a < ∞
(3, 3, 5)
(a, 3, a+1) ; 4 ≤ a <∞
(∞, 3, 4)
(3, 4, 6)
(∞, 4, 5)
(a, 4, a+1) ; 4 ≤ a <∞
(a, ∞, a+1) ; 2 ≤ a <∞
(∞, ∞, 3)

FIG. 2.3

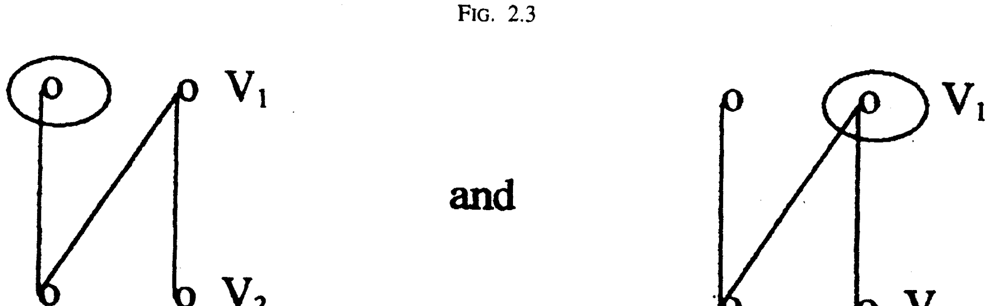

V1
V2
and

FIG. 2.4

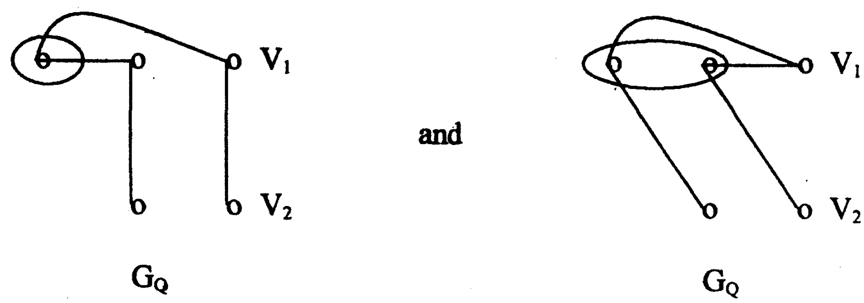



FIG. 2.5

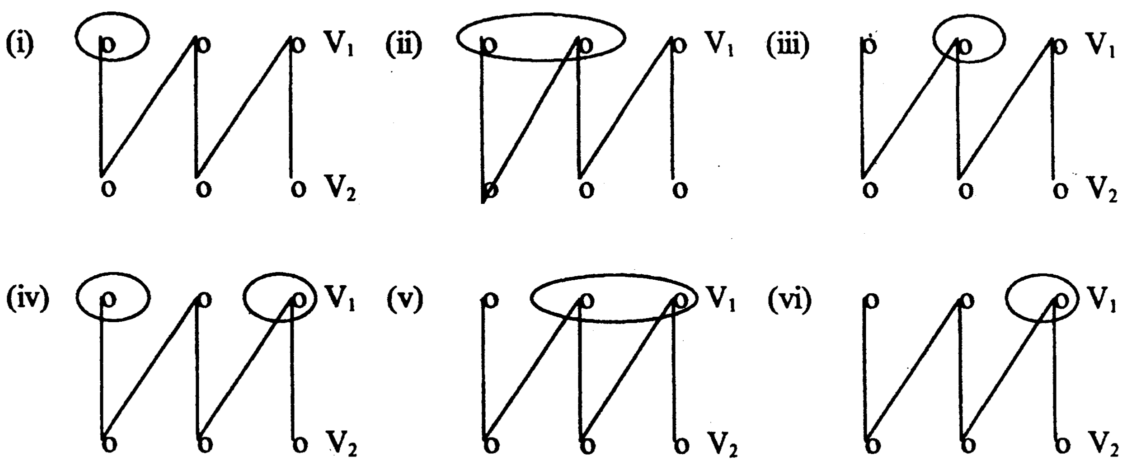



FIG. 2.6

*Step* 2 — (2, $b$, $p$) is not $g_1$-realizable for $3 \le b < \infty$ for any $p \ge 3$.

If $a = 2$ then $G$ must be a complete bipartite graph and hence in Fig. 2.1 $H_1$ and $H_2$ are complete in $G$. Thus in $G_Q, \overline{H}_1$ is null and $H_2$ is complete. But $G_Q(W_1 \mid V_1 - W_1)$ is complete and hence $b = 2$.

*Claim B* — If a positive integer $p^*$ corresponds to a pair ($a$, $b$) in Table 2.1, then ($a$, $b$, $p^*$) is $g_1$-realizable.

To prove claim $B$, we exhibit, in Fig. 2.3, a realization of ($a$, $b$, $p^*$). We also write the triple realized by the graph below it.

*Claim C* — If a positive integer $p^*$ corresponds to a pair ($a$, $b$) in Table 2.1 then ($a$, $b$, $p$) is realizable only if $p \ge p^*$ i.e. $p^*$ is the smallest such integer.

If in Table 2.1 ($a$, $b$, $p^*$) is $g_1$-realizable then we will show that ($a$, $b$, $p$) is not $g_1$-realizable for any $p < p^*$. Towards this we prove the following steps.

*Step 1* — (3, 3, 4) is not $g_1$-realizable. This is because the only possibilities for the choice of $W_1$ in $G$ are as shown in Fig. 2.4. This gives $d(G_Q)$ as 2 and $\infty$ respectively.

*Step 2* — (3, 4, 5) is not $g_1$-realizable.

Here we use the fact that $G_Q$ must be a path and then the only choices for $W_1$ are as shown in Fig. 2.5, which give $d(G)$ to be 4 and $\infty$ respectively.

*Step* 3 — (5, 2, 6), (6, 2, 7) and (7, 2, 8) are not $g_1$-realizable.

We will show that (5, 2, 6) is not realizable and similarly it can be shown that (6, 2, 7) and (7, 2, 8) are not realizable. Here we use the fact that $G$ must be a path on 6 vertices and the only possibilities for the choice of $W_1$ in $G$ are shown in Fig. 2.6. These give $d(G_Q)$ to be either 3 or 4 or $\infty$.

## 3. Enumeration of $g_2$-Realizable Triples

In this section we enumerate all $g_2$-realizable triples. Since $W_i \cap V_j \neq \phi$ for any $i$ and $j$; $i = 1, 2$; $j = 1, 2$, we must have $p \geq 4$.

Now, consider Fig. 3.1

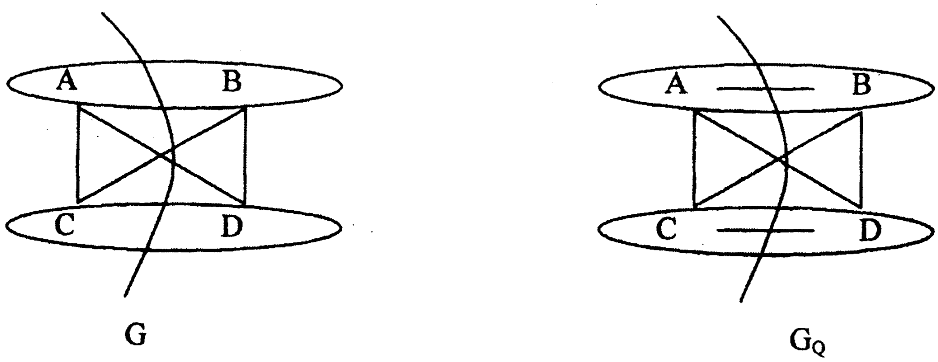



Fig. 3.1

If $G(A \mid C) = H_1$ then $G_Q(A \mid C) = H_1$.

If $G(A \mid D) = H_2$ then $G_Q A \mid D) = \overline{H}_2$.

If $G(B \mid D) = H_3$ then $G_Q(B \mid D) = H_3$.

If $G(B \mid C) = H_4$ then $G_Q(B \mid C) = \overline{H}_4$. Also, since $G(A \mid B) = Null$, $G_Q(A \mid B) = K$, full and similarly $G_Q(C \mid D) = K'$, full.

Throughout this section $W_1 = A \cup C$ and $W_2 = B \cup D$.

*Observation* 3.1 — $(\infty, 1, 4)$ is the only $g_2$-realizable triple (see Fig. 3.2).

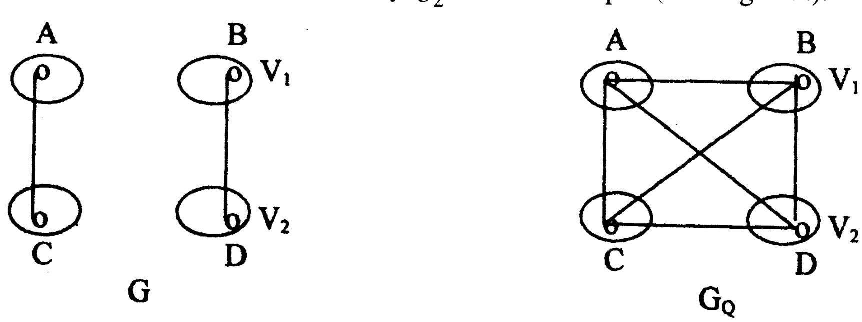



Fig. 3.2

***Theorem* 3.1** — *Let min $(a, b) \geq 2$, the smallest value of p, if it exists, for which (a, b, p) is $g_2$-realizable is given in Table 3.1.*

PROOF : *Claim A* — If in Table 3.1 '-', corresponds to a pair $(a, b)$ then $(a, b, p)$ is not realizable for any $p$.

We prove this claim in 4 steps.

*Step* 1 — If $a = 2$ then $b$ must be 2. Since $a = 2$, $G$ must be a complete bipartite graph and from Fig. 3.1, it is clear that $d(G_Q) = 2$.

*Step* 2 — $b \leq 5$ or $b = \infty$.

From Fig. 3.1, $G_Q$ is $H_1 \cup \overline{H}_2 \cup H_3 \cup \overline{H}_4 \cup K \cup K'$. If all of $H_1, \overline{H}_2, H_3$ and $\overline{H}_4$ are null then $G_Q$ becomes disconnected giving $b = \infty$. If any of $H_1, \overline{H}_2, H_3$ or $\overline{H}_4$ is non-null then maximum value of $b$ is 5 as can be seen from the example shown in Fig. 3.3 when $H_1$ is non-null.

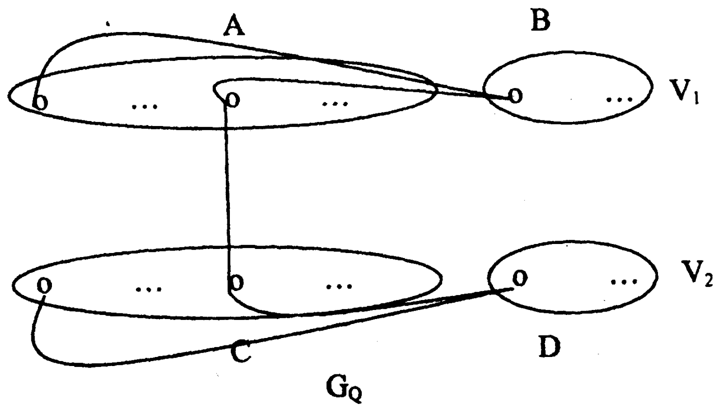



FIG. 3.3

*Step* 3 — If $b = 5$ then $a = 5$ or $\infty$.

For $b$ to be 5 we must have exactly one of $H_1, \overline{H}_2, H_3$ and $\overline{H}_4$ to be non-null and all others must be null in $G_Q$. In addition if either when $H_1$ or $H_3$ is non-null, it must be non full as well, for otherwise $b < 5$ and then $a = 5$. If either of $\overline{H}_2$ or $\overline{H}_4$ is non-null then $G$ is disconnected. Hence $a = 5$ or $a = \infty$.

*Step* 4 — If $b = \infty$ then $a$ must be $\infty$.

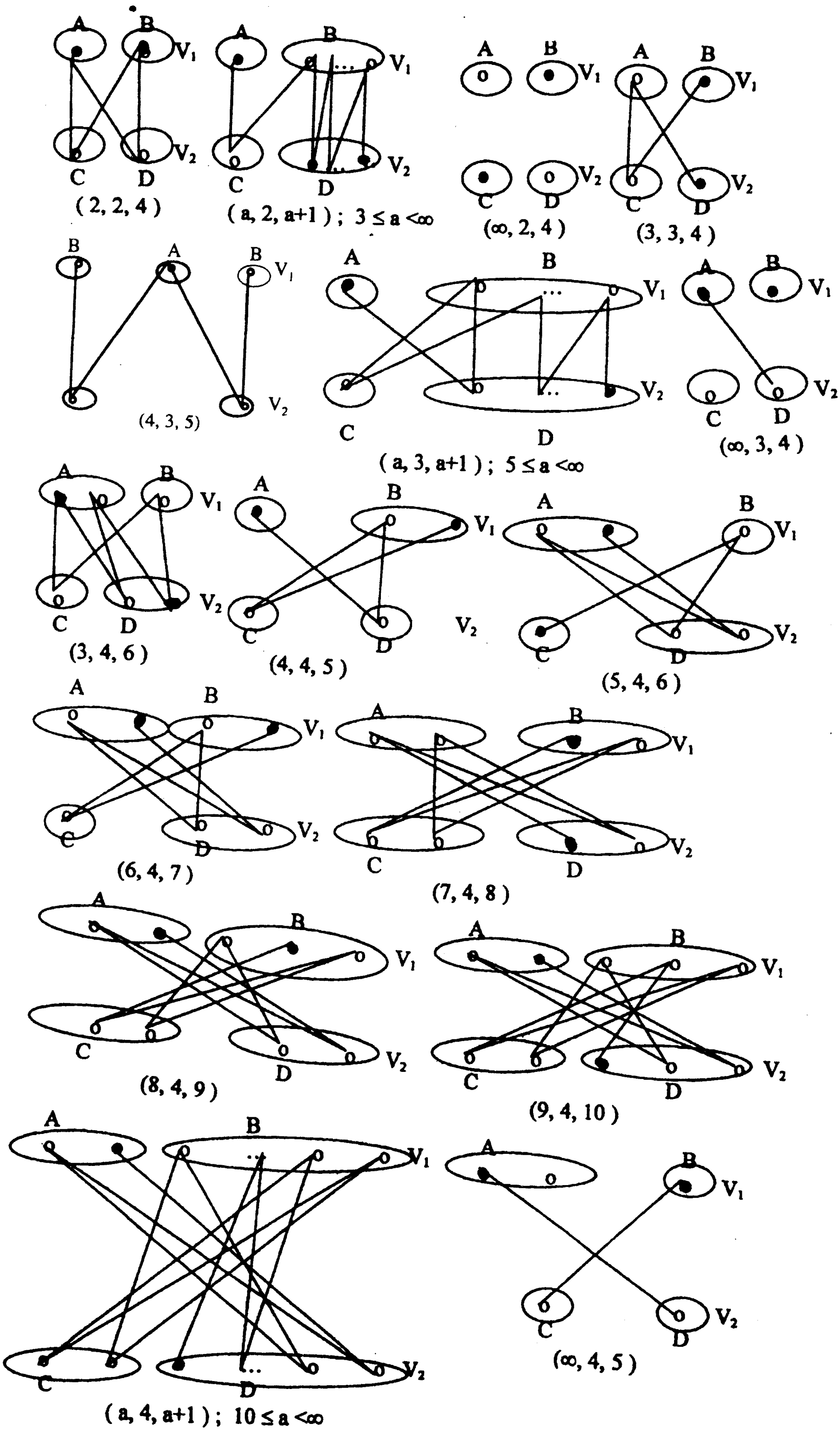
Since $b = \infty$, all of $H_1, \overline{H}_2, H_3$ and $\overline{H}_4$ in Fig. 3.1 are null and hence $G$ becomes disconnected giving $a = \infty$.

*Claim B* — If a positive integer $p^*$ corresponds to a pair $(a, b)$ in Table 3.1 then $(a, b, p^*)$ is realizable.

To prove claim $B$, we exhibit in Fig. 3.4 a realization of $(a, b, p^*)$. We also write the triple realized by the graph below it and the darkened vertices represent a pair that is farthest in the graph.

*Claim C* — If a positive integer $p^*$ corresponds to a pair $(a, b)$ in Table 3.1 then $(a, b, p)$ is not realizable for any $p < p^*$ i.e. $p^*$ is the smallest such integer.

We prove claim C using the following two step.



A B V1 V2 C D
( 2, 2, 4 )
A B V1 V2 C D
( a, 2, a+1 ) ; 3 ≤ a <∞
A B V1 V2 C D
(∞, 2, 4 )
A B V1 V2 C D
(3, 3, 4 )
B A B V1 V2
(4, 3, 5)
A B V1 V2 C D
( a, 3, a+1 ) ; 5 ≤ a <∞
A B V1 V2 C D
(∞, 3, 4 )
A B V1 V2 C D
(3, 4, 6 )
A B V1 V2 C D
(4, 4, 5 )
A B V1 V2 C D
(5, 4, 6 )
A B V1 V2 C D
(6, 4, 7 )
A B V1 V2 C D
(7, 4, 8 )
A B V1 V2 C D
(8, 4, 9 )
A B V1 V2 C D
(9, 4, 10 )
A B V1 V2 C D
( a, 4, a+1 ) ; 10 ≤ a <∞
A B V1 V2 C D
(∞, 4, 5 )

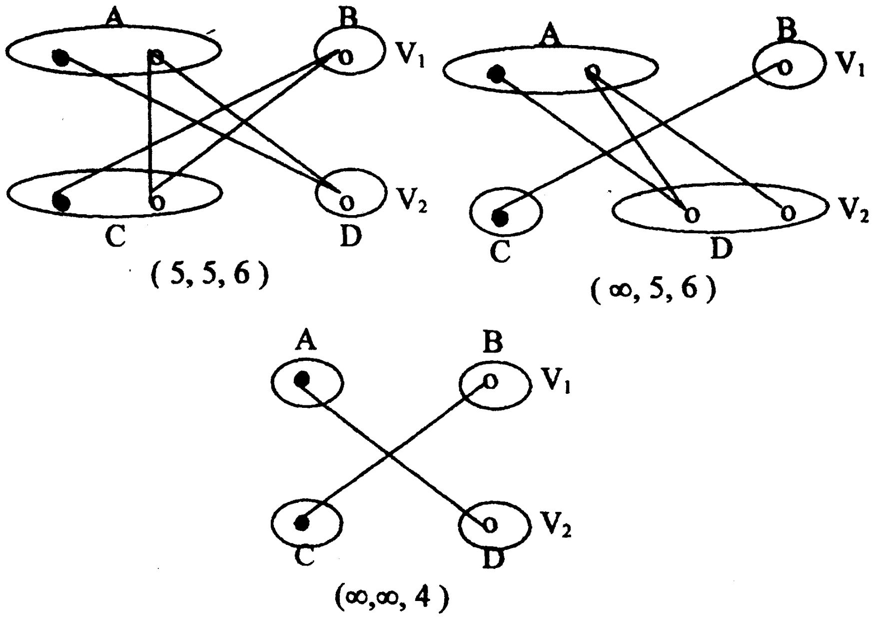



FIG. 3.4

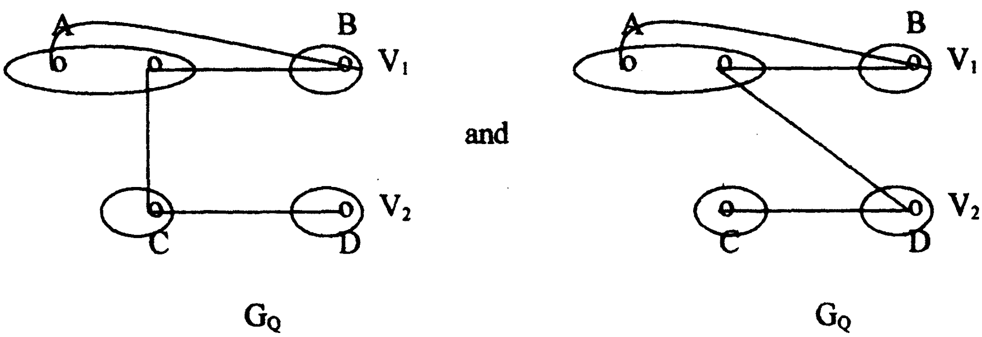



FIG. 3.5

*Step* 1 — (3, 4, 5) is not realizable. Since $b = 4$, $G_Q$ must be a path and the only possibilities are as shown in Fig. 3.5. This gives $a = 4$ and $\infty$ respectively.

*Interchanging* $C$ and $D$ in each of the above exhausts all the possibilities for the choice of $W_1$ and we get $d(G_Q)$ to be either 2 or 4. □

TABLE 2.1

| $a \setminus b$ | 2 | 3 | 4 | $5 \leq b < \infty$ | $\infty$ |
|---|---|---|---|---|---|
| 2 | 3 | - | - | - | 3 |
| 3 | 4 | 5 | 6 | - | 4 |
| 4 | 5 | 5 | 5 | - | 5 |
| 5 | 7 | 6 | 6 | - | 6 |
| 6 | 8 | 7 | 7 | - | 7 |
| 7 | 9 | 8 | 8 | - | 8 |
| $8 \leq a < \infty$ | $a + 1$ | $a + 1$ | $a + 1$ | - | $a + 1$ |
| $\infty$ | 3 | 4 | 5 | - | 3 |

TABLE 3.1

| $a$ / $b$ | 2 | 3 | 4 | 5 | $6 \leq b < \infty$ | $\infty$ |
|---|---|---|---|---|---|---|
| 2 | 4 | - | - | - | - | - |
| 3 | 4 | 4 | 6 | - | - | - |
| 4 | 5 | 5 | 5 | - | - | - |
| 5 | 6 | 6 | 6 | 6 | - | - |
| $6 \leq a < \infty$ | $a + 1$ | $a + 1$ | $a + 1$ | - | - | - |
| $\infty$ | 4 | 4 | 5 | 6 | - | 4 |

## REFERENCES


1. C. R. J. Clapham, *Discrete Math.*, **B** (1974), 251-55.
2. T. Gangopadhyay and S. P. Rao, Hebbare, *Discrete Mathematics*, **32** (1980), 245-55.
3. T. Gangopadhyay, *ARS Combinatoria*, Vol. **13** (1982), 245-56.
4. R. A. Gibbs, *J. Comb. Theory*, **16** (1974), 106-123.
5. F. Harary, *Graph theory*, Narosa Publishing House, New Delhi (1988).
6. E. A. Nordhaus and J. W. Gaddum, *Amer. math. Monthly*, **63** (1956), 175-77.
7. S. B. Rao, *Explored, semi-explored and unexplored territories in the structure theory of self-complementary graphs and digraphs, proceedings of the symposium on Graph Theory*, I.S.I. lecture notes 4, MacMillan Co. (Ed. A.R. Rao).
8. G. Ringel, *Arch. math.*, **14** (1963), 354-58.
9. H. Sachs, *Pub. Math. Debrecen*, **9** (1962), 270-88.
10. E. Sampathkumar and L. Pushpalatha, *Graph and Combinatorics*, **14** (1998), 377-92.