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# ON SINGULARITY OF SPHERICALLY SYMMETRIC SPACE TIMES

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In this note the definition of the singularity is proposed on the basis of the characteristic system of a spherically symmetric space time containing two orthogonal vectors and nine scalars and then concluded that the so called Schwarzschild singularity is physical.

**Key Words : Duplexes; Singular Space Time; Nonsingular Space Time; Schwarzschild Singularity**

## 1. INTRODUCTION

The issue of the Schwarzchild singularity at $r = 2m$ has created interest but of differing opinions: some believe that $r = 2m$ is a mathematical singularity while the others opine that it is a real or physical singularity [see Eddington and Finkelstein (1958), Kruskal (1960), Synge (1964), Penrose (1965), Hilton (1965), Geroch (1966), (1968), Bel (1969), Rosen (1970), Finley III (1971), Karade and Rao (1975), (1976), Karade (1975), (1976) and (1978)]. In this note we discuss this issue purely from mathematical standpoint of view basing the argument on the theory of analytical invariants involving two orthogonal unit vectors and nine scalars (for details one may refer to Takeno (1966), Karade and Borkar (2000)). We have deduced a theorem on singularity with regard to the spherically symmetric space time (SSST)

$$ds^2 = Adr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) + Bdt^2 \qquad \ldots (1)$$

where $A$ and $B$ are the functions of $r$ and $t$ and then applied it to the Schwarzschild space time

$$ds^2 = -\left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) + \left(1 - \frac{2m}{r}\right) dt^2 \qquad \ldots (2)$$

and the Reissner-Nordstrom (RN) space time

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) + \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dt^2. \qquad \ldots (3)$$

The characteristic system consists of space-like vector $\alpha_i$, time-like vector $\beta_i$ and nine scalars $\rho^a$, $a = 1, \ldots, 5$, $\sigma, \bar{\sigma}, \kappa, \bar{\kappa}$ (as defined in Takeno (1966)).

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We postulate some defintions as follows :

*I. Singular Characteristic System (CS)*

If there is one member of the CS which is singular or nonexistent at $x=c \in V_4$ then we say that the CS is singular at $x=c$ for some duplex $(\alpha_m, \beta_n)$.

*II. Nonsingular Characteristic System*

If none of the members of the CS is singular at $x=c \in V_4$ then we term the CS tp be nonsingular at $x = c$ for all existing duplexes $(\alpha_m, \beta_n)$.

*III. Singular Space Time $V_4$*

A space time $V_4$ is said to be singular at $x = c \in V_4$ if its CS is singular at $x = c$.

*IV. Nonsingular Space Time $V_4$*

A space time $V_4$ is said to be nonsingular at $x=c \in V_4$ iff its CS is nonsingular at $x = c$.

*V. Mathematical Singularity*

If the space time breaks down at $x=c \in V_4$ i.e., it is not defined at $x = c$ but the CS is nonsingular then we call $x = c$, a mathematical or a coordinate singularity.

*VI. Physical Singularity*

If the space time $V_4$ is singular at $x=c \in V_4$ then $x = c$ is called the physical singularity.

We adopt the notation $(\alpha_m, \beta_n)$ (See Karade and Borker 2000) for the duplex of the CS. The possible duplexes for the general spherically symmetric space time $ds^2 = g_{ij}\, dx^i\, dx^j$ are

$$\left.\begin{array}{l}(\alpha_1, \beta_1), (\alpha_1, \beta_2), (\alpha_1, \beta_3), (\alpha_1, \beta_4), (\alpha_2, \beta_1), (\alpha_2, \beta_2), (\alpha_2, \beta_3), (\alpha_2, \beta_4) \\ (\alpha_3, \beta_1), (\alpha_3, \beta_2), (\alpha_3, \beta_3), (\alpha_3, \beta_4), (\alpha_4, \beta_1), (\alpha_4, \beta_2), (\alpha_3, \beta_3), (\alpha_4, \beta_4)\end{array}\right\} \quad \ldots (4)$$

It was proved that $(\alpha_m, \beta_m)$, $m = 1, 2, 3, 4$ are nonexistent (see Karade and Borkar 2000).

Here and hereafter we consider the SSST (1) with $AB = 1$ unless otherwise specified.

For the metric (1) with $AB = 1$, the nonzero components of the curvature tensor $K_{ijlm}$ are

$$\left.\begin{array}{ll} K_{1212} = \dfrac{K_{1313}}{\sin^2\theta} = \dfrac{-rA'}{2A}, & K_{1414} = \left(\dfrac{A''}{2A^2} - \dfrac{A'^2}{A^3} + \dfrac{\ddot{A}}{2}\right), \\ K_{2323} = (A^{-1} - 1)\, r^2 \sin^2\theta & K_{2424} = \dfrac{K_{3434}}{\sin^2\theta} = \dfrac{rA'}{2A^3}. \\ K_{2124} = \dfrac{K_{3134}}{\sin^2\theta} = \dfrac{-r\dot{A}}{2A} & \end{array}\right\} \quad \ldots (5)$$

Here (*) and (·) denote the derivatives with respect ro $r$ and $t$ respectively.

From the definition of singular space time and the scalars $\rho^a, a = 1, ..., 5$ obtained from the components (see Karade and Borkar 2000) of the curvature tensor $K_{ijlm}$, we immediately arrive at the theorem :

*Theorem* 1 — *The space time (1) is singular for all the duplexes* $(\alpha_{m'}, \beta_{n'})$, $m, n = 1, 2, 3, 4$ except the pairs $(m, n) = (1, 4), (4, 1), (2, 3), (3, 2)$.

***Theorem*** 2 — *For the SSST (1), the scalars* $\rho^a$, $a = 1, ..., 5$ $\tau^b$, $b = 1, 2, 3$ *retain the same values for* $(\alpha_{m'}, \beta_{n'})$ *and* $(\alpha_{m'}, \beta_{m'})$ *whereas* $\sigma \Leftrightarrow i\,\bar{\sigma}$, $\bar{\sigma} \Leftrightarrow \frac{1}{i}\,\sigma$, $\kappa \Leftrightarrow i\,\bar{\kappa}$ *and* $\bar{\kappa} \Leftrightarrow \frac{1}{i}\,\kappa$ *under* $(\alpha_{m'}, \beta_{n'}) \Leftrightarrow (\alpha_{m'}, \beta_{m'})$, $(m, n) = (1, 4)$ *and* $(2, 3)$, *where* $\tau^b$, $b = 1, 2, 3$ *are the scalars given in Takeno (1966).*

## 2. THE REISSNER — NORDSTROM SPACE TIME

The RN space time (3) has singularities at $r = 0$ and $r = m \pm \sqrt{m^2 - e^2}$. Applying the Theorem 1 to RN field, the duplexes $(\alpha_{m'}, \beta_{n'})$ are nonexistent for $(m, n) \neq (1, 4), (4, 1), (2, 3), (3, 2)$.

For $(\alpha_1, \beta_4)$, we deudce

$$\alpha_1 = \sqrt{\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)}, \qquad \beta_4 = \sqrt{\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)}, \qquad \text{... (6)}$$

$$\rho^1 = -24\left(\frac{m}{r^3} - \frac{e^2}{r^4}\right), \; \rho^2 = \rho^3 = \left(\frac{12m}{r^3} - \frac{8e^2}{r^4}\right), \; \rho^4 = -2\left(\frac{2m}{r^3} - \frac{e^2}{r^4}\right), \rho^5 = 0. \qquad \text{... (7)}$$

$$\tau^1 = \left(\frac{2m}{r^3} - \frac{3e^2}{r^4}\right), \quad \tau^2 = \tau^3 = \frac{-e^2}{r^4} \qquad \text{... (8)}$$

and

$$\sqrt{g_{44}}\;\sigma = 0, \quad \sqrt{g_{44}}\;\bar{\sigma} = -\left(\frac{m}{r^2} - \frac{e^2}{r^3}\right), \quad \kappa = \frac{-\sqrt{g_{44}}}{r}, \quad \sqrt{g_{44}}\;\bar{\kappa} = 0. \qquad \text{... (9)}$$

For $(\alpha_2, \beta_3)$, we have

$$\alpha_2 = \sqrt{r^2}, \quad \beta_3 = \sqrt{-r^2 \sin^2\theta}, \qquad \text{... (10)}$$

$$\rho^1 = -24\left(\frac{m}{r^3} - \frac{e^2}{r^4}\right), \; \rho^2 = \rho^3 = \left(\frac{12m}{r^3} - \frac{16e^2}{r^4}\right),$$

$$\rho^4 = -2\left(\frac{2m}{r^3} - \frac{3e^2}{r^4}\right), \rho^5 = 0. \qquad \text{...(11)}$$

$$\tau^1 = \left( \frac{2m}{r^3} - \frac{e^2}{r^4} \right), \quad \tau^2 = \tau^3 = \frac{e^2}{r^4} \qquad \ldots (12)$$

$$\sigma = 0, \quad \bar{\sigma} = \frac{-\cot\theta}{r}, \quad \kappa = 0, \quad \bar{\kappa} = 0. \qquad \ldots (13)$$

## 3. The Schwarzschild Space Time

Puttine $e = 0$ in (3) we get the Schwarszchild space time (2). Thus the results corresponding to (2) can easily be deduced from those of (3).

For the Schwarzschild space time we get:

For $(\alpha_2, \beta_3)$ :

$$\alpha_1 = \sqrt{\left(1 - \frac{2m}{r}\right)^{-1}}, \qquad \beta_4 = \sqrt{\left(1 - \frac{2m}{r}\right)}, \qquad \ldots (14)$$

$$\rho^1 = \frac{-24m}{r^3}, \quad \rho^2 = \rho^3 = \frac{12m}{r^3}, \quad \rho^4 = \frac{-4m}{r^3}, \quad \rho^5 = 0. \qquad \ldots (15)$$

$$\tau^1 = \frac{2m}{r^3}, \quad \tau^2 = \tau^3 = 0. \qquad \ldots (16)$$

$$\sqrt{g_{44}}\ \sigma = 0, \quad \sqrt{g_{44}}\ \bar{\sigma} = \frac{-m}{r^2}, \quad \kappa = \frac{-\sqrt{g_{44}}}{r}, \quad \sqrt{g_{44}}\ \bar{\kappa} = 0. \qquad \ldots (17)$$

For $(\alpha_2, \beta_3)$ :

$$\alpha_2 = \sqrt{r^2}, \quad \beta_3 = \sqrt{-r^2 \sin^2\theta}, \qquad \ldots (18)$$

$$\rho^1 = \frac{-24m}{r^3}, \quad \rho^2 = \rho^3 = \frac{12m}{r^3}, \quad \rho^4 = \frac{-4m}{r^3}, \quad \rho^5 = 0. \qquad \ldots (19)$$

$$\tau^1 = \frac{2m}{r^3}, \quad \tau^2 = \tau^3 = 0. \qquad \ldots (20)$$

$$\sigma = 0, \quad \bar{\sigma} = \frac{-\cot\theta}{r}, \quad \kappa = 0, \quad \bar{\kappa} = 0. \qquad \ldots (21)$$

*Theorem* 3 — *All the duplexes* $(\alpha_1, \beta_4), (\alpha_4, \beta_1), (\alpha_2, \beta_3), (\alpha_3, \beta_2)$ *of the CS of the Schwarzschild space time yield the same values of the scalars* $\rho^a$, $a = 1, \ldots, 5$.

As regard to the singularities we put them in the tabular form as below :

| | | RN space time | | Schwarzschild space time | |
|---|---|---|---|---|---|
| Singularities → | | $r=0$ | $r=m\pm\sqrt{ms-e^2}$ | $r=0$ | $r=2m$ |
| $(\alpha_1, \beta_4)$ | Singular → | $\rho^1, \rho^2, \rho^3, \rho^4, \tau^b, \bar{\sigma}, \kappa$ | $\bar{\sigma}$ | $\rho^1, \rho^2, \rho^3, \rho^4, \tau^b, \bar{\sigma}, \kappa$ | $\bar{\sigma}$ |
| | Nonsingular → | $\rho^5, \sigma, \bar{\kappa}$ | $\rho^a, \tau^b, \sigma, \kappa, \bar{\kappa}$ | $\rho^5, \tau^2, \tau^3\ \sigma, \bar{\kappa}$ | $\rho^a, \tau^b, \sigma, \kappa, \bar{\kappa}$ |
| $(\alpha_4, \beta_1)$ | Singular → | $\rho^1, \rho^2, \rho^3, \rho^4, \tau^b, \sigma, \bar{\kappa}$ | $\sigma$ | $\rho^1, \rho^2, \rho^3, \rho^4, \tau^1, \sigma, \bar{\kappa}$ | $\sigma$ |
| | Nonsingular → | $\rho^5, \bar{\sigma}, \kappa$ | $\rho^a, \tau^b, \bar{\sigma}, \kappa, \bar{\kappa}$ | $\rho^5, \tau^2, \tau^3\ \bar{\sigma}, \kappa$ | $\rho^a, \tau^b, \bar{\sigma}, \kappa, \bar{\kappa}$ |
| $(\alpha_2, \beta_3)$ | Singular → | $\rho^1, \rho^2, \rho^3, \rho^4, \tau^b, \bar{\sigma}$ | — | $\rho^1, \rho^2, \rho^3, \rho^4, \tau^1, \bar{\sigma}$ | — |
| | Nonsingular → | $\rho^5, \sigma, \kappa, \bar{\kappa}$ | $\rho^a, \tau^b, \sigma, \bar{\sigma}\ \kappa, \bar{\kappa}$ | $\rho^5, \tau^2, \tau^3\ \sigma, \kappa, \bar{\kappa}$ | $\rho^a, \tau^b, \sigma, \bar{\sigma}\ \kappa, \bar{\kappa}$ |
| $(\alpha_3, \beta_2)$ | Singular → | $\rho^1, \rho^2, \rho^3, \rho^4, \tau^b, \sigma$ | — | $\rho^1, \rho^2, \rho^3, \rho^4, \tau^1, \sigma$ | — |
| | Nonsingular → | $\rho^5, \bar{\sigma}, \kappa, \bar{\kappa}$ | $\rho^a, \tau^b, \sigma, \bar{\sigma}\ \kappa, \bar{\kappa}$ | $\rho^5, \tau^2, \tau^3\ \bar{\sigma}, \kappa, \bar{\kappa}$ | $\rho^a, \tau^b, \sigma, \bar{\sigma}\ \kappa, \bar{\kappa}$ |

In the above table : $a = 1, ..., 5$; $b = 1, 2, 3$.

In view of our postulated definitions and the computation of the CS we conclude that the singularities at $r = 2m$ of the Schwarzschild space time and $r=m\pm\sqrt{m^2-e^2}$ of the RN space time are the physical singularities. This is differing from those of Finklestein (1958), Fronsdal (1959), Graves and Brill (1960) but confirming the vies of Rosen (1970), Hilton (1965), Bel (1969), Karade (1975, 1078) etc.

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