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TRIANGULAR NUMBERS IN THE GENERALIZED ASSOCIATED PELL SEQUENCE

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1. INTRODUCTION

It is well known that a positive integer n is called a triangular number if it is of the form $m(m+1)/2$ where m is an integer greater than zero.

The Pell sequence $\{P_n\}$ defined by

$$P_0 = 0,\ P_1 = 1 \text{ and } P_{n+2} = 2P_{n+1} + P_n \text{ for } n \geq 0. \qquad \ldots(1.1)$$

The Associated Pell sequence $\{Q_n\}$ defined by the recursive relation

$$Q_0 = Q_1 = 1 \text{ and } Q_{n+2} = 2Q_{n+1} + Q_n \text{ for } n \geq 0. \qquad \ldots (1.2)$$

Now we define for a fixed integer $\alpha > 0$, a new sequence called Generalized Pell sequence $\{P_n^{(\alpha)}\}$, defined by

$$P_0^{(\alpha)} = 0,\ P_1^{(\alpha)} = 1 \text{ and } P_{n+2}^{(\alpha)} = (\alpha+1)\, P_{n+1}^{(\alpha)} + \frac{\alpha(\alpha+1)}{2} P_n^{(\alpha)} \text{ for } n \geq 0 \qquad \ldots (1.3)$$

and Generalized Associated Pell sequence $\{Q_n^{(\alpha)}\}$, defined by the recursive relation

$$Q_0^{(\alpha)} = Q_1^{(\alpha)} = 1 \text{ and } Q_{n+2}^{(\alpha)} = (\alpha+1)\, Q_{n+1}^{(\alpha)} + \frac{\alpha(\alpha+1)}{2} Q_n^{(\alpha)} \text{ for } n \geq 0. \qquad \ldots (1.4)$$

Note that $P_n^{(1)} = P_n$ and $Q_n^{(1)} = Q_n$ for $n = 0, 1, 2, 3, \ldots$

In this paper, we prove that at $n = 0$, 1 or 2 there exists triangular numbers in the sequence $\{Q_n^{(\alpha)}\}$.

In section 2 we present some preliminary results on sequence $\{Q_n\}$ defined in (1.2) which are needed for our purpose. Since an integer N is triangular if and only if $8N + 1$ is a perfect square, to find the triangular numbers in the Generalized Associated Pell sequence $\{Q_n^{(\alpha)}\}$ one has to identify those n for which $8Q_n^{(\alpha)} + 1$ is a perfect square. This is shown in section 3.

The Pell-Lucas sequence $\{q_n\}$ defined by

$$q_0 = q_1 = 2 \text{ and } q_{n+2} + 2q_{n+1} + q_n \text{ for } n \geq 0. \qquad \text{... (1.5)}$$

In section 4 we identify the pronic numbers (that is, the number is of the form $m(m+1)$ where m is an integer) in the Generalized Pell-Lucas sequence $\{q_n^{(\alpha)}\}$ defined by

$$q_0^{(\alpha)} = q_1^{(\alpha)} = 2 \text{ and } q_{n+2}^{(\alpha)} = (\alpha+1)q_{n+1}^{(\alpha)} + \frac{\alpha(\alpha+1)}{2} q_0^{(\alpha)} \text{ for } n \geq 0.$$

Where α is a fixed integer greater than zero.

Note that $q_n^{(1)} = q_n$ for n = 0, 1, 2, 3, ...

2. Preliminary Results

For the first few values of n, the terms of Generalized Pell sequence $\{P_n^{(\alpha)}\}$ and Generalized Associated Pell sequence $\{Q_n^{(\alpha)}\}$ are given below:

TABLE 2.1

N	$P_n^{(\alpha)}$	$Q_n^{(\alpha)}$
0	0	1
1	1	1
2	$\alpha+1$	$(\alpha^2+3\alpha+2)/2$
3	$(3\alpha^2+5\alpha+2)/2$	$(\alpha^3+5\alpha^2+6\alpha+2)/2$
4	$(4\alpha^3+10\alpha^2+8\alpha+2)/2$	$(3\alpha^4+16\alpha^3+27\alpha^2+18\alpha+4)/4$
5	$(11\alpha^4+36\alpha^3+43\alpha^2+22\alpha+4)/4$	$(4\alpha^5+25\alpha^4+54\alpha^3+53\alpha^2+24\alpha+4)/4$

For certain values of α, the values of $Q_n^{(\alpha)}$ and $8Q_n^{(\alpha)}$ + 1 is given in the following table:

TABLE 2.2

α	1	2	3	4	5	6	7	8	9	10
$Q_0^{(\alpha)}$	1	1	1	1	1	1	1	1	1	1
$Q_1^{(\alpha)}$	1	1	1	1	1	1	1	1	1	1
$Q_2^{(\alpha)}$	3	6	10	15	21	28	36	45	55	66
$Q_3^{(\alpha)}$	7	21	46	85	141	217	316	441	595	781
$Q_4^{(\alpha)}$	17	81	244	575	1161	2107	3536	5589	8425	12221
$Q_5^{(\alpha)}$	41	306	1252	3725	9081	19306	37136	66177	111025	177386
$8Q_0^{(\alpha)}+1$	9	9	9	9	9	9	9	9	9	9

$8Q_1^{(\alpha)}+1$	9	9	9	9	9	9	9	9	9	9
$8Q_2^{(\alpha)}+1$	25	49	81	121	169	225	289	361	441	529
$8Q_3^{(\alpha)}+1$	57	169	369	681	1129	1737	2529	3529	4761	6249
$8Q_4^{(\alpha)}+1$	137	649	1953	4601	9289	16857	28289	44713	67401	97769
$8Q_5^{(\alpha)}+1$	329	2449	10017	29801	72649	154449	297089	529417	888201	1419089

The following properties of the sequences $\{P_n\}$ and $\{Q_n\}$ given in (1.1) and (1.2) are well known :

$$P_{-n} = (-1)^{n+1} P_n \text{ and } Q_{-n} = (-1)^n Q_n \qquad \text{... (2.3)}$$

$$P_n = \frac{a^n - b^n}{2\sqrt{2}} \text{ and } Q_n = \frac{a^0 + b^n}{2} \qquad \text{... (2.4)}$$

where $a = 1 + \sqrt{2}$ and $b = 1 - \sqrt{2}$

$$Q_n^2 = 2P_n^2 + (-1)^n \qquad \text{... (2.5)}$$

$$Q_{2n} = 2Q_n^2 - (-1)^n \qquad \text{... (2.6)}$$

As a direct consequence of (2.4) we have

$$Q_{m+n} = 2Q_m Q_n - (-1)^n Q_{m-n} \text{ for all integers } m \text{ and } n. \qquad \text{... (2.7)}$$

Now we present the proof of the Lemma, which we use at the later stage.

Lemma 2.8 : If m is even and n, k are any integers then

$$Q_{n+2km} \equiv (-1)^k Q_n \pmod{Q_m}$$

PROOF : For $k = 0$, the lemma is trivial. We prove this lemma for $k > 0$ by using induction on k. By (2.7),

$$Q_{n+2m} = 2Q_{n+m} Q_m - (-1)^m Q_n.$$

Because m is even, this gives the lemma for $k = 1$.

Assume that the lemma holds all integers $\leq k$. Again by (2.7) and the induction hypothesis, we get

$$Q_{n+2(k+1)m} = 2Q_{n+2km} Q_{2m} - Q_{n+2(k-1)m} \qquad \text{... (2.9)}$$

$$\equiv 2(-1)^k Q_n Q_{2m} - (-1)^{k-1} Q_n \pmod{Q_m}$$

$$\equiv (-1)^k (2Q_{2m} - 1)\, Q_n \pmod{Q_m}.$$

But since m is even it follows from (2.6) that

$$2Q_{2m} + 1 \equiv -1 \pmod{Q_m} \qquad \ldots (2.10)$$

Now (2.9) and (2.10) together prove the lemma for $k + 1$.

Hence by induction the lemma holds for $k > 0$.

If $k < 0$, say $k = -r$ where $r > 0$, we have by (2.7) and (2.3), that

$$Q_{n+2km} = Q_{n-2rm} = 2Q_n Q_{-2rm} - (-1)^{-2rm} Q_{n+2rm}$$

$$= 2Q_n Q_{2rm} - Q_{n+2rm}$$

$$\equiv 2Q_n(-1)^r - (-1)^r Q_n \pmod{Q_m}$$

$$\equiv (-1)^r Q_n \pmod{Q_m}$$

$$\equiv (-1)^k Q_n \pmod{Q_m}.$$

Which completes the proof of the Lemma.

3. Triangular Numbers in the Generalized Associated Pell Sequence $\{Q_n^{(\alpha)}\}$

In this section first we present those n for which $8Q_n^{(1)} + 1$ is a perfect square. i.e., $8Q_n^{(1)} + 1$ is a perfect square only when $n = 0$, 1 or 2. So, $Q_0^{(1)}, Q_1^{(1)}, Q_2^{(1)}$ are the only triangular numbers in $Q_n^{(1)}$.

Next we prove that $Q_0^{(\alpha)}, Q_1^{(\alpha)}$ and $Q_2^{(\alpha)}$ are the triangular numbers in the Generalized Associated Pell sequence $\{Q_n^{(\alpha)}\}$ by using mathematical induction on α. This is shown in Theorem 3.15.

In the course of the proofs of the results to follow we present the period k of the sequence $\{Q_t^{(1)}\}$ modulo certain integer $M > 0$. That is for all integers $u \geq 0$, $Q_{t+ku}^{(1)} \equiv Q_t^{(1)} \pmod{M}$. Also if modulo M, the sequence $\{Q_t^{(1)}\}$ has period k, we have R_t and U_t for $t = 0, 1, 2, \ldots, k - 1$ where $Q_t^{(1)} \equiv R_t \pmod{M}$ and $8Q_t^{(1)} + 1 \equiv U_t \pmod{M}$. For certain values of $M > 0$, the period of k of $\{Q_t^{(1)}\}$, the numbers R_t ($t = 0, 1, 2, \ldots, k - 1$) and the numbers U_t(t 0, 1, 2, ..., $k - 1$) are given in Table 3.1.

For example Table (3.1) shows that modulo $M = 7$, the sequence $\{Q_t^{(1)}\}$ has period 6 and $R_t(t = 0, 1, 2, ..., k-1)$ are respectively $R_0 = 1$, $R_1 = 1$, $R_2 = 3$, $R_3 = 0$, $R_4 = 3$ and $R_5 = 6$ (given in coloumn III), and the respective U_t are given in column IV.

TABLE 3.1

I Mod M	II Period K	III R_t $(t = 0, 1, 2, ..., k-1)$ where $Q_t^{(1)} \equiv R_t \pmod M$	IV U_t $(y = 0, 1, 2, ..., k-1)$ where $8Q_t^{(1)} + 1 \equiv U_t \pmod M$
7	6	1, 1, 3, 0, 3, 6	2, 2, 4, 1, 4, 0
9	24	1, 1, 3, 7, 5, 0, 5, 1, 7, 6, 1, 8, 8, 6, 2, 1, 4, 4, 8, 2, 3, 8	0, 0, 7, 3, 2, 5, 1, 5, 0, 3, 4, 0, 2, 2, 4, 8, 0, 6, 1, 6, 2, 8, 7, 2
10	12	1, 1, 3, 7, 7, 1, 9, 9, 7, 3, 3, 9	9, 9, 5, 7, 7, 9, 3, 3, 7, 5, 5, 3
11	24	1, 1, 3, 7, 6, 8, 0, 8, 5, 7, 8, 1, 10, 10, 8, 4, 5, 3, 0, 3, 6, 4, 3, 10	9, 9, 3, 2, 5, 10, 1, 10, 8, 2, 10, 9, 4, 4, 10, 0, 8, 3, 1, 3, 5, 0, 3, 4
23	22	1, 1, 3, 7, 17, 18, 7, 9, 2, 13, 5, 0, 5, 10, 2, 14, 7, 5, 17, 16, 3, 22	9, 9, 2, 11, 22, 7, 11, 4, 17, 13, 18, 1, 18, 12, 17, 21, 11, 18, 22, 14, 2, 16
73	72	1, 1, 3, 7, 17, 41, 26, 20, 66, 6, 5, 16, 37, 17, 71, 13, 24, 61, 0, 61, 49, 13, 2, 17, 36, 16, 68, 6, 7, 20, 47, 41, 56, 7, 70, 1, 72, 72, 70, 66, 56, 32, 47, 53, 7, 67, 68, 57, 36, 56, 2, 60, 49, 12, 0, 12, 24, 60, 71, 56, 37, 57, 5, 67, 66, 53, 26, 32, 17, 66, 3, 72	9, 9, 25, 57, 64, 37, 63, 15, 18, 49, 41, 56, 5, 64, 58, 32, 47, 51, 1, 51, 28, 32, 17, 64, 70, 56, 34, 49, 57, 15, 12, 37, 11, 57, 50, 9, 66, 66, 50, 18, 11, 38, 12, 60, 57, 26, 34, 19, 70, 11, 17, 43, 28, 24, 1, 24, 47, 43, 58, 11, 5, 19, 41, 26, 18, 60, 63, 38, 64, 18, 25, 66
197	36	1, 1, 3, 7, 17, 41, 99, 42, 183, 14, 14, 42, 98, 41, 180, 7, 194, 1, 196, 196, 194, 190, 180, 156, 198, 155, 14, 183, 185, 155, 99, 156, 17, 190, 3, 196	9, 9, 25, 57, 137, 132, 5, 140, 86, 113, 113, 140, 194, 132, 62, 57, 174, 9, 190, 190, 174, 142, 62, 67, 194, 59, 113, 86, 86, 59, 5, 67, 137, 142, 25, 190
199	18	1, 1, 3, 7, 17, 41, 99, 40, 179, 0, 179, 159, 99, 158, 17, 192, 3, 198	9, 9, 25, 57, 137, 130, 196, 122, 40, 1, 40,78, 196, 71, 137, 144, 25, 192

Now we need the following Lemmas:

Lemma 3.2 : Suppose $n \equiv 0$ or $1 \pmod 4$. Then $8Q_n^{(1)} + 1$ is a perfect square if and only if $n = 0$ or 1.

PROOF : We know that $Q_n^{(1)} = Q_n$.

If $n = 0$ or 1, then $8Q_n + 1 = 3^2$, by Table (2.2).

Conversely, suppose $n \equiv 0$ or 1 (mod 4) and $n \notin \{0, 1\}$. Then n can be written as $n = 2km + \varepsilon$, $m = 2^r$, $r \geq 1$, k is odd and $\varepsilon = 0$ or 1. Therefore, by (2.8) and Table (2.2), we get

$$Q_n = 2_{2km+\varepsilon} \equiv (-1)^k Q_\varepsilon \pmod{Q_m} \equiv -1 \pmod{Q_m}$$

so that $8Q_n + 1 \equiv -7 \pmod{Q_m}$. Hence the Jacobi symbol

$$\left(\frac{8Q_n+1}{Q_m}\right) = \left(\frac{-7}{Q_m}\right) = \left(\frac{-1}{Q_m}\right) = \left(\frac{7}{Q_m}\right) \quad \text{... (3.3)}$$

$$= \left(\frac{-1}{Q_m}\right) \cdot \left(\frac{Q_m}{7}\right) \cdot \left(\frac{-1}{Q_m}\right) = \left(\frac{Q_m}{7}\right).$$

Now we first note that

modulo 7, the sequence $\{Q_n\}$ is periodic with period 6. In fact, by (2.7) and Table (2.2), we have ... (3.4)

$$Q_{n+6} = 2Q_3 \cdot Q_{n+3} + Q_n = 2(7)Q_{n+3} + Q_n \equiv Q_n \pmod 7.$$

Also, since $m = 2^r \equiv \pm 2 \pmod 6$, we have by (3, 4), (2, 3) and Table (3.1), that $Q_m \equiv Q_{\pm 2} \pmod 7 \equiv Q_2 \pmod 7 \equiv 3 \pmod 7$.

Therefore

$$\left(\frac{Q_m}{7}\right) = \left(\frac{3}{7}\right) = \left(\frac{-4}{7}\right) = \left(\frac{-1}{7}\right) \cdot \left(\frac{4}{7}\right) = -1. \quad \text{... (3.5)}$$

Now (3.3) and (3.5) give $\left(\frac{8Q_{n+1}}{Q_m}\right) = -1$, providng that $8Q_n + 1$ cannot be a square.

Which completes the proof of the Lemma.

Lemma 3.6 — Suppose $n \equiv \pm 2 \pmod{36}$. Then $8Q_n^{(1)}+1$ is a perfect square if and only if $n = \pm 2$.

PROOF : We know that $Q_n^{(1)} = Q_n$.

If $n = \pm 2$ then $8Q_n + 1 = 5^2$, by (2.3) and Table (2.2).

Conversely, suppose $n \equiv \pm 2 \pmod{36}$ and $n \notin \{-2, 2\}$. Then n can be written as $n = 2 \cdot 3^2 \cdot 2^r \cdot g \pm 2$ where $r \geq 1$ and g is odd.

Write

$$m = \begin{cases} 3^2 . 2^r \text{ if } r \equiv 3 \pmod{10} \\ 3 . 2^r \text{ if } r \equiv 1 \text{ or } 6 \pmod{10} \\ 2^r \text{ otherwise} \end{cases} \quad \text{.. (3.7)}$$

so that $n = 2km \pm 2$, where k is odd (in fact, $k = g$, $3g$ or $3^2 g$). Also, since $2^{t+10} \equiv 2^t \pmod{22}$ for $t \geq 1$, it follows that m, defined in (3.7), is such that

$$m \equiv \pm 4, \pm 6, \pm 10 \pmod{22}. \qquad \ldots (3.8)$$

For instance, if $r \equiv 6 \pmod{10}$ then $r = 10u + 6$ for some integer u and in this case, by (3.7), $m = 3 \cdot 2^r = 3 \cdot 2^{10u+6} \equiv 3 \cdot 2^6 \pmod{22} \equiv 6 \pmod{22}$.

Now, by Lemma 2.8, (2.3) and Table (2.2) we have

$$Q_n = Q_{2km \pm 2} \equiv (-1)^k Q_2 \pmod{Q_m} \equiv -3 \pmod{Q_m}$$

so that $8Q_n + 1 \equiv -23 \pmod{Q_m}$. Therefore

$$\left(\frac{8Q_n + 1}{Q_m}\right) = \left(\frac{-23}{Q_m}\right) = \left(\frac{-1}{Q_m}\right) \cdot \left(\frac{23}{Q_m}\right) \qquad \ldots (3.9)$$

$$= \left(\frac{-1}{Q_m}\right) \cdot \left(\frac{Q_m}{23}\right) \cdot \left(\frac{-1}{Q_m}\right) = \left(\frac{Q_m}{23}\right).$$

Note that modulo 23, the sequence $\{Q_j\}$ is periodic with period 22. That is

$$Q_{j+22i} \equiv Q_j \pmod{23} \text{ for all integers } i \geq 0. \qquad \ldots (3.10)$$

Now (3.8), (3.9) and (2.3) imply that $Q_m \equiv Q_4, Q_6$ or $Q_{10} \pmod{23}$. That is $Q_m \equiv 17$, 7 or 5 (mod 23), by Table (3.1).

Therefore (3.9) gives

$$\left(\frac{8Q_n + 1}{Q_m}\right) = \left(\frac{17}{23}\right), \left(\frac{7}{23}\right) \text{ or } \left(\frac{5}{23}\right) \text{ showing } \left(\frac{8Q_n + 1}{Q_m}\right) = -1$$

in any case, since $\left(\frac{17}{23}\right) = \left(\frac{7}{23}\right) = \left(\frac{5}{23}\right) = -1$

For example,

$$\left(\frac{17}{23}\right) = \left(\frac{23}{17}\right) \cdot (-1)^{\frac{23-1}{2} \cdot \frac{17-1}{2}} = \left(\frac{6}{17}\right) = \left(\frac{2}{17}\right) \cdot \left(\frac{3}{17}\right)$$

$$= (-1)^{\frac{17^2-1}{8}} \cdot \left(\frac{3}{17}\right)$$

$$= (+1) \cdot \left(\frac{17}{3}\right) \cdot (-1)^{\frac{17-}{2} \cdot \frac{3-1}{2}} = \left(\frac{2}{3}\right) = (-1)^{\frac{3^2-1}{8}} = -1.$$

Which completes the proof of the Lemma.

An immediate consequence of Lemma 3.2 and Lemma 3.6 is the following :

Lemma 3.11 — Suppose $n \equiv 0, 1, \pm 2 \pmod{72}$. Then $8Q_n^{(1)} + 1$ is a perfect square if and only if $n = 0, 1, \pm 2$.

PROOF : We know that $Q_n^{(1)} = Q_n$.

If $n \equiv 0$ or 1 (mod 72), then $n \equiv 0$ or 1 (mod 4), while when $n \equiv \pm 2$ (mod 72), we have $n \equiv \pm 2$ (mod 36). Now the Lemma follows from Lemma 3.2 and Lemma 3.6.

Lemma 3.12 — $8Q_n^{(1)}+1$ is not a perfect square if $n \not\equiv 0, 1, \pm 2$ (mod 72).

PROOF : We know that $Q_n^{(1)}=Q_n$.

We prove this lemma in different steps eliminating at each stage certain integers n modulo 72 for which $8Q_n + 1$ is not a square. In each step we choose an integer m such that the period k (of the sequence $\{Q_n\}$ mod m) is a divisor of 72 and there by eliminating certain residue class modulo k.

Step I : Note that modulo 10, the sequence $\{Q_n\}$ is periodic with period 12. That is, $Q_{n+12u} \equiv Q_n$ (mod 10) for all integers $u \geq 0$. Therefore, if $n \equiv 3$, 4, 6, 7, 8 or 11 (mod 12) then we respectively have $8Q_n + 1 \equiv 8Q_3 + 1$, $8Q_4 + 1$, $8Q_6 + 1$, $8Q_7 + 1$, $8Q_8 + 1$ or $8Q_{11} + 1$ (mod 10) so that by Table (3.1), $8Q_n+1 \equiv 3$ or 7 (mod 10) for these values of n, showing $8Q_n + 1$ is not a square, since $m^2 \equiv 0$, 1, 4, 5, 6 or 9 (mod 10) for any integer $m \geq 1$. Therefore, for the sequence in the form $8Q_n + 1$ we have to search those n for which $n \equiv 0$, 1, 2, 5, 9, or 10 (mod 12) or equivalently among $n \equiv 0$, 1, 2, 5, 9, 10, 12, 13, 14, 17, 21 or 22 (mod 24).

Step II : Modulo 9, the sequence $\{Q_n\}$ is periodic with period 24 (that is, $Q_{n+24u} \equiv Q_n$ (mod 9) for all integers $u \geq 0$) so that when $n \equiv 5$, 9, 12, 13, 17 or 21 (mod 24) we respectively have $Q_n \equiv Q_5, Q_9, Q_{12}, Q_{13}, Q_{17}$ or Q_{21} (mod 9) and therefore, in view of Table (3.1), $8Q_n + 1 \equiv 2$, 3, 5, 6 or 8 (mod 9), showing $8Q_n+1$ is not a square, since $m^2 \equiv 0$, 1, 4, or 7 (mod 9) for any integer m or ≥ 1.

Thus, there remain $n \equiv 0$, 1, 2, 10, 14 or 22 (mod 24).

Step III : Modulo 11, also the sequence $\{Q_n\}$ is periodic with period 24, so that for $n \equiv 10$ or 14 (mod 24) we have $Q_n \equiv Q_{10}$ or Q_{14} (mod 11), showing $8Q_n + 1 \equiv 2$ or 10 (mod 11), by Table (3.1). Therefore $8Q_n + 1$ is not a square if $n \equiv 10$, or 14 (mod 24), since 2 and 10 are quadratic nonresidues modulo 11.

Thus there remain $n \equiv 0$, 1, 2, or 22 (mod 24) or equivalently, $n \equiv 0$, 1, 2, 22, 24, 25, 26, 46, 48, 49, 50 or 70 (mod 72).

Step IV : Modulo 199, the sequence $\{Q_n\}$ has period 18, so that if $n \equiv 4$, 11, 13, 14 or 17 (mod 18) then by Table (3.1), we respectively have $8Q_n + 1 \equiv 137$, 78, 71, 37, 192(mod 199) giving $8Q_n +1$ is not a square, since 71, 78, 137 and 192 are quadratic nonresidues modulo 199.

Hence we eliminate $n \equiv 22$, 49 and 50 (mod 72).

Step V: Modulo 197, the sequence $\{Q_n\}$ has period 36, if $n \equiv \pm 10, \pm 12 \pmod{36}$ then by Table (3.1), we respectively have $8Q_n + 1 \equiv 113$ or $194 \pmod{197}$, showing these n can be eliminated. Thus we can eliminate $n \equiv 24, 26, 46$ and $48 \pmod{72}$.

Step VI : Modulo 73 the sequence $\{Q_n\}$ is periodic with period 72. Therefore, if $n \equiv 25 \pmod{72}$, then $8Q_n + 1 \equiv 56 \pmod{73}$ by Table (3.1), showing $8Q_n + 1$ is not a square, since

$$\left(\frac{56}{73}\right) = -1$$

Finally, there remain $n \equiv 0, 1, 2$ or $70 \pmod{72}$.

***Theorem* 3.13** — *$Q_n^{(1)}$ is triangular number if and only if $n = 0, 1, \pm 2$*

PROOF : From Lemma 3.11 and Lemma 3.12 the theorem follows :

Observations 3.14 — (*i*) From Table 2.1 and Theorem 3.13 we observe that $Q_n^{(2)}$ is triangular if $n = 0, 1, 2,$ or 3.

(*ii*) Similarly, we observe that $Q_n^{(3)}$ is triangular if $n = 0, 1$ or 2; $Q_n^{(4)}$ is triangular if $n = 0, 1$ or 2 and so on.

***Theorem* 3.15** — *$Q_n^{(\alpha)}$ is triangular number if $n = 0, 1$ or 2.*

PROOF : We prove the theorem by the principle of mathematical induction on α. i.e., we have to prove $8Q_n^{(\alpha)} + 1$ is a perfect square if $n = 0, 1$ or 2 for all integers $\alpha > 0$.

We have

$$Q_0^{(\alpha)} = 1,$$

$$Q_1^{(\alpha)} = 1,$$

and
$$Q_2^{(\alpha)} = \frac{\alpha^2 + 3\alpha + 2}{2}$$

Put $\alpha = 1$.

From the Theorem 3.13, $8Q_n^{(1)} + 1$ is a perfect square if $n = 0, 1$ or 2.

Therefore, the result is true for $\alpha = 1$.

Assume that it is true for $\alpha = m$.

Note that $8Q_0^{(\alpha)} + 1$ and $8Q_1^{(\alpha)} + 1$ are clearly perfect squares for all integers $\alpha > 0$.

Now we prove that $8Q_2^{(\alpha)} + 1$ is also perfect square.

Let
$$f(m) = 8Q_2^{(m)} + 1 = \frac{8(m^2 + 3m + 2)}{2} + 1$$

then
$$f(m) = 4m^2 + 12m + 9$$

$= (2m + 3)^2$ which is a perfect square.

Now $f(m+1) = [2(m + 1) + 3]^2$

$= (2m + 5)^2$ which is a perfect square.

Therefore, the result is true for $\alpha = m + 1$.

Therefore, by the principle of the mathematical induction $8Q_n^{(\alpha)} + 1$ is a perfect square if $n = 0, 1$ or 2.

i.e., $Q_n^{(\alpha)}$ is triangular number if $n = 0, 1$ or 2.

Which completes the proof of the theorem.

4. Pronic Numbers in the Generalized Pell-Lucas Sequence

A pronic number is one which is the product of two consecutive integers. That is, a number is of the form $m(m + 1)$ where m is an integer.

the Pell-Lucas sequence $\{q_n\}$ defined in (1.5).

Now we prove that at $n = 0, 1$ or 2 there exists pronic numbers in the Generalized Pell-Lucas sequence $\{q_n^{(\alpha)}\}$ defined in (1.6).

The first few terms of the Generalized Pell-Lucas sequence $\{q_n^{(\alpha)}\}$ are given in Table 4.1

TABLE 4.1

α	1	2	3	4	5	6	7	8	9	10
$q_0^{(\alpha)}$	2	2	2	2	2	2	2	2	2	2
$q_1^{(\alpha)}$	2	2	2	2	2	2	2	2	2	2
$q_2^{(\alpha)}$	6	12	20	30	42	56	72	90	110	132
$q_3^{(\alpha)}$	14	42	92	170	282	434	632	882	1190	1562
$q_4^{(\alpha)}$	34	162	488	1150	2322	4214	7072	11178	16850	24442
$q_5^{(\alpha)}$	82	612	2504	7450	18162	38612	74272	132354	222050	354772

The explicit Binet form of $q_n^{(1)}$ defined in (1.6) is

$$q_n^{(1)} = a^n + b^n \quad \text{where} \quad a = 1 + \sqrt{2} \quad \text{and} \quad b = 1 - \sqrt{2} \qquad \text{... (4.2)}$$

It follows from (2.4) and (4.2) that

$$q_n^{(1)} = 2q_n^{(1)} \quad \text{for all integers } n. \qquad \text{... (4.3)}$$

Also (4.3) exists for the Generalized Pell-Lucas sequence $\{q_n^{(\alpha)}\}$.

We need the following proof of the Theorem.

***Theorem* 4.4** — *The Pell-Lucas number* $q_n^{(1)}$ *is pronic if and only if* $n = -2, 0, 1$ *or* 2.

PROOF : We know that $q_n^{(1)} = q_n$.

In view of (4.2), we have

$$q_n \text{ is pronic} \Leftrightarrow 2Q_n = m(m+1) \text{ for some integer } m,$$

$$\Leftrightarrow Q_n \text{ is a triangular number.}$$

Now, by Theorem 3.13, Q_n is triangular if and onlyif $n = 0, 1, \pm 2$.

Hence q_n is pronic if and only if $n = 0, 1, \pm 2$.

***Theorem* 4.5** — $q_n^{(\alpha)}$ *is pronic if* $n = 0, 1,$ *or* 2.

PROOF : We prove the theorem by the principle of mathematical induction on α.

We have

$$q_0^{(\alpha)} = q_1^{(\alpha)} = 2$$

and

$$q_2^{(\alpha)} = \alpha^2 + 3\alpha + 2$$

Put $\alpha = 1$.

From the Theorem 4.4, $q_n^{(1)} + 1$ is pronic number.

Therefore, the result is true for $\alpha = 1$.

Assume that it is true for $\alpha = m$.

Note that $q_0^{(\alpha)}$, and $q_1^{(\alpha)}$ are clearly pronic numbers for all integers $\alpha > 0$.

Now we prove that $q_2^{(\alpha)}$ is also pronic number

$$\text{Let } f(m) = q_2^{(m)} = m^2 + 3m + 2$$

then $f(m) = (m+1)(m+2)$ which is pronic number.

Now $f(m+1) = [(m+1)+1]\,[(m+1)+2]$

$= (m+2)\,(m+3)$ which is pronic number.

Therefore, the result is true for $\alpha = m + 1$.

Therefore, by the principle of the mathematical induction $q_n^{(\alpha)}$ is a pronic number if $n = 0$, 1 or 2.

i.e., $q_n^{(\alpha)}$ is triangular number if $n = 0, 1$ or 2.

Which completes the proof of the Theorem.

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