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OSCILLATION OF HIGHER ORDER DELAY DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO HYPERBOLIC EQUATIONS

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In this paper we shall consider an odd order delay differential equation with positive and negative coefficients. New sufficient conditions which guarantee the osciallation of all solutions are presented. Our results extend and improve some known results in the literature. Next, these results are applied to establish the oscillation criteria for hyperbolic delay differential equations with positive and negative coefficients corresponding three sets of boundary conditions.

Key Words : Oscillation; Delay Differential Equations; Hyperbolic

1. INTRODUCTION

In recent years oscillation theory of delay differential equations has received considerable atention. This is largely due to the fact that delay differential equations find various interesting applications in the real world problem, see e.g. [15].

In this paper we shall consider the following higher-order delay differential equation with variable coefficients

$$x^{(n)}(t) + P(t)x(t-\sigma) - Q(t)x(t-\tau) = 0, \qquad t \geq t_0, \quad n \in N. \qquad \text{... (1.1)}$$

where

$$P, Q \in C([t_0, \infty), R^+), \ \sigma \geq \tau \geq \sigma/n > 0 \text{ and } n > 1 \text{ is odd}, \qquad \text{... (1.2)}$$

$$P(t) \geq Q(t+\tau-\sigma) \text{ for } t \geq t_0 + \sigma - \tau, \text{ and not identically zero for large } t, \qquad \text{... (1.3)}$$

and

$$1 - \int_t^{\infty} \frac{(s-t)^{n-2}}{(n-2)!} \int_{s-\sigma+\tau} Q(u)du\, ds \geq 0, \text{ for all sufficiently large } t. \qquad \text{... (1.4)}$$

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By a solution of eq. (1.1) we mean a function $x \in C[[t_1 - \sigma, \infty), R]$, for some $t_1 \geq t_0$, such that $x(t)$ is n times continuously differentiable on $[t_1, \infty)$ and such that eq. (1.1) is satisfied for $t \geq t_1$. Let $\phi \in C[[t_1 - \sigma, t_1], R]$ be a given initial funciton, and let y_k, $k = 0, 1, \ldots, n-1$ be given initial constants. By using the method of steps one can see that eq. (1.1) has a unique solution $x \in C[[t_1 - \sigma, \infty), R]$ such that

$$x(t) = \phi(t) \qquad \text{for } t \in [t_1 - \sigma, t_1] \qquad \ldots (1.5)$$

and

$$\left[\frac{d^k}{dt^k} x(t) \right]_{t = t_1} = y_k \text{ for } k = 0, 1, \ldots, n-1. \qquad \ldots (1.6)$$

Definition 1 — We say that (1.1) is oscillatory if every solution of eq. (1.1) is oscillatory, i.e., for each initial point $t_1 \geq t_0$ the unique solution of (1.1) and (1.5) has arbitrarily large zeros. Otherwise, it is called nonoscillatory.

We also consider the hyperbolic delay differential equation

$$\frac{\partial^n u(x, t)}{\partial^n t} = a(t)\, \Delta u - P(t) u(x, t - \sigma) + Q(t) u(x, t - \tau) \qquad \ldots (E)$$

where $(x, t) \in \Omega \times [t_0, \infty) \equiv G$, and Ω is a bounded set in R^n with a piecewise smooth boundary $\partial\Omega$, and $\Delta u(x, t) = \sum_{i=1}^{n} \frac{\partial_i^2 u(x, t)}{\partial x_i^2}$ together with two kinds of boundary conditions

$$\frac{\partial u(x, t)}{\partial N} = 0, \qquad \text{on} \qquad (x, t) \in \partial\Omega \times [t_0, \infty), \qquad \ldots (B1)$$

$$u(x, t) = 0, \qquad \text{on} \qquad (x, t) \in \partial\Omega \times [t_0, \infty), \qquad \ldots (B2)$$

and

$$\frac{\partial u(x, t)}{\partial N} + \gamma u = 0, \qquad \text{on } (x, t) \in \partial\Omega \times [t_0, \infty), \qquad \ldots (B3)$$

where N is the unit exterior normal vector to $\partial\Omega$, $\gamma(x, t)$ is a nonnegative continuous function on $\partial\Omega \times [t_0, \infty)$. The related results can refer to [5, 19].

Definition 2 — A function $u(x, t) \in C^2(G) \cap C^1(\bar{G})$ is said to be a solution of the problem (E), (B_1) $[(B_2), (B_3)]$ if it satisfies (E) in the domain G and satisfies the boundary condition (B_1) $[(B_2), (B_3)]$.

Definition 3 — The solution $u(x, t)$ of the problem (E), $(B1)$, $[B2)$, $(B3)]$ is said to be oscillatory in the domain $G \equiv \Omega \times [t_0, \infty)$ if for any positive number μ there exists a point $(x_1, {}_1) \in \Omega \times [\mu, \infty)$ such that the equality $u(x_1, t_1) = 0$ holds.

Definition 4 — A function $U(t)$ is called eventually positive (negative) if there exists a number $t_1 \geq t_0$ such that $U(t) > 0(<0)$ holds for all $t_1 \geq t_0$.

For the oscillation of (1.1) when $n = 1$ the work of Qian and Ladas[21] provides some finite sufficient conditions for the oscillation of all solutions, and Elabbasy and Saker[10] have extended the results in[21] to the delay differential equations with several positive and negative coefficients. However, for the same case Agarwal and Saker[3], Elabbasy *et al.*[9] present some new infinite sufficient conditions, which extend and improve the results in[21]. Recently, in the same case, Lakrib[20] by the same procedure used in[9] and obtained the same condition for oscillation of delay differential equations with positive and negative coefficients. Except for some speical case of (1.1) when $n > 1$, to the best of our knowledge, the only oscillation results for (1.1) is due to Li[18]. He has proved that if (1.2)-(1.3) hold, then every solution of (1.1) oscillates if the inequality

$$z^{(n)}(t) + [P(t) - Q(t + \tau - \sigma)]\, z(t - \sigma) \leq 0 \qquad \text{... (1.7)}$$

has no eventually positive solution. The oscillation of various other functional differential equations has been investigated by several authors. For some of these contibutions we refer to the monographs [1, 2, 11, 12, 17] and for other oscillation of equations with positive and negative coefficients we refer to the paper [6, 7, 8, 9, 10, 19, 23].

The plain of the paper is as follows: In Section 2, we shall present oscillation criteria for eq. (1.1). The obtained results improve and extend those established in Li [18]. These results will be used in Section 3, 4 and 5 to establish the oscillation of (E)-$(B1)$, (E)-$(B2)$ and (E)-$(B3)$ respectively.

In what follows, a functional inequality will be assumed to hold for all sufficiently large values of t.

2. Main Results

To prove our main results we shall need the following lemmas :

Lemma 2.1[14] — If $w(t)$ is a function such that it and its all derivatives up to $(n - 1)$ inclusive are absolutely continuous and of constant sign in the interval $[t_0, \infty)$ satisfy $w(t) \neq 0$ for $t \geq t_0$ and $w(t)w^{(n)}(t) \leq 0$. Then there exists an even integer $k, 0 \leq k \leq n - 1$, such that for $t \geq t_1$

$$\left.\begin{array}{ll} w(t)\, w^{(i)}(t) > 0, & \text{for } i = 0, 1, \ldots, k, \\ (-1)^{n+i-1}\, w(t)\, w^{(i)}(t) > 0, & \text{for } i = k + 1, \ldots n, \end{array}\right\}$$

and

$$|\, w(t)\, | \geq \left| \frac{(t - t_1)^{n-1}}{(n-1) \ldots (n-k)}\, w^{(n-1)}\, (2^{n-k-1}\, t) \right| .$$

Lemma 2.2[5] — Suppose that $f \in C^{(n)}[[T, \infty), R^{+,} T \geq 0$, such that $f^{(i)}(t)$ $(i < n)$ is of one sign in $[T, \infty)$ and $f^{(n)}(t) \leq 0$ for $t \geq T$. Then $\alpha > 0$ implies that

$$f(t \ \alpha) \geq \frac{\alpha^{n-1}}{(n-1)!} f^{(n-1)}(t), \quad t \geq T + 2\alpha.$$

Lemma 3.2[18] — Assume that (1.2)-(1.4) hold. Let $x(t)$ be an eventually positive solution of eq. (1.1) and set

$$z(t) = x(t) - \int_t^{\infty} \frac{(s-t)^{n-2}}{(n-2)!} \int_{s-(\sigma \ \tau)}^{s} Q(\theta)\, x(\theta - \tau)\, d\theta\, ds, \ t \geq t_0 + \sigma - \tau, \quad \text{... (2.1)}$$

then

$$z^n(t) \leq 0, \ \ z^{(n-1)}(t) > 0, \ldots, z'(t) < 0, \ \ z(t) > 0. \quad \text{... (2.2)}$$

and $z(t)$ satisfy the equation (1.7).

***Theorem* 2.1** — *Assume that (1.2) - (1.4) hold. Then every solution of eq. (1.1) oscillates if the delay differential equation*

$$y'(t) + R(t)\, y(t - \sigma) = 0, \quad \text{... (2.3)}$$

where

$$R(t) = [P(t) - Q(t + \sigma - \sigma)] \frac{(t - t_2 \ \sigma)^{n-1}}{(n-1)!}.$$

has no eventually positive solution.

PROOF : Assume for the sake of contradiction that the equation (1.1) has an eventually positive solution $x(t)$. By Lemma 2.3 it follows that the function $z(t)$ satisfies (2.2) and the inequality (1.7) holds for $t \geq t_2$. By Lemma 2.1 for $z(t - \sigma)$ for we have

$$z(t - \sigma) \geq \frac{(t - t_2 - \sigma)^{n-1}}{(n-1)!} z^{(n-1)}(t - \sigma).$$

Thus (1.7) leads to

$$z^{(n)}(t) + [P(t) - Q(t + \tau - \sigma)] \frac{(t - t_2 - \sigma)^{n-1}}{(n-1)!} z^{(n-1)}(t - \sigma) \leq 0.$$

Let $y(t) = z^{(n \ 1}(t)$ then $y(t) > 0$ and satisfies the inequality

$$y'(t) + R_1 y(t - \sigma_1) \leq 0.$$

But, then by Corollary 3.2.2 in [12] the delay differential equation (2.3) has an eventually positive solution also. This contradiction ensure that every solution of (1.1) is oscillator ■

***Theorem* 2.3** — *Assume that (1.2)-(1.4) hold. Then, either*

$$\lim_{t \to \infty} \inf \int_{t-\sigma}^{t} R(s)\, ds > \frac{1}{e} \quad \text{... (2.5)}$$

or

$$\lim_{t \to \infty} \sup \int_{t-\sigma}^{t} R(s)\, ds > 1 \quad \text{... (2.6)}$$

implies that every solution of eq. (1.1) oscillates.

PROOF : It is well known that (2.5) or (2.6) implies that (2.3) has no eventually positive solution (see, for example, Theorem 2.3.3 and 3.4.3 in [12]). This contradiction completes the proof ■

***Theorem* 2.4** — *Assume that (1.2)-(1.4) hold. Then, either*

$$\lim_{t \to \infty} \inf \int_{t-\sigma_1}^{t} R_1 (s)ds > \frac{1}{e} \quad \text{... (2.7)}$$

or

$$\lim_{t \to \infty} \inf \int_{t-\sigma_1}^{t} R_1 (s)ds > 1 \quad \text{... (2.8)}$$

implies that every solution of eq. (1.1) oscillates.

PROOF : It is well known that (2.7) or (2.8) implies that (2.4) has no eventually positive solution (see, for example, Theorem 2.3.3 and 3.4.3 in [12]). This contradiction completes the proof ■

It is clear that there is a gap between the conditions (2.5), (2.6)[(2.7), (2.8)]. The problem of filling this gap for the equation (2.3)[(2.4)] when the limit $\lim_{t \to \infty} \int_{t-\sigma}^{t} R(s) \left[\lim_{t \to \infty} \int_{t-\sigma_1}^{t} R_1(s) \right]$ does not hold has been addressed by several authors, e.g., see[13, 15]. In view of these works and the fact that every solution of (1.1) oscillates when (2.3)[(2.4)] has no eventually positive solution one can state several finite sufficient conditions for oscillation of all solutions.

In the following theorems we infinite sufficient conditions for the oscillation of (1.1) which shows that the condition (2.5) and (2.7) are not necessaries.

***Theorem* 2.5** — *Assume that (1.2)-(1.4) hold,*

$$0 < d \leq \lim_{t \to \infty} \int_{t-\sigma}^{t} R(s)\, ds \quad \text{... (2.9)}$$

and

$$\int_{t_0}^{\infty} R(t)\left[\ln \int_{t}^{t+\sigma} R(s)\,ds + 1\right] dt = \infty \qquad \text{... (2.10)}$$

Then every solution of eq. (1.1) oscillates.

PROOF : Assume for the sake of contradiction, that the eq. (1.1) has an eventually positive solution $x(t)$. Then from Theorem 2.1, $y(t)$ is positive solution of eq. (2.3). Let

$$\lambda(t) = -\,y'(t)/y(t) \qquad \text{... (2.11)}$$

Clearly, then $\lambda(t)$ is non-negative and continuous, and there exists $t_1 \geq t_0$ such that $y(t_1) > 0$ and $y(t) = y(t_1)\exp\left(-\int_{t_1}^{t} \lambda(s)\,ds\right)$. Furthermore, $\lambda(t)$ satisfies the generalized characteristic equation,

$$\lambda(t) = R(t)\exp\left(\int_{t-\sigma}^{t} \lambda(s)\,ds\right). \qquad \text{... (2.12)}$$

As, integration (2.3) from t $t + \sigma$, we find

$$y(t+\sigma) - y(t) + \int_{t}^{t+\sigma} R(s)\,y(s-\sigma)\,ds = 0,$$

and hence in view of $y(t)$ being positive and decreasing, we find

$$y(t) > \int_{t}^{t+\sigma} R(s)\,y(t-\sigma)\,ds > y(t)\int_{t}^{t+\sigma} R(s)\,ds$$

which implies that

$$\int_{t}^{t+\sigma} R(s)\,ds < 1. \qquad \text{... (2.13)}$$

Now, using the inequality (cf. [11, p.32])

$$e^{rx} \geq x + \frac{\ln r + 1}{r} \quad \text{for } x, \text{ and } r > 0 \qquad \text{... (2.14)}$$

in the right side of the function

$$\lambda(t) = R(t)\left[\frac{1}{A(t)} A(t)\int_{t-\sigma}^{t} \lambda(s)ds\right], \qquad \text{... (2.15)}$$

we obtain

$$A(t)\lambda(t) - R(t) \int_{t-\sigma}^{t} \lambda(s)ds \geq R(t) \ln [A(t)+1]. \quad \text{.. (2.16)}$$

Then for $N > T$,

$$\int_{T}^{N} \lambda(t)A(t)dt - \int_{T}^{N} R(t) \int_{t-\sigma}^{t} \lambda(s)ds\, dt \geq \int_{T}^{N} R(t)\ln [A(t)+1]\, dt. \quad \text{... (2.17)}$$

Interchanging the order of integration, we find

$$\int_{T}^{N} R(t) \left(\int_{t-\sigma}^{t} \lambda(s)ds \right) dt \geq \int_{T}^{N-\sigma} \lambda(t) \left(\int_{to}^{t+\sigma} R(s)\, ds \right) dt.$$

Hence, it follows that

$$\int_{T}^{N} \lambda(t)A(t)dt - \int_{T}^{N-\sigma} \lambda(t) \left(\int_{t}^{t+\sigma} R(s)ds \right) dt$$

$$\geq \int_{T}^{N} \lambda(t)A(t)dt - \int_{T}^{N} R(t) \int_{t-\sigma}^{t} \lambda(s)ds\, dt. \quad \text{... (2.18)}$$

From (2.17) and (2.18), it follows that

$$\int_{T}^{N} \lambda(t)A(t)dt - \int_{T}^{N-\sigma} \lambda(t) \left(\int_{t}^{t+\sigma} R(s)ds \right) dt \geq \int_{T}^{N} R(t) \ln [A(t)+1]\, dt. \quad \text{... (2.19)}$$

Combining (2.13) and (2.19), we find

$$\int_{N-\sigma}^{N} \lambda(t)dt \geq \int_{T}^{N} R(t) \ln [A(t)+1]\, dt,$$

or

$$\ln \frac{y(N-\sigma)}{y(N)} \geq \int_{T}^{N} R(t) \ln [A(t)+1]\, dt.$$

Thus, from (2.14), we obtain

$$\lim_{t\to\infty} \frac{y(t-\sigma)}{y(t)} = \infty. \quad \text{... (2.20)}$$

Now, since $d \leq \int_{t}^{t+\sigma} R(s)ds$ there exist a sequence $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$ and $\zeta_k \in (t_k, t_k + \sigma)$ for every k such that

$$\int_{t_k}^{\zeta_k} R(s)ds \geq \frac{d}{2} \text{ and } \int_{\zeta_k}^{t_k+\sigma} R(s)ds \geq \frac{d}{2}. \qquad \dots (2.21)$$

Integrating both sides of (2.3) over the intervals $[t_k, \zeta_k]$ and $[\zeta_k, t_k + \sigma]$, we have

$$y(\zeta_k) - y(t_k) + \int_{t_k}^{\zeta_k} R(s)y(t-\sigma)ds = 0. \qquad \dots (2.22)$$

and

$$y(t_k+\sigma) - y(\zeta_k) + \int_{\zeta_k}^{t_k+\sigma} R(s)y(t-\sigma)\, ds = 0. \qquad \dots (2.23)$$

From (2.21), (2.22) and (2.23), it follows that

$$-y(t_k) + \frac{d}{2}y(\zeta_k - \sigma) \leq 0 \quad \text{and} \quad -y(\zeta_k) + \frac{d}{2}y(t_{kk}) \leq 0,$$

which implies that

$$\frac{y(\zeta_k - \sigma)}{y(\zeta_k)} \leq \left(\frac{2}{d}\right)^2.$$

But, this contradicts (2.21). Hence, every solution of (1.1) oscillates. ■

Similar to Theorems 2.5 one can use eq. (2.4) and proves the following theorem.

***Theorem* 2.6** — *Assume that (1.2)-(1.4) hold,*

$$0 < d \leq \liminf_{t\to\infty} \int_t^{t+\sigma_1} R_1(s)ds$$

and

$$\int_{t_0}^{\infty} R_1(t)\left[\ln \int_t^{t+\sigma_1} R_1(s)ds + 1\right]dt = \infty$$

Then every solution of eq. (1.1) oscillates.

3. Oscillation of Boundary Value Problem (E), (B1)

The following results provide some sufficient conditions for the oscillation of (E), (B1).

***Theorem* 3.1** — *Assume that (1.2)-(1.4) hold. Then every solution of (E), (B1) oscillates if every solution of eq. (2.3) or eq. (2.4) oscillates.*

PROOF : Assume for the sake of contradiction that (E), (B1) has a nonoscillatory solution. Since the negative solution of (E), (B1) is also a solution, then without loss of generality we assume that (E (B1) has a solution $u(x, t) > 0, u(x, t-\sigma)$ and $u(x, t-\tau)$ in $\Omega + [t_1, \infty)$ for some $t_1 \geq t_0$. Set

$$U(t) = \int_{\Omega} u(x, t)dx, \quad t \geq t_1 \qquad \text{... (3.1)}$$

then $U(t) > 0$ for $t \geq t_1$. Integrating (E) with respect to x over the domain Ω, we get

$$\frac{d^n}{dt^n}\left[\int_{\Omega} u(x, t)\, dx\right] = a(t) \int_{\Omega} \Delta u(x, t)dx$$

$$- \int_{\Omega} P(t)u(x, t-\sigma)dx - \int_{\Omega} Q(t)u(x, t-\tau)dx. \qquad \text{... (3.2)}$$

From Green's formula and the boundary condition (B1), we have

$$\int_{\Omega} \Delta u(x, t)dx = \int_{\partial\Omega} \frac{\partial u(x, t)}{\partial N} dS = 0, \quad t \geq t_1, \qquad \text{... (3.3)}$$

where dS is the surface element on $\partial\Omega$. Then (3.2) reduces to

$$\frac{d^n}{dt^n}\left[\int_{\Omega} u(x, t)\, dx\right] = -\int_{\Omega} p(x, t)u(x, t-\sigma)dx - \int_{\Omega} q(x, t)u(x, t-\tau)dx. \qquad \text{... (3.2)}$$

which in view of the definition of $U(t)$, can be rewritten as

$$\frac{d^n}{dt^n} U(t) + P(t)U(t-\sigma) - Q(t)U(t-\tau) \leq 0, \quad t \geq t_1. \qquad \text{... (3.5)}$$

Set

$$z(t) = U(t) - \int_t^{\infty} \frac{(s-t)^{n-2}}{(n-2)!} \int_{s-\sigma+\tau}^{s} Q(u)U(u-\tau)du\, ds, \; t \geq t_2 = t_1 + \sigma - \tau. \qquad \text{... (3.6)}$$

Then $$z^{(n)}(t) + [P(t) - Q(t+\tau-\sigma)]\, U(t-\sigma) \leq 0, \quad t \geq t_2. \qquad \text{... (3.7)}$$

The remainder of the proof is similar to the proof of Theorems 2.1 and 2.2 and will be omitted.

Theorem 3.1 shows that the oscillation of (E), (B1) is equivalent to oscillation of the delay differential eq. (3.1) or (32.). Thus, we can use the results of Section 2 to obtain some oscillation criteria for the problem (E), (B1), we state two such results in the following theorems:

***Theorem* 3.2** — *Assume that (1.2)-(1.4) and (3.1) hold. Then, either (2.4) or (2.5) implies that every solution of (E), (B1) oscillates.*

***Theorem* 3.3** — *Assume that all the assumption of Theorem 2.6 hold, and (3.1) holds. Then every solution of (E), (B1) oscillates.*

4. Oscillation of Boundary Value Problem (E), (B2)

In the following theorems we provide some sufficient conditions for oscillation of all solutions of (E) and $(B2)$ and the following fact will be used. For the following Dirichlit problem in the domain Ω

$$\Delta u + \alpha u = 0 \qquad \text{in } (x, t) \in \Omega \times [t_1, \infty] \qquad \text{... (4.1)}$$

$$u = 0 \qquad \text{on } (x, t) \in \partial\Omega \times [t_1, \infty) \qquad \text{... (4.2)}$$

in which α is a constant. It is well known[22] that the smallest eigenvalue α_1 of problem (4.1), (4.2) is positive and the corresponding eigenfunction $\Phi(x)$ is also positive on $x \in \Omega$. With each solution $u(x, t)$ of problem (1.3), (B2) we associate a function $V(t)$ defined by

$$V(t) = \int_\Omega u(x, t)\Phi(x)dx, \quad t \geq t_1.$$

***Theorem* 4.1** — *Assume that (1.2)-(1.4) and (3.1) hold. Then every solution of (E), (B2) oscillates, if every solution of the delay differential equation*

$$z^{(n)}(t) + [P_1(t) - Q_1(t + \tau - \sigma)] \frac{(t - t_2 - \sigma)^{n-1}}{(n-1)!} z(t - \sigma) = 0, \qquad \text{... (4.3)}$$

or

$$z^{(n)}(t) + [P_1(t) - Q_1(t + \tau - \sigma)] \frac{\sigma^{n-1}}{(n-1)!} z(t - \sigma_1) = 0, \qquad \text{... (4.4)}$$

where

$$P_1(t) = P(t) \exp\left(\alpha_1 \int_{t-\sigma}^{t} a(s)ds \right) \quad \text{and} \quad Q_1(t) = Q(t) \exp\left(\alpha_1 \int_{t-\tau}^{t} a(s)ds \right).$$

oscillates.

Proof : Assume for the sake of contradiction that (E), (B2) has a nonoscillatory solution. Since the negative solution of (1.3), (B2) is also a solution, then without loss of generality we assume that (E), (B2) has a solution $u(x, t) > 0, u(x, t - \sigma)$ and $u(x, t - \sigma)$ in $\Omega \times [t_1, \infty)$ for some $t_1 \geq t_0$. Multiplying (E) by $\Phi(x)$ and integrate with respect to x over the domain Ω, we have

$$\frac{d^n}{dt^n}\left[\int_\Omega u(x, t)\, \Phi(x)dx \right] = a(t) \int_\Omega \Delta u(x, t)\Phi(x)dx - \int_\Omega P(t)u(x, t - \sigma)\Phi(x)dx$$

$$- \int_\Omega Q(t)u(x, t - \tau)\, \Phi(x)dx. \qquad \text{... (4.5)}$$

Using Green's formula and boundary condition (B2), we obtain

$$\int_{\Omega} \Delta u(x,t)\, \Phi(x)dx = \int_{\partial\Omega} \left(\Phi(x)\frac{\partial u}{\partial N} - u\frac{\partial \Phi(x)}{\partial N} \right) dS + \int_{\Omega} u(x,t)\, \Delta\Phi(x)dx$$

$$= -\alpha_1 \int_{\Omega} u(x,t)\Phi(x)dx, \quad t \geq t_1,$$

where dS is the surface element on $\partial\Omega$. From the definitions of $V(t)$, we get

$$V^{(n)}(t) + \alpha_1 a(t)V(t) + P(t)V(t-\sigma) - Q(t)V(t-\tau) \leq 0, \quad t \geq t_1. \qquad \text{... (4.6)}$$

Set

$$V(t) = v(t)\exp\left(-\alpha_1 \int_{t_0}^{t} a(s)ds \right),$$

which is oscillation invariant transformation, reduces inequality (4.6) to

$$v^{(n)}(t) + P_1(t)v(t-\sigma) - Q_1(t)v(t-\tau) \leq 0, \quad t \geq t_1. \qquad \text{... (4.7)}$$

Set

$$z(t) = v(t) - \int_{t}^{\infty} \frac{(s-t)^{n-2}}{(n-2)!} \int_{s-\sigma+\tau}^{s} Q(u)v(s-\tau)du\, ds, \; t \geq t_2.$$

Then

$$z^{(n)}(t) + [P_1(t) - Q_1(t+\tau-\sigma)]\, v(t-\sigma) \leq 0, \quad t \geq t_2$$

The remainder of the proof is similar to the proof the Theorem 2.1 and 2.2 and will be omitted.

Theorem 3.2 follows that the oscillation of (E) and (B2) is equivalent to oscillation of the delay differential eq. (4.3) or (4.4). So one can apply the results in section 2, and obtain some succifient conditions for oscillation of all solution problem (E) and (B2). The details are left to the reader.

5. Oscillation of Boundary Value Problem (E), (B3)

In the following theorem we establish some sufficient conditions for oscillation of all solutions of (E) and (B3).

***Theorem* 5.1** — *Assume that (1.2)-(1.4) and (3.1) hold. If every solution of (2.3) or (2.4) oscillates, then every solution of (E) and (B3) oscillates.*

PROOF : Assume for the sake of contradiction that (E), (B3) has a nooscillatory solution. Since the negative solution of (*E*), (*B*3) is also a solution, then without loss of generality we assume that (E), (B3) has a solution $u(x,t) > 0$, $u(x, t-\sigma)$ and $u(x\, t-\tau)$ in $\Omega \times [t_1, \infty)$ for some $t_1 \geq t_0$. Set

$$U(t) = \int_{\Omega} u(x, t)dx, \quad t \geq t_1,$$

then $U(t) > 0$ for $t \geq t_1$. Integrating the (E) with respect to x over the domain Ω, we have

$$\frac{d^n}{dt^n}\left[\int_{\Omega} u(x, t)\, dx\right] = a(t) \int_{\Omega} \Delta u(x, t)dx$$

$$- \int_{\Omega} P(t)u(x, t-\sigma)dx - \int_{\Omega} Q(t)u(x, t-\tau)dx. \quad \text{... (5.1)}$$

From Green's formula and boundary condition (B3), it follows that

$$\int_{\Omega} \Delta u(x, t)dx = -\int_{\partial\Omega} -vudS \leq 0, \quad t \geq t_1,$$

where dS is the surface element on $\partial\Omega$. Then (5.1) reduces to

$$\frac{d^n}{dt^n}\left[\int_{\Omega} u(x, t)\, dx\right] = -\int_{\Omega} P(t)u(x, t-\sigma)dx - \int_{\Omega} Q(t)u(x, t-\tau)dx, \quad \text{... (5.2)}$$

by using the definiton of $U(t)$, and substitute in (5.1), we have

$$U^{(n)}(t) + P(t)U(t-\sigma) - Q(t)U(t-\tau) \leq 0, \quad t \geq t_1. \quad \text{... (5.3)}$$

The remainder of the proof is now similar to the proof of Theorem 3.1 and will be omitted.

Theorem 3.3 shows that the oscillation of problem (E) and (B3) is equivalent to oscillation of the delay differential eq. (2.3) or (2.4). One can apply the results in Section 2 to obtain some sufficient conditions for oscillation of all the problem (E) and (B3). The details are left to the reader.

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