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THE SOLUTION SETS OF @-FUZZY RELATIONAL EQUATIONS IN FINITE DOMAINS AND ON A COMPLETE BROUWERIAN LATTICE*

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In this paper, @-Fuzzy relational equations (where, "@" is the inf - α composite operator) in finite domains and on a complete Brouwerian lattice are discussed by utilizing the good properties of an α-operator. The solution sets of three @-Fuzzy relational equations are given respectively.

Key Words : Brouwerian Lattice; Fuzzy Relation; @-Fuzzy Relational Equations; Solution Sets

1. INTRODUCTION

Let $\underline{n} = \{1, 2, ..., n\}$ and $\underline{m} = \{1, 2, ..., m\}$ be the index sets, $A = (a_{ij})_{n \times m}$ is a coefficient matrix, $B = (b_i)_{i \in \underline{n}}$ is a constant column vector. Then

$$A \ @ \ X = B \qquad \text{... (I)}$$

is called an @-Fuzzy relational equation assigned on a complete Brouwerian lattice L, where @ denotes the inf-α composition, and all a_{ij}, b_i, x_j's are on a complete Brouwerian Lattice L. An X which satisfies (I) is called a solution of (I), the solution set of (I) is denoted by $\mathcal{X}_1 = \{X : A \ @ \ X = B\}$. A special case of ($I$) is as follows :

$$A \ @ \ X = b \qquad \text{... (II)}$$

or

$$\bigwedge_{i \in \underline{n}} (a_i \, \alpha \, x_i) = b,$$

where $b \in L, A = (a_i)_{i \in \underline{n}}$ is a row vector. The solution set of (II) is denoted by

$$\mathcal{X}_2 = \{X : A \ @ \ X = b\}.$$

Another form of eq. (II) is as follows :

$$A \ @ \ X = B. \qquad \text{... (III)}$$

Where $A = (a_i)_{i \in \underline{n}}$ and $B = (b_j)_{j \in \underline{m}}$ are two constant row vectors, $X = (\mathcal{X}_{ij})_{n \times m}$ is an unknown matrix. The solution set of (III) is denoted by $\mathcal{X}_3$ $\{X : A \ @ \ X = B\}$.

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In 1976, Sanchez[1] first introduced fuzzy relational equations. Furtherly, in 1985, Di Nola, Pedrycz and Sessa[2] introduced an @-Fuzzy relational equation (i.e. the eq. (III) in this paper) and had proved that $\mathcal{X}_3 \neq \phi$ if and only if $A \times B \in \mathcal{X}_3$ (where, $A \times B = (a_i \wedge b_j)_{n \times m}$). Further, $A \times B \leq X$ for all $X \in \mathcal{X}_3$. In particular, in 1989, Di Nola, Sessa, Pedrycz and Sanchez[3] discussed eq. (III) in linear lattices and constructed its maximal solutions when the universe of discourse is finite. In this paper, not only the maximal solutions of (I), (II) and (III) are constructed, but also the solution sets of the three equations are given clearly on a complete Brouwerian lattice.

Throughout this paper, we assume that $L = (L, \leq, \wedge, \vee)$ is a complete Brouwerian lattice with universal bounds 0 and 1, where $a \vee b$ = sup $\{a, b\}$, $a \wedge b$ = inf $\{a, b\}$, "$\leq$" stands for the partial ordering of L. We also assume that all the elements of a vector or a matrix are on L. The other definitions and symbols beyond special statements can be found in [3].

Definition 1 (Birkhoff[4]) — A Brouwerian lattice is a lattice L in which, for any given elements a and b, the set of all $x \in L$ such that $a \wedge x \leq b$ contains a largest element, dented by $a\alpha b$, the relative pseudocomplement of a in b. If the lattice L is also complete, we call L a complete Brouwerian lattice.

Definition 2 (Di Nola *et al.*[3]) — Let $A = (a_{ij})_{n \times m}$ and $B = (b_{jk})_{m \times k}$, define $A \ @ \ B = C = (c_{ij})_{n \times k}$ and $A \odot B = D = (d_{ij})_{n \times k}$ as follows :

$$c_{ij} = \bigwedge_{r=1}^{m} (a_{ir} \alpha b_{rj}) \text{ for all } i \in \underline{n} \text{ and } j \in \underline{k};$$

$$d_{ij} = \bigvee_{r=1}^{m} (a_{ir} \wedge b_{rj}) \text{ for all } i \in \underline{n} \text{ and } j \in \underline{k}.$$

Definition 3 (Di Nola *et al.*[3]) — Let $A = (a_i)_{i \in \underline{n}}$, $B = (b_i)_{i \in \underline{n}}$, $C = (c_j)_{j \in \underline{m}}$ and $b \in L$, then the partial order $\leq$, the meet $\wedge$, the Cartesian product $\times$ are defined as follows :

$A \leq B$ if and only if $a_i \leq b_i$ for all $i \in \underline{n}$;

$A \wedge B = (a_i \wedge b_i)_{i \in \underline{n}}$;

$A \wedge b = (a_i \wedge b)_{i \in \underline{n}}$;

$A \times C = (a_i \wedge c_j)_{n \times m}$.

Definition 4 (Crawley and Dilworth[5]) — An element b of a lattice L is called meet-irreducible if $x \wedge y = b$ implies $x = b$ or $y = b$.

For the sake of convenience, we first recall some useful properties of the α-operator.

Lemma 1.1 — For a, b, $c \in L$, we have:

(1) (Zhao[6]) $a\alpha b = a\alpha(a \wedge b)$;

(2) (Zhao[6]) If $b \le c$, then $a\alpha b \le a\alpha c$;

(3) (Di Nola *et al.*[3]) $a \wedge (a\alpha b) \le b$; or $a \wedge (a\alpha b) = a \wedge b$

(4) (Di Nola *et al.*[3]) $a\alpha (b \vee c) = (a\alpha b) \vee (a\alpha c)$;

(5) (Di Nola *et al.*[3]) $a\alpha (a \wedge b) \ge b$.

Lemma 1.2 (Zhao[6]) — Let $A = (a_{ij})_{n \times m}$, $B = (b_{ij})_{m \times r}$ and $C = (c_{ij})_{m \times r}$, if $B \le C$, then $A @ B \le A @ C$.

2. THE SOLUTION SET OF EQUATION (II)

In this section, we first discuss equation $b = a\alpha x$ (where $a, b \in L$) and get its solution set $\mathcal{X}_{21} = \{x : b = a\alpha x\}$. Then, on the basis of this discussion, equation (II) is investigated. When b is a meet-irreducible element of L, $\mathcal{X}_2$ is characterized entirely.

We have:

Proposition 2.1 (Di Nola *et al.*[2]) — $\mathcal{X}_{21} \ne \phi$ if and only if $a \wedge b \in \mathcal{X}_{21}$. Further, $a \wedge b \le x$ for all $x \in \mathcal{X}_{21}$.

Proposition 2.2 — $\mathcal{X}_{21} \ne \phi$ if and only if $b \in \mathcal{X}_{21}$.

PROOF : "⇒" According to Lemma 1.1 (1) and Proposition 2.1, $b \in \mathcal{X}_{21}$.

"⇐" If $b \in \mathcal{X}_{21}$, of course $\mathcal{X}_{21} \ne \phi$.

Proposition 2.3 — Let I be an index set, if $x_i \in \mathcal{X}_{21}$ for any $i \in I$, then $\bigvee_{i \in I} x_i \in \mathcal{X}_{21}$.

PROOF : By (1) and (5) of Lemma 1.1, $a\alpha x_i = a\alpha(a \wedge x_1) \ge x_i$, thus $\bigvee_{i \in I} (a\alpha x_i) \ge \bigvee_{i \in I} x_i$. Also by Lemma 1.1(2) and $x_i \in \mathcal{X}_{21}$ for any $i \in I$,

$$b = a\alpha x_i \le a\alpha\left(\bigvee_{i \in I} X_i\right) \le a\alpha\left[\bigvee_{i \in I} (a\alpha x_i)\right] = a\alpha\left[\bigvee_{i \in I} (a\,\alpha X_i)\right] = a\alpha b = b.$$

That is

$$a\alpha\,(\bigvee_{i \in I} x_i) = b. \text{ Hence, } \bigvee_{i \in I} x_i \in \mathcal{X}_{21}.$$

Proposition 2.4 — If $\mathcal{X}_{21} \ne \phi$, then b is the largest element of $\mathcal{X}_{21}$.

PROOF : If $\mathcal{X}_{21} \ne \phi$, suppose that $x \in \mathcal{X}_{21}$, and $x > b$. Since $x \le a\alpha x$ follows from $a \wedge x \le x$, then $b < x \le a\alpha x = b$ is *a* contridiction. This means that b is a maximal element of $\mathcal{X}_{21}$. Further, if there exists another maximal solution $x' \in \mathcal{X}_{21}$, then $x' \vee b \in \mathcal{X}_{21}$ by Proposition 2.3. So $x' \vee b \ge b$, and this implies $x' \vee b = b$ by the maximality of b in $\mathcal{X}_{21}$. Thus $b \ge x'$. Therefore, b is the largest element of $\mathcal{X}_{21}$.

By Proposition 2.1 and 2.4, the next proposition is easily seen:

Proposition 2.5 — If $\mathcal{X}_{21} \ne \phi$, then $\mathcal{X}_{21} = [a \wedge b, b]$.

Example — Let $3 = 2\alpha x$ on a complete Brouwerian lattice $(N, \leq \wedge, \vee)$. Where, $N = \{0, 1, 2, ..., n, ...\}$ and for any $a, b \in N$, $a \wedge b = l.c.m\{a, b\}$, $a \vee b = g.c.d\{a, b\}$, $a \leq b$ if and only if a is multiple of b.

The smallest solution of $3 = 2\alpha x$ is $x = 2 \wedge 3 = 6$. The largest solution of $3 = 2\alpha x$ is $x = 3$. So, its solution set is $\mathcal{X} = [6, 3] = \{6, 3\}$.

In the following of this section, we will discuss eq. (II). By denoting $G(b) = \{i \in \underline{n}; a_i\alpha b = b\}$, we have:

Proposition 2.6 — If b is a meet-irreducible element of L, then $\mathcal{X}_2 \neq \phi$ if and only if $G(b) \neq \phi$.

PROOF : "$\Rightarrow$" If $\mathcal{X}_2 \neq \phi$, suppose that $X \in \mathcal{X}_2$, then $b = A \circledast X = \bigwedge_{i \in \underline{n}} (a_i\alpha x_i)$. Since b is a meet-irriducible element, there exists an $i \in \underline{n}$ such that $b = a_i\alpha x_i$. By Proposition 2.2, $a_i\,\alpha b = b$, i.e. $i \in G(b)$, and $G(b) \neq \phi$.

"$\Leftarrow$" If $G(b) \neq \phi$, there is an $i \in G(b)$ such that $a_i\alpha b = b$. Let $X = (1, ..., \underset{i}{b}, ..., 1)^{\tau}$ (where, T denotes the transposition of X), then:

$$b = A \circledast X = (a_1\,\alpha 1) \wedge \ldots \wedge (a_i\alpha b) \wedge \ldots \wedge (a_n\alpha 1) = 1 \wedge b \wedge 1 = b,$$

Thus $X \in \mathcal{X}_2$, and $\mathcal{X}_2 \neq \phi$.

Proposition 2.7 (Di Nola *et al.*[2]) — $\mathcal{X}_2 \neq \phi$ if and only if $A \wedge b \in \mathcal{X}_2$. Further, $A \wedge b \leq X$ for all $X \in \mathcal{X}_2$.

Proposition 2.8 — If $\mathcal{X}_2 \neq \phi$ and b is a meet-irreducible element of L, then for any $i \in G(b)$, $X_i^* = (1, ..., \underset{i}{b}, ..., 1)^T$ is a maximal element of $\mathcal{X}_2$.

PROOF : If $\mathcal{X}_2 \neq \phi$, according to the proof of Proposition 2.6, $X_i^* \in \mathcal{X}_2$. Then we only need verify the maximality of X_i^* in $\mathcal{X}_2$. If $X \in \mathcal{X}_2$ and $X \geq X_i^*$, then when $j \neq i$, of course $x_j = x_j^* = 1$; when $j = i, x \geq b$ since $X \geq X_i^*$. On the other hand,

$$b = A \circledast X = (a_1\alpha 1) \wedge \ldots \wedge (a_i\alpha x_i) \wedge \ldots \wedge (a_n\alpha 1) = 1 \wedge (a_i\alpha x_i) \wedge 1 = a_i\alpha x_i,$$

i.e. $b = a_i\alpha x_i, x_i \leq b$ by Proposition 2.4. So, $x_i = b$. Therefore, $X = X_i^*$, i.e. X_i^* is maximal in $\mathcal{X}_2$.

Proposition 2.9 — If $\mathcal{X}_2 \neq \phi$, and b is a meet-irreducible element of L, then every maximal element of $\mathcal{X}_2$ has the form of X_i^* (where, $X_i^* = (1, ..., \underset{i}{b}, ..., 1)^{\tau}$ with $i \in G(b)$). Further, the number of maximal elements of $\mathcal{X}_2$ is $|\,G(b)\,|$.

PROOF : If $\mathcal{X}_2 \neq \phi$, let $X = (X_i)^T_{i \in \underline{n}}$ be any maximal element of $\mathcal{X}_2$. Since b is a meet-irreducible element, there is an $i \in \underline{n}$ such that $b = a_i \alpha x_i$ follows from $b = A @ C = \vee_{x \in \underline{n}} (a_i \alpha x_i)$, thus $x_i \leq b$ by Proposition 2.4. Set $X_i^* = (1, ..., \underset{i}{b}, ..., 1)^\tau$, obviously $X_i^* \geq X$ and $X_i^* \in \mathcal{X}_2$ by Proposition 2.8. Thus $X_i^* = X$ by the maximality of X in $\mathcal{X}_2$. This means that every maximal element of $\mathcal{X}_2$ has the form of X_i^*. Further by Proposition 2.8, the number of maximal elements of $\mathcal{X}_2$ is $| G(b) |$.

Proposition 2.10 — If $\mathcal{X}_2 \neq \phi$, and b is a meet-irreducible element of L. Then for each $X \in \mathcal{X}_2$, there exists a maximal element X_i^* in $\mathcal{X}_2$ such that $X_i^* \geq X$.

PROOF : If $\mathcal{X}_2 \neq \phi$, let $X \in \mathcal{X}_2$, then $b = A @ X = \vee_{i \in \underline{n}} (a_i \alpha x_i)$. Since b is a z meet-irreducible element, there exists an $i \in \underline{n}$ such that $b = a_i \alpha x_i$, i.e. $x_i \leq b$ by Proposition 2.4. Let $X_i^* = (1, ..., \underset{i}{b}, ..., 1)^\tau$, then $X_i^* \geq X$, X_i^* is a maximal element of $\mathcal{X}_2$ by Proposition 2.9.

Proposition 2.11 — If $X_1, X_2 \in \mathcal{X}_2$ (resp. $\mathcal{X}_1$, $\mathcal{X}_3$), then $X_1 \leq X \leq X_2$ implies $X \in \mathcal{X}_2$ (resp. $\mathcal{X}_1, \mathcal{X}_3$).

PROOF : Since $X_1 \leq X \leq X_2$ and $X_1, X_2 \in \mathcal{X}_2$, $b = A @ X_1 \leq A @ X \leq A @ N_2 = b$ by Lemma 1.2. Thus $A @ X = b$, i.e. $X \in \mathcal{X}_2$.

Remark 2.1 : According to Proposition 2.11, if $\mathcal{X}_2 \neq \phi$, then $X_1 \wedge X_2 \in \mathcal{X}_2$ for any $X_1, X_2 \in \mathcal{X}_2$, i.e. $\mathcal{X}_2$ is a meet semilattice with respect to $\wedge$.

PROOF : If $\mathcal{X}_2 \neq \phi$, then $A \vee b \in \mathcal{X}_2$ by Proposition 2.7, and $A \wedge b \leq X_1 \wedge X_2 \leq X_1$. Thus $X_1 \wedge X_2 \in \mathcal{X}_2$ by Proposition 2.11.

On the basis of Proposition 2.6-2.11 and Remark 2.1, the next proposition is easy to be shown.

Proposition 2.12 — If $\mathcal{X}_2 \neq \phi$, and b is a meet-irreducible element of L, then $\mathcal{X}_2 = \bigcup_{i \in G(b)} [A \wedge b, X_i^*]$, where, X_i^* with $i \in G(b)$ is a maximal element of $\mathcal{X}_2$.

Remark 2.2 — If $\mathcal{X}_2 \neq \phi$, and b is a meet-irreducible element of L, then by Proposition 2.6 - 2.12, the solution set of eq. (II) is characterized completely.

3. The Solution Set of Equation (I)

In this section, when b_i is a meet-irreducible element of L for any $i \in \underline{n}$, we construct the maximal solutions of equation (I) and $\mathcal{X}_1$ is determined.

Suppose that $A_i = (a_{i1}, a_{i2}, ..., a_{im})$, then equation (I) is equivalent to

$$\begin{cases} A_1 @ X = b_1, \\ A_2 @ X = b_2, \\ \cdots \\ A_n @ X = b_n. \end{cases}$$

Let $\mathcal{X}_{2i} = \{X : A_i @ X = b_i\}$ be the solution set of $A_i @ X = b_i$ for any $i \in \underline{n}$. Then we have:

Proposition 3.1 — $\mathcal{X}_1 \neq \phi$ if and only if $\bigcap_{i \in \underline{n}} \mathcal{X}_{2i} \neq \phi$. Further, $\mathcal{X}_1 = \bigcap_{i \in \underline{n}} \mathcal{X}_{2i}$.

PROOF : "$\Leftarrow$" If $\bigcap_{i \in \underline{n}} \mathcal{X}_{2i} \neq \phi$, then there exists an $X \in \mathcal{X}_{2i}$ such that $A_i @ X = b_i$ for all $i \in \underline{n}$. So, $A @ X = B$. Hence $X \in \mathcal{X}_1$, and $\mathcal{X}_1 \neq \phi$.

"$\Rightarrow$" If $\mathcal{X}_1 \neq \phi$, let $X \in \mathcal{X}_1$, then $A @ X = B$, i.e. $A_i @ X = b_i$ for all $i \in \underline{n}$. So, $X \in \mathcal{X}_{2i}$ for all $i \in \underline{n}$. That is $X \in \bigcap_{i \in n} \mathcal{X}_{2i} \neq \phi$. Furtherly, $\mathcal{X}_1 = \bigcap_{i \in \underline{n}} \mathcal{X}_{2i}$.

Proposition 3.2 (Di Nola *et al.*[3]) — If $\mathcal{X}_1 \neq \phi$, then $A^T \odot B \in \mathcal{X}_1$. Further, $A^T \odot B \leq X$ for all $X \in \mathcal{X}_1$.

Set $\mathcal{M}_i = \{M_j^* \in \mathcal{X}_{2i} : M_j^* \geq A^T \odot B$ and M_j^* is a maximal element of $\mathcal{X}_{2i}\}$, $i \in \underline{n}$. Then we have:

Proposition 3.3 — If $\mathcal{X}_1 \neq \phi$, and b_i is a meet-irreducible element of L for any $i \in \underline{n}$, then $\mathcal{M} = \{M^* : M^* = \bigwedge_{i \in \underline{n}} M_i^*$, and $M_i^* \in \mathcal{M}_i$, $i \in \underline{n}\}$ is a finite subset of $\mathcal{X}_1$. Further, an element M is maximal in $\mathcal{M}$ if an only if it is maximal in $\mathcal{X}_1$.

PROOF: We first prove that $\mathcal{M}$ is a subset of $\mathcal{X}_1$. Let $M^* = \bigwedge_{i \in \underline{n}} M_i^* \in \mathcal{M}$, then $A^\tau \odot B \leq M^* \leq M_i^*$ for all $i \in \underline{n}$. Since M_i^*, $A^\tau \odot B \in \mathcal{X}_{2i}$ for all $i \in \underline{n}$, then $M^* \in \mathcal{X}_{2i}$ by Proposition 2.11 for all $i \in \underline{n}$. Therefore $M^* \in \mathcal{X}_1$ by Proposition 3.1. By Proposition 2.9, $\mathcal{M}_i$ is a finite set, thus $\mathcal{M}$ is a finite poset and hence possesses maximal elements.

Let M be maximal in $\mathcal{M}$, and $X \in \mathcal{X}_1$ such that $X \geq M$. Since $X \in \mathcal{X}_{2i}$ for all $i \in \underline{n}$ by Proposition 3.1, then there exists a maximal element $M_i^* \in \mathcal{X}_{2i}$ such that $M_i^* \geq X$ for any $i \in \underline{n}$ by

Proposition 2.10. Hence $M^* = \underset{i \in \underline{n}}{\wedge} M_i^*$ is an element of $\mathcal{M}$, and $M^* \geq X \geq M$, then $M^* = M$ since M is maximal in $\mathcal{M}$. Therefore, $X = M$, and M is maximal in $\mathcal{X}_1$.

Vice versa, Let X be maximal in $\mathcal{X}_1$, and $M \in \mathcal{M}$ such that $M \geq X$. Since $\mathcal{M}$ is a subset of $\mathcal{X}_1$, then $M \in \mathcal{X}_1$. So $M = X$ by the maximality of X in $\mathcal{X}_1$. Hence, X is maximal in $\mathcal{M}$.

Proposition 3.4 — If $\mathcal{X}_1 \neq \phi$, and b_i is a meet-irreducible element of L for any $i \in \underline{n}$. Then for each $X \in \mathcal{X}_1$, there exists a maximal element X^* in $\mathcal{X}_1$ such that $X^* \geq X$.

PROOF : If $\mathcal{X}_1 \neq \phi$ and $X \in \mathcal{X}_1$, then $X \in \mathcal{X}_{2i}$ for all $i \in \underline{n}$. Then there exists an $M_i^* \in \mathcal{M}_i$ such that $M_i^* \geq X$ for any $i \in \underline{n}$ by Proposition 2.10. Let $M^* = \underset{i \in \underline{n}}{\wedge} M_i^*$, then $M^* \in \mathcal{M}$. Moreover, there must be a maximal element M in $\mathcal{M}$ such that $M \geq M^*$ by Proposition 3.3, and $M^* \geq X$. Therefore there exists a maximal element M in $\mathcal{X}_1$ such that $M \geq X$.

Let $\underline{\mathcal{M}}$ be a set of all the maximal elements of $\mathcal{M}$. By Proposition 31-3.4, we can easily see:

Proposition 3.5 — If $\mathcal{X}_1 \neq \phi$, b_i is a meet-irreducible element of L for any $i \in \underline{n}$, then $\mathcal{X}_1 = \underset{M^* \in \mathcal{M}}{\bigcup} [A^T \odot B, M^*]$.

Remark 3.1 — If $\mathcal{X}_1 \neq \phi$, and b_i is a meet-irreducible element of L for any $j \in \underline{n}$, then by Proposition 3.1-3.5, we can determine the soluion set of equation (I) entirely.

4. THE SOLUTION SET OF EQUATION (III)

In this section, when b_j is a meet-irreducible element of L for any $j \in \underline{m}$, we construct maximal solutions of equation (III) and $\mathcal{X}_3$ is given.

For every $j \in \underline{m}$, let $X_j = (x_{1j}, x_{2j}, ..., x_{nj})^{\tau}$, then equation (III) is equivalent to

$$\begin{cases} A \; @ \; X_1 = b_1, \\ A \; @ \; X_2 = b_2, \\ \cdots \\ A \; @ \; X_m = b_m. \end{cases}$$

Let $\mathcal{X}_{2j} = \{X_j : A \; @ \; X_j = b_j\}$ be the solution set of $A \; @ \; X_j = b_j$ for every $j \in \underline{m}$, and denoting $G(b_j) = | \, i \in \underline{n} : a_i \alpha b_j = b_j \, |$ for every $j \in \underline{m}$. We have:

Proposition 4.1 — If $\mathcal{X}_3 \neq \phi$, and b_j is a meet-irreducible element of L for any $j \in \underline{m}$, then:

(1) $\mathcal{X}_3 = \{| \, X = (X_1, X_2, ..., X_m) : X_j \in \mathcal{X}_{2j}, j \in \underline{m}\}$;

(2) $X^* = (X_1^*, X_2^*, ..., X_m^*)$ is maximal in $\mathcal{X}_3$ if and only if for every $j \in \underline{m}$, X_j^* is maximal in $\mathcal{X}_{2j}$. Further, the number of maximal elements of $\mathcal{X}_3$ is $|G(b_1)| \times |G(b_2)| \times \ldots \times |G(b_m)|$.

PROOF : (1) Since $X \in \mathcal{X}_3$ if and only if $B @ A @ X$ if and onlyif $b_j = A @ X_j$ for every $j \in \underline{m}$ if and only if $X_j \in \mathcal{X}_{2j}$ for any $j \in \underline{m}$, so that $\mathcal{X}_3 = \{X = (X_1, X_2, ..., X_m) : X_j \in \mathcal{X}_{2j}, j \in \underline{m}\}$.

(2) "$\Leftarrow$" If $\mathcal{X}_3 \neq \phi$ and X_j^* is maximal in $\mathcal{X}_{2j}$ for every $j \in \underline{m}$. Let $X = (X_1, X_2, ..., X_m) \in \mathcal{X}_3$ and for every $j \in \underline{m}$, then $X_j = X_j^*$ for every $j \in \underline{m}$ by the maximality of X_j^* in $\mathcal{X}_{2j}$. Hence, $X = X^*$, i.e. X^* is a maximal element of $\mathcal{X}_3$.

"$\Rightarrow$" If $\mathcal{X}_3 \neq \phi$, $X^* = (X_1^*, X_2^*, ..., X_m^*)$ is maximal in $\mathcal{X}_3$. Then for every $j \in \underline{m}$, $X_j^* \in \mathcal{X}_{2j}$, and there exists a maximal element X_j in $\mathcal{X}_{2j}$ such that $X_j \geq X_j^*$ by Proposition 2.10. Let $X = (X_1, X_2, \ldots, X_m)$, then $X \in \mathcal{X}_3$. That is $X_j = X_j^*$ for every $j \in \underline{m}$. Further by Proposition 2.9, the number of maximal elements of $\mathcal{X}_3$ is $|G(b_1)| \times |G(b_2)| \times \ldots \times |G(b_m)|$.

Proposition 4.2 (Di Nola *et al.*[2]) — $\mathcal{X}_3 \neq \phi$ if and only if $A \times B \in \mathcal{X}_3$. Further, $A \times B \leq X$ for all $X \in \mathcal{X}_3$.

Similar to Proposition 2.11 and Proposition 3.4, we have :

Proposition 4.3 — If $\mathcal{X}_3 \neq \phi$, and b_j is a meet-irreducible element of L for any $j \in \underline{m}$. Then for each $X \in \mathcal{X}_3$, there exists a maximal element X^* in $\mathcal{X}_3$ such that $X^* \geq X$.

PROOF : If $\mathcal{X}_3 \neq \phi$, suppose hat $X = (X_1, X_2, ..., X_m) \in \mathcal{X}_3$, then $X_j \in \mathcal{X}_{2j}$ for every $j \in \underline{m}$. Thus there must be a maximal element X_j^* in $\mathcal{X}_{2j}$ such that $X_j^8 \geq X_j$ by Proposition 2.10. Let $X^* = (X_1^*, X_2^* ..., X_m^*)$, then X^* is a maximal element of $\mathcal{X}_3$ by Proposition 4.1(2),.and $X^* \geq X$.

Let $\mathcal{X}^*$ be a set of all maximal elements of $\mathcal{X}_3$. By Proposition 4.1-4.3, the next Proposition is easy to be shown.

Proposition 4.4 — If $\mathcal{X}_3 \neq \phi$, and b_j is a meet-irreducible element of L for any $j \in \underline{m}$, then

$$\mathcal{X}_3 = \bigcup_{X^* \in \mathcal{X}^*} [A \times B, X^*].$$

Remark 4.1 : If $\mathcal{X}_3 \neq \phi$, and b_j is a meet-irreducible element of L for any $j \in \underline{m}$, then the solution set of eq. (III) is determined by Proposition 4.1-4.4.

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