

THE PSEUDO-DIFFERENTIAL OPERATOR $h_{\mu, a}$ ON SOME GEVREY SPACES

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Continuity of the pseudo-differential operator $h_{\mu, a}$ is investigated on certain Gevrey spaces when the symbol of the pseudo-differential operator also belongs to a certain Gevrey class.

Key Words : Pseudo-Differential Operator; Hankel Transform; Bessel-Differential Operator; Gevrey Space

1. INTRODUCTION

The Hankel transformation

$$\Phi(y) = (h_{\mu} \phi)(y) = \int_0^{\infty} \sqrt{xy} J_{\mu}(xy) \phi(x) dx, \quad \mu \geq -1/2. \quad \dots (1.1)$$

is known to be an automorphism on the Zemanian space $H_{\mu}(I)$, $I = (0, \infty)$, consisting of all complex-valued infinitely differentiable functions ϕ on I which satisfy :

$$\gamma_{m, k}^{\mu}(\phi) = \sup_{x \in I} \left| x^m (x^{-1} D)^k x^{-\mu-1/2} \phi(x) \right| < \infty \quad \forall m, k \in \mathbb{N}_0 \quad \dots (1.2)$$

Using theory of H_{μ} space [7] Pathak and Pandey [4] investigated the pseudo-differential operator $h_{\mu, a}$ defined as follows :

$$(h_{\mu, a} \phi)(x) = \int_0^{\infty} \sqrt{xy} J_{\mu}(xy) a(x, y) (h_{\mu} \phi)(y) dy \quad \mu \geq -1/2. \quad \dots (1.3)$$

With symbol $a(x, y)$ satisfying certain growth conditions $h_{\mu, a}$ is a continuous linear map of $H_{\mu}(I)$ into itself [4].

In this paper we assume that the symbol $a(x, y)$ belongs to a certain Gevrey class $S_{\rho, \delta}^{\infty, \omega}$ and study the pseudo-differential operator $h_{\mu, a}$ on Gevrey function spaces introduced by Pathak and Shrestha [3].

2. THE SPACE $H_{\mu}(\omega)$

Let ω be a continuous real-valued function defined on $I = (0, \infty)$ such that $\omega(0) = 0$ and

$$(a) \quad 0 \leq \omega(x+y) \leq \omega(x) + \omega(y) \quad \forall x, y \in I \quad \dots (2.1)$$

$$(b) \int_0^\infty \frac{\omega(x)dx}{1+x^2} < \infty \quad \dots (2.2)$$

$$(c) \omega(x) \geq a + b \log(1+x) \text{ for some real } a \text{ and } b > 0. \quad \dots (2.3)$$

The class of all such ω functions is denoted by M .

Now assume that ω is a function in M . A function ϕ is said to be in the space $H_\mu(\omega)$, where ϕ and $h_\mu(\phi)$ are smooth functions and for every $\mu \in \mathbf{R}, n \in \mathbf{N}_0$ and m a positive real number,

$$\alpha_{m,n}^\mu(\phi) = \sup_{x \in (0, \infty)} e^{m\omega(x)} \left| (x^{-1}D)^n x^{-\mu-1/2} \phi(x) \right| < \infty \quad \dots (2.4)$$

and

$$\beta_{m,n}^\mu(\phi) = \sup_{x \in (0, \infty)} e^{m\omega(x)} \left| (x^{-1}D)^n x^{-\mu-1/2} (h_\mu\phi)(x) \right| < \infty. \quad \dots (2.5)$$

On $H_\mu(\omega)$ we consider the topology generated by the family

$$\left\{ \alpha_{m,n}^\mu, \beta_{m,n}^\mu \right\}_{m \in \mathbf{R}_+, n \in \mathbf{N}_0}$$

of seminorms. Various properties of the space $H_\mu(\omega)$ can be found in [3]. In what follows we study the pseudo-differential operator $h_{\mu,a}$ of infinite order on $H_\mu(\omega)$. For this purpose we define the symbol class $S_{\rho,\delta}^{\infty,\omega}$.

Definition 2.1 — The function $a(x, y) : C^\infty(I \times I) \rightarrow \mathbf{C}$ belongs to class $S_{\rho,\delta}^{\infty,\omega}$ if and only if $\forall p, q \in \mathbf{N}_0, m \in \mathbf{R}_+, \text{ and } \sigma > 0,$

$$\begin{aligned} & e^{m\omega(x)} \left| (x^{-1} \partial/\partial x)^p (y^{-1} \partial/\partial y)^q y^{2\mu+1} a(x, y) \right| \\ & \leq c(\sigma, m) C^{p+q} p!q! (1+y)^{-\rho q + \delta p} e^{\sigma\omega(y)}, \end{aligned} \quad \dots (2.6)$$

where C is a constant, ρ, δ are real numbers such that $0 \leq \delta < \rho < 1$.

The previous Definition 2.1 is suggested by similar classes of the symbols in the literature, See⁵. Here we shall limit ourselves to give two theorems of continuity on $H_\mu(\omega)$ and related spaces $G_\mu(\omega)$, which are valid under assumptions somewhat weaker than (2.6), cf. the next proofs.

Theorem 2.2 — The pseudo-differential operator $h_{\mu,a}$ is a continuous linear map of $H_\mu(\omega)$ into itself for $\mu \geq -1/2$.

PROOF : (i) Let $\phi \in H_\mu(\omega)$. Then in view of Definition (1.3) using Leibnitz type formula [2, p. 242]

$$(x^{-1}D)^k (x^{-\mu-1/2} \psi \phi) = \sum_{\nu=0}^k \binom{k}{\nu} (x^{-1}D)^\nu \psi (x^{-1}D)^{k-\nu} x^{-\mu-1/2} \phi \quad \dots (2.7)$$

we have

$$\begin{aligned}
 & e^{m\alpha(x)} \left| (x^{-1} D)^n x^{-\mu-1/2} (h_{\mu,a} \phi)(x) \right| = e^{m\alpha(x)} \\
 & \times \left| (x^{-1} D)^n x^{-\mu-1/2} \int_0^\infty (xy)^{1/2} J_\mu(xy) a(x,y) (h_\mu \phi)(y) dy \right| \\
 & = e^{m\alpha(x)} \left| (x^{-1} D)^n \int_0^\infty (xy)^{-\mu} J_\mu(xy) y^{2\mu+1} a(x,y) y^{-\mu-1/2} (h_\mu \phi)(y) dy \right| \\
 & = \int_0^\infty \sum_{r=0}^n \binom{n}{r} \\
 & \times \left| (x^{-1} D)^{n-r} [(xy)^{-\mu} J_\mu(xy)] e^{m\alpha(x)} (x^{-1} D)^r [y^{2\mu+1} a(x,y)] y^{-\mu-1/2} (h_\mu \phi)(y) \right| dy.
 \end{aligned}$$

Using inequality (2.6), the last expression can be estimated by

$$\begin{aligned}
 & \int_0^\infty \sum_{r=0}^n \binom{n}{r} y^{2(n-r)} \left| (xy)^{-\mu-n+r} J_{\mu+n-r}(xy) \right| \\
 & \times c(\sigma, m) C^r r! (1+y)^{r\delta} e^{\sigma\alpha(y)} \\
 & \times e^{-m\alpha(y)} \left| e^{m\alpha(y)} y^{-\mu-1/2} (h_\mu \phi)(y) \right| dy. \tag{2.8}
 \end{aligned}$$

By property (c) of the function $\alpha(x)$ we have

$$(1+y) \leq e^{-a/b} e^{\alpha(y)/b}, \tag{2.9}$$

and from [2, p. 247] we know that there exists a constant Q_μ independent of r such that

$$\left| z^{-\mu-n+r} J_{\mu+n-r}(z) \right| \leq Q_\mu.$$

Therefore (2.8) can be bounded by

$$\begin{aligned}
 & Q_\mu c(\sigma, m) \sum_{r=0}^n \binom{n}{r} C^r r! \\
 & \times \int_0^\infty e^{(\sigma-m)\alpha(y)} e^{-ar\delta/b} e^{r\delta\alpha(y)/b} y^{2(n-r)} e^{m\alpha(y)} \\
 & \times \left| y^{-\mu-1/2} (h_\mu \phi)(y) \right| dy \\
 & \leq Q_\mu c(\sigma, m) \sum_{r=0}^n \binom{n}{r} C^r r!
 \end{aligned}$$

$$\int_0^\infty e^{(\sigma-m)\omega(y)} y^{2(n-r)} e^{-ar\delta/b} \sup_{y \in I} e^{(m+r\delta/b)\omega(y)} \left| y^{-\mu-1/2} (h_\mu \phi)(y) \right| dy.$$

Using (2.5), we have

$$\begin{aligned} & \sup_{x \in I} e^{m\omega(x)} \left| (x^{-1} D)^n x^{P-\mu-1/2} (h_{\mu,a} \phi)(x) \right| \\ & \leq Q_\mu c(\sigma, m) \sum_{r=0}^n \binom{n}{r} C^r r! e^{-ar\delta/b} \\ & \times \beta_{(m+r\delta/b), 0}^\mu(\phi) \int_0^\infty e^{(\sigma-m)\omega(y)} y^{2(n-r)} dy < \infty \end{aligned}$$

by choosing $m > \sigma$. Therefore, $\alpha_{m,n}^\mu(h, \mu, a \phi) < \infty$.

(ii) Let $\Phi(x) = (h_{\mu,a} \phi)(x)$. Then to complete the proof of the theorem we need to show that $\Phi(x)$ satisfies (2.5). From property (b) of the function $\omega(x)$ it follows that to every $\varepsilon > 0$ there exists a constant $c(\varepsilon)$ such that

$$\omega(x) \leq \varepsilon x + c(\varepsilon); \tag{2.10}$$

so that

$$e^{m\omega(x)} \leq e^{mc(\varepsilon)} \sum_{v=0}^\infty \frac{(m\varepsilon)^v}{v!} x^v. \tag{2.11}$$

Therefore, following the technique of Zemanian [7, p. 141] we can write

$$\begin{aligned} & e^{m\omega(x)} \left| (x^{-1} D)^n x^{-\mu-1/2} (h_\mu \Phi)(x) \right| \\ & \leq e^{mc(\varepsilon)} \sum_{v=0}^\infty \frac{(m\varepsilon)^v}{v!} x^v \left| (x^{-1} D)^n x^{-\mu-1/2} (h_\mu \Phi)(x) \right| \\ & \leq e^{mc(\varepsilon)} \sum_{v=0}^\infty \frac{(m\varepsilon)^v}{v!} \\ & \left| \int_0^\infty y^{2\mu+2n+v+1} (y^{-1} D)^v [y^{-\mu-1/2} \Phi(y)] (xy)^{-(\mu+n)} J_{\mu+v+n}(xy) dy \right|. \end{aligned} \tag{2.12}$$

Since for $\mu \geq -1/2$, $\left| (xy)^{-(\mu+n)} J_{\mu+v+n}(xy) \right|$ is bounded on $0 < x, y < \infty$ by Q_μ ,

applying definition (1.3), right-hand side of the eq. (2.12) can be estimated by

$$\begin{aligned}
 & e^{mc(\epsilon)} Q_\mu \sum_{\nu=0}^{\infty} \frac{(m\epsilon)^\nu}{\nu!} \int_0^\infty y^{2\mu+2n+\nu+1} \\
 & \times \left| (y^{-1} D)^\nu \left[y^{-\mu-1/2} \int_0^\infty (yt)^{1/2} J_\mu(yt) a(y, t) (h_\mu \phi)(t) dt \right] \right| dy \\
 & \leq e^{mc(\epsilon)} Q_\mu^2 \sum_{\nu=0}^{\infty} \frac{(m\epsilon)^\nu}{\nu!} \int_0^\infty \int_0^\infty y^{2\mu+2n+\nu+1} \\
 & \times \sum_{s=0}^{\nu} \binom{\nu}{s} \left| (y^{-1} D)^{\nu-s} \left[(yt)^{-\mu} J_\mu(yt) \right] \right| \\
 & \times \left| (y^{-1} D)^s \left[t^{2\mu+1} a(y, t) \right] t^{-\mu-1/2} (h_\mu \phi)(t) \right| dt dy \\
 & \leq e^{mc(\epsilon)} Q_\mu^2 \sum_{\nu=0}^{\infty} \frac{(m\epsilon)^\nu}{\nu!} \int_0^\infty \int_0^\infty y^{2\mu+2n+\nu+1} \\
 & \times \sum_{s=0}^{\nu} \binom{\nu}{s} t^{2(\nu-s)} \left| (y^{-1} D)^s t^{2\mu+1} a(y, t) \right| \left| t^{-\mu-1/2} (h_\mu \phi)(t) \right| dt dy. \quad \dots (2.13)
 \end{aligned}$$

Suppose N is an integer no less than $2\mu + 2n + 1$, then

$$y^{2\mu+2n+\nu+1} \leq (1+y)^{N+\nu}. \quad \dots (2.14)$$

Using (2.14), right-hand side of eq. (2.13) can be bounded by

$$\begin{aligned}
 & Q_\mu^2 e^{mc(\epsilon)} \sum_{\nu=0}^{\infty} \frac{(m\epsilon)^\nu}{\nu!} \int_0^\infty \int_0^\infty (1+y)^{N+\nu+2} \\
 & \times \sum_{s=0}^{\nu} \binom{\nu}{s} t^{2(\nu-s)} \left| (y^{-1} D)^s t^{2\mu+1} a(y, t) \right| \\
 & \times \left| t^{-\mu-1/2} (h_\mu \phi)(t) \right| dt \frac{dy}{(1+y)^2}.
 \end{aligned}$$

Using the inequality (2.3) and $(1+y) \leq e^{-a/b} e^{\omega(y)/b}$, the right-hand side can be bounded by

$$Q_\mu^2 e^{mc(\epsilon)} \sum_{\nu=0}^{\infty} \frac{(m\epsilon)^\nu}{\nu!} \int_0^\infty \int_0^\infty \exp(-a(N+\nu+2)/b) \exp[(N+\nu+2)\omega(y)/b]$$

$$\begin{aligned}
 & \times \sum_{s=0}^{\nu} \binom{\nu}{s} t^{2(\nu-s)} \left| (y^{-1} D)^s t^{2\mu+1} a(y, t) \right| \\
 & \times \left| t^{-\mu-1/2} (h_{\mu} \phi)(t) \right| dt \frac{dy}{(1+y)^2} \\
 & \leq Q_{\mu}^2 e^{mc(\epsilon)} \sum_{\nu=0}^{\infty} \frac{(m\epsilon)^{\nu}}{\nu!} \int_0^{\infty} \exp[-a(N+\nu+2)/b] \\
 & \sum_{s=0}^{\nu} \binom{\nu}{s} t^{2(\nu-s)} \sup_{y \in I} \exp[(N+\nu+2)\alpha(y)/b] \\
 & \times \left| (y^{-1} D)^s t^{2\mu+1} a(y, t) \right| \left| t^{-\mu-1/2} (h_{\mu} \phi)(t) \right| dt \int_0^{\infty} \frac{dy}{(1+y)^2}. \quad \dots (2.15)
 \end{aligned}$$

Using inequality (2.6) the right-hand side can be bounded by

$$\begin{aligned}
 & Q_{\mu}^2 e^{mc(\epsilon)} \sum_{\nu=0}^{\infty} \frac{(m\epsilon)^{\nu}}{\nu!} \int_0^{\infty} \int_0^{\infty} \exp(-a(N+\nu+2)/b) \\
 & \times \sum_{s=0}^{\nu} \binom{\nu}{s} t^{2(\nu-s)} \int_0^{\infty} c(\sigma, (N+\nu+2)/b) C^s s! (1+t)^{s\delta} \\
 & \times e^{\sigma\alpha(t)} \left| t^{\mu-1/2} (h_{\mu} \phi)(t) \right| dt.
 \end{aligned}$$

Since $s\delta \leq \nu$ the above expression can be estimated by

$$\begin{aligned}
 & Q_{\mu}^2 e^{mc(\epsilon)} \exp(-a(N+2)/b) \\
 & \times \sum_{\nu=0}^{\infty} \frac{(m\epsilon)^{\nu}}{\nu!} e^{-a\nu/b} c(\sigma, (N+\nu+2)/b) \sum_{s=0}^{\nu} C^s s! \\
 & \times \sup_{t \in I} \left| (1+t)^{3\nu+2} e^{\sigma\alpha(t)} \left[t^{-\mu-1/2} (h_{\mu} \phi)(t) \right] \right| \int_0^{\infty} \frac{dt}{(1+t)^2} \\
 & \leq Q_{\mu}^2 \exp(mc(\epsilon) - a(N+2)/b) \sum_{\nu=0}^{\infty} \frac{(m\epsilon)^{\nu}}{\nu!} e^{-a\nu/b} \\
 & \times c(\sigma, (N+\nu+2)/b) \sum_{s=0}^{\nu} \binom{\nu}{s} C^s s!
 \end{aligned}$$

$$\times e^{-a(3v+2)/b} \sup_{t \in I} \left| \exp(((3v+2)/b + \sigma) \omega(t)) \left[t^{-\mu-1/2} (h_\mu \phi)(t) \right] \right|. \quad \dots (2.16)$$

Using property (2.5) we can estimate the right-hand side of (2.16) by

$$\begin{aligned} & Q_\mu^2 \exp(mc(\varepsilon) - a(N+4)/b) \sum_{v=0}^\infty \frac{(m\varepsilon)^v}{v!} e^{-4av/b} \\ & \times c(\sigma, (N+v+2)/b) \sum_{s=0}^v \binom{v}{s} C^s v! \\ & \times \beta_{\sigma+(3v+2)/b, 0}^\mu (h_\mu \phi) \\ & \leq Q_\mu^2 \exp(mc(\varepsilon) - a(N+4)/b) \sum_{v=0}^\infty (m\varepsilon)^v \\ & \times c(\sigma, (N+v+2)/b) e^{-4av/b} \sum_{s=0}^v \binom{v}{s} C^s \\ & \times \beta_{\sigma+(3v+2)/b, 0}^\mu (h_\mu \phi) \\ & \leq Q_\mu^2 \exp(mc(\varepsilon) - a(N+4)/b) \sum_{v=0}^\infty [(m\varepsilon) e^{-4a/b} (1+C)]^v \\ & \times \left\{ \beta_{\sigma+(3v+2)/b, 0}^\mu (h_\mu \phi) c(\sigma, (N+v+2)/b) \right\}^{1/v}]^v. \end{aligned}$$

Now choosing

$$\varepsilon < \left\{ \beta_{\sigma+(3v+2)/b, 0}^\mu (h_\mu \phi) c(\sigma, N+v+2)/b \right\}^{-1/v} (me^{-4a/b} (1+C))^{-1},$$

we find that the last series is convergent. This concludes the proof.

3. THE SPACE $G_\mu(\omega)$

The Bessel-differential operator S_μ is defined by

$$S_{\mu, x} = d^2/dx^2 + (1 - 4\mu)^2/4x^2. \quad \dots (3.1)$$

From [7, p. 139] we know that for any $\phi \in H_\mu(I)$,

$$h_\mu(S_\mu \phi) = -y^2 h_\mu \phi \quad \dots (3.2)$$

and

$$S_{\mu, x}^n \phi(x) = \sum_{j=0}^n b_j x^{2j+\mu+1/2} (x^{-1} D)^{n+j} (x^{-\mu-1/2} \phi(x)) \quad \dots (3.3)$$

where b_j are constants depending only on μ .

Now assume that ω is a function in M . A function $\phi \in C^\infty(I)$ is said to be in the space $G_\mu(\omega)$ if for every $\mu \in \mathbf{R}$, $n \in \mathbf{N}_0$, and m a positive real number,

$$A_{m, n}^\mu(\phi) = \sup_{x \in (0, \infty)} e^{m\omega(x)} \left| S_\mu^n \phi(x) \right| < \infty \quad \dots (3.4)$$

and

$$A_{m, n}^\mu(\phi) = \sup_{x \in (0, \infty)} e^{m\omega(x)} \left| S_\mu^n (h_\mu \phi)(x) \right| < \infty. \quad \dots (3.5)$$

The family $\left\{ A_{m, n}^\mu, B_{m, n}^\mu \right\}_{m \in \mathbf{R}, n \in \mathbf{N}_0}$ of seminorms generates the topology of $G_\mu(\omega)$ can be in¹

Theorem 3.1 — *The pseudo-differential operator $h_{\mu, a}$ is a continuous linear map of $G_\mu(\omega)$ into itself for $\mu \geq -1/2$.*

PROOF (i) : Let $\phi \in G_\mu(\omega)$ then in view of formula (3.3) we have

$$e^{m\omega(x)} \left| S_\mu^n (h_{\mu, a} \phi)(x) \right| = e^{m\omega(x)} \times \left| \sum_{j=0}^n b_j x^{2j+\mu+1/2} (x^{-1} D)^{n+j} x^{-\mu-1/2} (h_{\mu, a} \phi)(x) \right|.$$

Using definiton (1.3) we can show that the right-hand side is bounded by

$$\sum_{j=0}^n |b_j| x^{2j+\mu+1/2} \int_0^\infty \sum_{r=0}^{n+j} \binom{n+j}{r} \times \left| (x^{-1} D)^{n+j-r} \left[(xy)^{-\mu} J_\mu(xy) \right] \right| e^{m\omega(x)} \left| (x^{-1} D)^r y^{2\mu+1} a(x, y) \right| \times \left| y^{-\mu-1/2} (h_\mu \phi)(y) \right| dy.$$

Since by property (c), $x \leq (1+x) \leq e^{-a/b} e^{\omega(x)/b}$, using inequality (2.6) the above expression can be estimated by

$$c(\sigma, m) e^{-(\mu+1/2)a/b} Q_\mu \int_0^\infty \sum_{j=0}^n \sum_{r=0}^{n+j} |b_j| \times \binom{n+j}{r} e^{-2aj/b} y^{2(n+j-r)} C^r r! e^{-r\delta a/b}$$

$$\begin{aligned}
 & \times e^{2n\delta\alpha(y)/b} e^{(\sigma-m)\alpha(y)} e^{m\alpha(y)} \left| y^{-\mu-1/2} (h_\mu \phi)(y) \right| dy \\
 & \leq c(\sigma, m) e^{-(\mu+1/2\gamma b)} Q_\mu \int_0^\infty \sum_{j=0}^n \sum_{r=0}^{n+j} |b_j| \\
 & \times \binom{n+j}{r} e^{-2aj/b} e^{-r\delta\alpha/b} C^r r! \beta_{m+2n\delta/b, 0}^\mu(\phi) \\
 & \times \int_0^\infty e^{(\sigma-m)\alpha(y)} y^{2(n+j-r)} dy < \infty, \quad \dots (3.6)
 \end{aligned}$$

choosing $m > \sigma$. Therefore,

$$A_{m, n}^\mu (h_{\mu, a} \phi) < \infty.$$

(ii) Let $\Phi(x) = (h_{\mu, a} \phi)(x)$, then as in the proof of Theorem 2.2 (ii), using inequality (2.11), we have

$$\begin{aligned}
 & e^{m\alpha(x)} \left| S_\mu^n (h_\mu \Phi)(x) \right| \leq e^{mc(\epsilon)} \sum_{v=0}^\infty \sum_{j=0}^n |b_j| \\
 & \times \frac{(m\epsilon)^v}{v!} x^{2j+v+\mu+1/2} \left| (x^{-1} D)^{n+j} x^{-\mu-1/2} (h_\mu \Phi)(x) \right| \quad \dots (3.7)
 \end{aligned}$$

Assume that $0 \leq \mu + 1/2 \leq p$, where p is positive integer then

$$x^{\mu+1/2} \leq (1+x)^{\mu+1/2} \leq (1+x)^p,$$

and the right-hand side of the eq. (3.7) is bounded by

$$\begin{aligned}
 & e^{mc(\epsilon)} \sum_{v=0}^\infty \sum_{j=0}^n |b_j| \frac{(m\epsilon)^v}{v!} (1+x)^{2j+v+p} \left| (x^{-1} D)^{n+j} x^{-\mu-1/2} (h_\mu \Phi)(x) \right| \\
 & \leq e^{mc(\epsilon)} \sum_{v=0}^\infty \sum_{j=0}^n |b_j| \frac{(m\epsilon)^v}{v!} \sum_{k=0}^{2j+v+p} \binom{2j+v+p}{k} x^k \\
 & \left| (x^{-1} D)^{n+j} x^{-\mu-1/2} (h_\mu \Phi)(x) \right|.
 \end{aligned}$$

Using Zemanian's technique and eq. (1.3) and the Leibnitz type formula (2.7), the last expression can be estimated by

$$e^{mc(\epsilon)} \sum_{v=0}^\infty \sum_{j=0}^n |b_j| \frac{(m\epsilon)^v}{v!} \sum_{k=0}^{2j+v+p} \binom{2j+v+p}{k}$$

$$\begin{aligned}
 & \times \int_0^\infty y^{2\mu+2(n+j)+k+1} \left| (y^{-1} D)^k \left[y^{-\mu-1/2} \Phi(y) \right] \right. \\
 & \times (xy)^{-(\mu+n+j)} J_{\mu+k+(n+j)}(xy) \left. \right| dy \\
 & \leq e^{mc(\epsilon)} \sum_{v=0}^\infty \sum_{j=0}^n |b_j| \sum_{k=0}^{2j+v+p} \frac{(m\epsilon)^v}{v!} \binom{2j+v+p}{k} Q_\mu \int_0^\infty y^{2\mu+2(n+j)+k+1} \\
 & \times \left| (y^{-1} D)^k \left[y^{-\mu-1/2} \int_0^\infty (yt)^{1/2} J_\mu(yt) a(y, t) (h_\mu \phi)(t) dt \right] \right| dy \\
 & \leq e^{mc(\epsilon)} Q_\mu^2 \sum_{v=0}^\infty \sum_{j=0}^n \sum_{k=0}^{2j+v+p} |b_j| \frac{(m\epsilon)^v}{v!} \binom{2j+v+p}{k} \int_0^\infty \int_0^\infty y^{2\mu+2(n+j)+k+1} \\
 & \times \sum_{s=0}^k \binom{k}{s} t^{2(k-s)} \left| (y^{-1} D)^s \left[t^{2\mu+1} a(y, t) \right] \right| \left| t^{-\mu-1/2} (h_\mu \phi)(t) \right| dt dy.
 \end{aligned}$$

Suppose that N is a positive integer no less than $2\mu + 6n + p + 1$, then the above expression can be estimated by

$$\begin{aligned}
 & e^{mc(\epsilon)} Q_\mu^2 \sum_{v=0}^\infty \sum_{j=0}^n \sum_{k=0}^{2j+v+p} |b_j| \frac{(m\epsilon)^v}{v!} \\
 & \times \binom{2j+v+p}{k} \int_0^\infty \int_0^\infty \sum_{s=0}^k \binom{k}{s} t^{2(k-s)} (1+y)^{N+v+2} \left| (y^{-1} D)^s \left[t^{2\mu+1} a(y, t) \right] \right| \\
 & \times \left| t^{-\mu-1/2} (h_\mu \phi)(t) \right| dt \frac{dy}{(1+y)^2}.
 \end{aligned}$$

Using the inequality $(1+y) \leq e^{-a/b} e^{\omega(y)/b}$ by property (c) and also inequality (2.6) the right-hand of the above expression can be bounded by

$$\begin{aligned}
 & e^{mc(\epsilon)} Q_\mu^2 \sum_{v=0}^\infty \sum_{j=0}^n \sum_{k=0}^{2j+v+p} \sum_{s=0}^k |b_j| \frac{(m\epsilon)^v}{v!} \binom{2j+v+p}{k} \\
 & \times \binom{k}{s} t^{2(k-s)} \int_0^\infty \exp(-(N+v+2)a/b) \\
 & \times c(\sigma, (N+v+2)/b) C^s s! (1+t)^{s+2} e^{\sigma\omega(t)} \left| t^{-\mu-1/2} (h_\mu \phi)(t) \right| \frac{dt}{(1+t)^2}
 \end{aligned}$$

$$\begin{aligned} &\leq e^{mc(\epsilon)} Q_\mu^2 \sum_{v=0}^\infty \sum_{j=0}^n \sum_{k=0}^{2j+v+p} \sum_{s=0}^k |b_j| \frac{(m\epsilon)^v}{v!} \\ &\times \binom{2j+v+p}{k} \binom{k}{s} \int_0^\infty \exp(-(N+v+2)a/b) \\ &\times c(\sigma, (N+v+2)a/b) C^s s!(1+t)^{2k+2} e^{\sigma\alpha t} \left| t^{-\mu-1/2} (h_\mu \phi)(t) \right| \frac{dt}{(1+t)^2}. \end{aligned}$$

Using again the property $(1+x) \leq e^{-a/b} e^{\alpha x/b}$ and writing $d(\sigma, v)$ for $c(\sigma, (N+v+2)/b)$, we can estimate the last expression by

$$\begin{aligned} &e^{mc(\epsilon)} Q_\mu^2 \exp[-a(N+2)/b] \sum_{v=0}^\infty \frac{(m\epsilon)^v}{v!} e^{-av/b} \sum_{j=0}^n |b_j| \sum_{k=0}^{2j+v+p} \binom{2j+v+p}{k} \\ &\times \sup_{k \in [0, 2+4n+2v+2p]} \beta_{\sigma+(2k+2/b), 0}^\mu (h_\mu \phi) d(\sigma, v) \sum_{s=0}^k \binom{k}{s} C^s s! \\ &\leq e^{mc(\epsilon)} Q_\mu^2 \exp[-a(N+2)/b] \sum_{v=0}^\infty \frac{(m\epsilon)^v}{v!} \\ &\times e^{-av/b} \sum_{j=0}^n |b_j| Q_{n, v, p}^\mu \sum_{k=0}^{2j+v+p} \binom{2j+v+p}{k} (1+C)^k k! \\ &\leq e^{mc(\epsilon)} Q_\mu^2 \exp[-a(N+2)/b] \\ &\sum_{v=0}^\infty \left[m\epsilon e^{-a/b} (Q_{n, v, p}^\mu)^{1/v} \right]^v \sum_{j=0}^n |b_j| (2+c)^{2j+v+p} \frac{(2j+v+p)!}{v!}, \quad \dots (3.8) \end{aligned}$$

where $Q_{n, v, p}^\mu = d(\sigma, v) \sup_{k \in [0, 2+4n+2v+2p]} \beta_{\sigma+(2k+2)/b, 0}^\mu (h_\mu \phi)$.

Setting $T = 2 + c$ and using the result $\Gamma(v + \alpha)/\Gamma(v + \beta) = O(v^{\alpha-\beta})$, $v \rightarrow \infty$, the expression (3.8) can also be estimated by

$$\begin{aligned} &Ae^{mc(\epsilon)} Q_\mu^2 \exp[-a(N+2)/b] \sum_{v=0}^\infty e^{(2n+p) \log v} \\ &\times \exp \left[v \log \left(m\epsilon T \left(Q_{n, v, p}^\mu \right)^{1/v} e^{-a/b} \right) \right] \sum_{j=0}^n |b_j| T^{2j+p} \end{aligned}$$

for some positive constant A .

Since $\log v < v$ we can further estimate it as follows:

$$\begin{aligned}
 & e^{mc(\varepsilon)} Q_{\mu}^2 \exp[-a(N+2)/b] \\
 & \times \sum_{v=0}^{\infty} \exp \left[v \left\{ \log \left(m\varepsilon T \left(Q_{n, v, p}^{\mu} \right)^{1/v} - a/b + 2n + p \right) \right\} \right] \\
 & \times \sum_{j=0}^n |b_j| T^{2j+p} < \infty
 \end{aligned}$$

as the infinite series can be made convergent by choosing

$$\varepsilon < (mT)^{-1} (Q_{n, v, p}^{\mu})^{-1/v} \exp(-(2n + p - a/b)).$$

This completes the proof of the theorem.

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